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THE NUMBER OF (d,k)-HYPERTREES

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Abstract. In this paper, we define and enumerate two tree-like hypergraph structures which we call them (d,k)-trees and d-trees, where $d \geq 2$ and $k > 0$ are integers. These new definitions generalize traditional and HP-hypertrees.

1. Introduction

Among the many interesting types of graphs that have been studied throughout the history of graph theory, the tree is the simplest and the most useful. Therefore, counting the number of trees (both labeled and unlabeled) attracted the attention of many researchers. For a comprehensive survey of techniques and results on tree enumeration, the reader can consult [11]. Besides developing the theory of graphs, mathematicians started to generalize the concept of graphs and trees to higher dimensions in the 1960's, when hypergraphs and hypertrees were introduced. Claude Berge was a pioneer in this field and the reader can consult his books [1], [2] and [3] for an introduction to hypergraphs. A graph G is a finite nonempty set of objects called vertices(the singular is vertex) togerther with a (possibly empty) set of unordered pairs of distinct vertices of G called edges. A *tree* can be defined in several equivalent ways, one of which is inductive : A graph consisting of a single vertex is a tree, and a tree with $n+1$ vertices is obtained from a tree with n vertices by adjoining a new vertex adjacent to one old vertex.

A *labelling* of a graph on n vertices is an assignment of the integers 1 to n to its vertices. Two labelings are *equivalent* if the same pairs of integers are adjacent in both. An old result of enumerative graph theory is that the number of labelled trees on n is n^{n-2}. This was first obtained by Cayley [7] in 1889, but there have since been many proofs. Moon

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[10] gives outlines of ten different approaches. The generalized concept of graphs and trees to higher dimensions has been studied since 1960's, when hypergraphs and hypertrees were introduced [7]. In this paper, we define and enumerate two tree-like hypergraph structures which we call (d, k)-trees and d-trees, where $d \geq 2$ and $k > 0$ are integers. These new definitions generalize traditional and HP-hypertrees.

2. Enumeration of (d, k)-hypertrees

For k fixed, with $1 \leq k \leq d-1$, a (d, k)*-tree* is defined inductively as follows :

(1) A single edge which is a d-subset of vertex set is a (d, k)-tree.

(2) Suppose T is a (d, k)-tree with m vertices, then the hypergraph formed by adding a new edge consisting of $d-k$ new vertices and any k vertices of any edge of T is also a (d, k)-tree.

Note that when $k = d-1$, these are the d-dimensional HP-hypertrees that first appeared in [8] and were subsequently studied in [4], [6] and [11]. Of course, when $d = 2$, then $k = 1$, we have the ordinary trees of graph theory with at least one edge. When $d = 3$ and $k = 1$, they are pure Husimi trees [9]. If we relax condition (2) and allow k to vary $(1 \leq k \leq d-1)$, we obtain a different kind of hypertree called a *d-tree*. For example, the 2-dimensional trees, also called $(3, 2)$-trees, were introduced in [8]. Like trees, they can be defined inductively : The graph consisting of two vertices joined by an edge is a $(3, 2)$-tree, and a $(3, 2)$-tree with $n + 1$ vertices is obtained from a $(3, 2)$-tree with n vertices by adjoining a new vertex to each of two adjacent vertices. It follows from this definition that a $(3, 2)$-tree with n vertices has $2n-3$ edges and $n-2$ triangles(Possible alternative definitions and properties of $(3, 2)-$trees are given by Beineke and Pippert [5]).

We begin by determining exponential generating functions, say egf, for various types of rooted (d, k)-trees. The relationships between these egf's are determined in the following lemmas, concluding with a specific generalization of Cayley's famous n^{n-2} formula. Since in the process for building a (d, k)-tree, we make a new edge everytime when we add $d-k$ vertices to the first d vetices, the number of vertices $|V(T)| = |V|$ and the number of edges $|E(T)| = |E|$ of a (d, k)-tree satisfy the equation

$$|V| - k = |E|(d-k) \tag{1}$$

which reduces to the familiar

$$|V| - 1 = |E| \tag{2}$$

for $(2, 1)$-trees. A *simply rooted* (d, k)-tree has as its root a linearly ordered k-subset of vertices which belongs to exactly one edge. Let y_m be the number of labelled simply rooted (d, k)-trees with m edges, whose vertices are labelled except for the k linearly ordered vertices(K_k that is complete graph of order k) of the root and let y be the egf for these labelled simply rooted (d, k)-trees. Then it follows from (1) that y has the form :

$$y = \sum_{m=1}^{\infty} y_m \frac{x^{m(d-k)}}{(m(d-k))!}. \tag{3}$$

Note that $y_1 = 1$ and $y_2 = \binom{2(d-k)}{(d-k)}(\binom{d}{k} - 1)$. Here is an explanation for this number. Since there are $\binom{d}{k} - 1$ ways for choosing k vertices for a new edge except the root where the d vertices are already labelled, and there are $d + d - k - k = 2(d - k)$ vertices except the root, we need the labelling for the new $d - k$ vertices after $d - k$ vertices in the first edge are already labelled. That is $\binom{2(d-k)}{(d-k)}$. Then we relabel the $d - k$ vertices in the first edge according to their first label.

Lemma 2.1. The egf y for simply rooted (d, k)-trees satisfies the functional equation

$$y = \frac{(e^y)^{\binom{d}{k}-1} x^{d-k}}{(d-k)!}. \tag{4}$$

Proof. Since y is the egf for the labelled simply rooted (d, k)-trees, e^y is the egf for the labelled (d, k)-trees which are rooted at an unlabelled, linearly ordered k-subset of an edge (the k-subset may belong to many edges because there is no labelling for the root and each of these labelled (d, k)-tree is called a special (d, k)-tree). Then, $(e^y)^{\binom{d}{k}-1}$ is the egf of $\binom{d}{k} - 1$ ordered copies of this kind of (d, k)-tree. Now, if we start with an ordered d-set of vertices, then the order of the vertices imposes a natural order on the $\binom{d}{k} - 1$ number of k-subsets of it. Next we use the orders to match up the $\binom{d}{k} - 1$ copies of special (d, k)-trees with the k-subsets, and then, identify the vertices in the (d, k)-trees with the vertices in the corresponding k-subsets using the orders except the leftover k vertices. The result is a tree like structure whose egf is $(e^y)^{\binom{d}{k}-1}$. Now, if we label the unlabelled vertices and remove the order on the d-set except for those vertices in the last k-subset (which has no special (d, k)-tree assigned to it), we get a labelled, simply rooted (d, k)-tree. But its egf is $\frac{(e^y)^{\binom{d}{k}-1} x^{d-k}}{(d-k)!}$ as in (4). □

A *rooted* (d,k)-tree has as its root a k-subset of vertices which belongs to at least one edge. We denote by Y the egf for rooted (d,k)-trees whose vertices are all labelled, even those which belong to the root. We define the coefficients of Y as follows :

$$Y = \sum_{m=1}^{\infty} Y_m \frac{x^{m(d-k)+k}}{(m(d-k)+k)!}\,. \tag{5}$$

So Y_m is the number of these trees with m edges, and we have $Y_1 = \binom{d}{k}$ and $Y_2 = \binom{d}{k}\binom{2(d-k)+k}{d-k,k,d-k}$.

Lemma 2.2. The egf Y for rooted (d,k)-trees can be expressed in terms of the egf y for simply rooted trees as follows :

$$Y = \begin{cases} e^y x, & d=2 \text{ and } k=1; \\ \frac{(e^y-1)x^k}{k!} & \text{otherwise.} \end{cases} \tag{6}$$

Proof. From the proof of Lemma 2.1, $e^y - 1$ is the egf for the labelled (d,k)-trees rooted at an unlabelled, linearly ordered k-subset of an edge (the k-subset may belong to many edges), with at least one edge. If we label the k vertices of the root and remove the order on them, we have a rooted (d,k)-tree. The egf of the rooted (d,k)-trees resulting from the above operations is $\frac{(e^y-1)x^k}{k!}$. When $d=2$ and $k=1$ (i.e. ordinary trees), unlike $d \geq 3$ cases, a single rooted vertex is considered as a rooted tree with no edge in $d=2$ and $k=1$. Hence $Y = e^y x$. □

An *edge rooted* (d,k)-tree has as its root a single edge. Let z be the egf for these. Then the coefficients of z are defined by

$$z = \sum_{m=1}^{\infty} z_m \frac{x^{m(d-k)+k}}{(m(d-k)+k)!}\,. \tag{7}$$

So we have $z_1 = 1$ and $z_2 = \binom{2(d-k)+k}{d-k,k,d-k}$ where z_m is the number of edge rooted (d,k)-trees with m edges.

Lemma 2.3. The egf for edge rooted (d,k)-trees can be expressed in terms of the egf for simply rooted trees as follows :

$$z = \frac{(e^y)^{\binom{d}{k}} x^d}{d!}\,. \tag{8}$$

Proof. We follow the proof of lemma 1, but use ordered $\binom{d}{k}$ copies of the special kind of (d,k)-trees and match them to all the k-subsets of the ordered d-set. Then we fill in labels for all d unlabelled vertices and remove the order on them to obtain an edge rooted (d,k)-tree. The egf of the edge rooted (d,k)-tree resulting from the above operations is $\frac{(e^y)^{\binom{d}{k}} x^d}{d!}$. $\square$

We denote the egf for labelled (d,k)-trees by Z and define its coefficients by

$$Z = \sum_{m=1}^{\infty} Z_m \frac{x^{m(d-k)+k}}{(m(d-k)+k)!}\,. \tag{9}$$

Then $Z_1 = 1$, $Z_2 = \frac{z_2}{2} = \frac{\binom{2(d-k)+k}{d-k,k,d-k}}{2}$ where Z_m is the number of labelled (d,k)-trees with m edges.

Lemma 2.4. The egf for (d,k)-trees is expressed in terms of Y and z as follows :

$$Z = Y - \left(\binom{d}{k} - 1\right) z. \tag{10}$$

Proof. Consider a (d,k)-tree T. By the inductive definition of (d,k)-trees, there is a way to construct T by adding edges one by one. Following the order of construction, we can order the edge set of T. By using the order of the edge set, we can construct a many-to-one mapping from the set of labelled rooted (d,k)-tree obtained from T to the set of labelled edge rooted (d,k)-trees obtained from T as follows : For a labelled rooted (d,k)-tree of T, we map it to a labelled edge rooted (d,k)-tree of T whose rooted edge contains the k-set which form the root of the labelled rooted (d,k)-tree of T. If the k-set is contained in more than one edge of T, we choose the labelled edge rooted (d,k)-tree of T whose rooted edge has the highest priority among the edges which contain the rooted k-set. It's easy to see that the above construction provides a many to one mapping. Futhermore, there are exactly $\binom{d}{k} - 1$ many labelled rooted (d,k)-trees of T mapped to every labelled edge rooted (d,k)-tree of T, except the one which is rooted at the edge of T with highest priority among all the edges of T. For the only exception here, it is mapped by exactly $\binom{d}{k}$ rooted (d,k)-trees. Therefore, we have :

$1 =$ (number of ways to root the (d,k)-tree T)
$-(\binom{d}{k} - 1)$(number of ways to root the (d,k)-tree T at an edge)

or

$$Z_m = Y_m - \left(\binom{d}{k} - 1 \right) z_m \tag{11}$$

where m is the number of edges of a (d,k)-tree. Multiplying both sides of (11) by $\frac{x^{m(d-k)+k}}{(m(d-k)+k)!}$ and summing over $m \geq 1$, we arrive the generating function equation (10). □

From (4), (6), (8) and (11), we obtain the following theorm:

Theorem 2.5. The number of labelled (d,k)-trees of order $n = m(d-k)+k$ is

$$Z_m = Y_m - \left(\binom{d}{k} - 1 \right) z_m \tag{12}$$

where

$$Y_m = \frac{((m\,(d-k)+k)!) \left(m \left(\binom{d}{k} - 1 \right) + 1 \right)^{m-1}}{m!k!\,((d-k)!)^m} \tag{13}$$

and

$$z_m = \frac{\binom{d}{k}\,((m\,(d-k)+k)!) \left(\left(\binom{d}{k} - 1 \right) m + 1 \right)^{m-2}}{d!\,((d-k)!)^{m-1}\,(m-1)!} \tag{14}$$

Proof. Let $n = m(d-k)+k$ and $\alpha = x^{d-k}$. If we write y in terms of α, then from (4), we find

$$y = \frac{(e^y)^{\binom{d}{k}-1} x^{d-k}}{(d-k)!} = \frac{(e^y)^{\binom{d}{k}-1} \alpha}{(d-k)!} = \phi(y)\alpha \tag{15}$$

where $\phi(y) = \frac{(e^y)^{\binom{d}{k}-1}}{(d-k)!}$.

The coefficients of Y are extracted from (6) as follows. Let

$$\sum a_n \frac{x^n}{n!} = \frac{(e^y - 1)}{k!}.$$

Then

$$\frac{(e^y-1)x^k}{k!} = \sum a_n \frac{x^{n+k}}{n!} = \sum a_{n-k} \frac{x^n}{(n-k)!} = \sum \frac{n!a_{n-k}}{(n-k)!} \frac{x^n}{n!}.$$

Hence

$$\left[\frac{x^n}{n!} \right] \frac{(e^y-1)x^k}{k!} = \frac{n!a_{n-k}}{(n-k)!} \tag{16}$$

and

$$\left[\frac{x^{n-k}}{(n-k)!}\right]\frac{e^y-1}{k!}=a_{n-k}. \tag{17}$$

By equations (16) and (17), we have

$$\left[\frac{x^n}{n!}\right]\frac{(e^y-1)x^k}{k!}=\frac{n!}{(n-k)!}\left[\frac{x^{n-k}}{(n-k)!}\right]\frac{e^y-1}{k!}.$$

Hence the coefficients of Y are

$$\begin{aligned}\left[\frac{x^n}{n!}\right]Y=\left[\frac{x^n}{n!}\right]\frac{(e^y-1)x^k}{k!}&=\frac{n!}{(n-k)!}\left[\frac{x^{n-k}}{(n-k)!}\right]\frac{e^y-1}{k!}\\ &=\frac{n!}{(n-k)!k!}\frac{(n-k)!}{m!}\left[\frac{\alpha^m}{m!}\right]e^y-1\\ &=\frac{(m(d-k)+k)!}{m!k!}\left[\frac{\alpha^m}{m!}\right]e^y-1.\end{aligned} \tag{18}$$

The third and fourth equalities above are obtained by similar process as in (16) and (17) above. On applying Lagrange's inversion formula to (15), we have

$$\begin{aligned}\left[\frac{\alpha^m}{m!}\right]e^y&=\left(\frac{d^{m-1}}{dy^{m-1}}\right)_{y=0}(e^y)'\,\alpha^m(y)\\ &=\frac{\left(m\left(\binom{d}{k}-1\right)+1\right)^{m-1}}{((d-k)!)^m}.\end{aligned}$$

Therefore, $Y_m=\frac{((m(d-k)+k)!)\left(m\left(\binom{d}{k}-1\right)+1\right)^{m-1}}{m!k!((d-k)!)^m}$. We can obtain (14) in a similar way. Note that the formula works for $d\geq 3$ with $m\geq 1$ and $d=2$ with $m\geq 0$. For $d\geq 3$ with $m=0$, the number should be 0. $\square$

To obtain Cayley's Theorem as a corollary, we take $d=2$, $k=1$ and $m=n-1$. Then Z_m is the number of labelled trees of order n, and the formulas in the Theorem yield

$$z_m=(n-1)n^{n-2} \tag{19}$$

$$Y_m=n^{n-1} \tag{20}$$

$$Z_m=n^{n-2}. \tag{21}$$

When $k = d-1$, The Theorem provides the formulas for the number of $(d-1)$-dimensional trees found by Beineke and Pippert [6]. In this case $m = n - d + 1$ and formulas (12), (13) and (14) give

$$z_m = \binom{n}{d}\left(d\left((n-d)(d-1)+d\right)^{n-d-1}\right), \tag{22}$$

$$Y_m = \binom{n}{d-1}\left(\left((n-d)(d-1)+d\right)^{n-d}\right), \tag{23}$$

and
(24)

$$Z_m = \binom{n}{d-1}\left((n-d)(d-1)+d\right)^{n-d-1} = \binom{n}{k}\left(k(n-k)+1\right)^{n-k-2}$$

Note that

$$mZ_m = z_m \tag{25}$$

provides the preferred formula

$$Z_m = \frac{\binom{d}{k}\left((m\,(d-k)+k)!\right)\left(\left(\binom{d}{k}-1\right)m+1\right)^{m-2}}{d!m!\left((d-k)!\right)^{m-1}} \tag{26}$$

for Z_m over (12).

Note that Beineke and Pippert found the formula for Z_m with $k = d-1$ by using a special case of (6) and did not extend the egfs to include our equations (8) and (10). We have done this to enable us to enumerate forests in the following paper.

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