Honam Mathematical J. **39** (2017), No. 4, pp. 503–514
https://doi.org/10.5831/HMJ.2017.39.4.503

THE BRIOT-BOUQUET DIFFERENTIAL SUBORDINATION ASSOCIATED WITH VERTICAL STRIP DOMAINS

Young Jae Sim, and Oh Sang Kwon*

Abstract. For real parameters α and β such that $\alpha < 1 < \beta$, we denote by $\mathcal{P}(\alpha, \beta)$ the class of analytic functions p, which satisfy $p(0) = 1$ and $\alpha < \Re\{p(z)\} < \beta$ in $\mathbb{D}$, where $\mathbb{D}$ denotes the open unit disk. Let $\mathcal{A}$ be the class of analytic functions in $\mathbb{D}$ such that $f(0) = 0 = f'(0) - 1$. For $f \in \mathcal{A}$, $\mu \in \mathbb{C} \setminus \{0\}$ and $\nu \in \mathbb{C}$, let $I_{\mu,\nu} : \mathcal{A} \to \mathcal{A}$ be an integral operator defined by

$$I_{\mu,\nu}[f](z) = \left(\frac{\mu + \nu}{z^{\nu}} \int_0^z f^{\mu}(t) t^{\nu-1} \mathrm{d}t \right)^{1/\mu}.$$

In this paper, we find some sufficient conditions on functions to be in the class $\mathcal{P}(\alpha, \beta)$. One of these results is applied to the integral operator $I_{\mu,\nu}$ of two classes of starlike functions which are related to the class $\mathcal{P}(\alpha, \beta)$.

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{P}$ be the class of functions $p \in \mathcal{H}$ such that $p(0) = 1$. For real numbers α and β such that $\alpha < 1 < \beta$, a function $p \in \mathcal{P}$ belongs to the class $\mathcal{P}(\alpha, \beta)$ if p satisfies the inequality $\alpha < \Re\{p(z)\} < \beta$ in $\mathbb{D}$.

We say that f is subordinate to F in $\mathbb{D}$, written as $f \prec F$ (or $f(z) \prec F(z)$) in $\mathbb{D}$, if and only if, $f(z) = F(w(z))$ for some Schwarz function $w(z)$, $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{D}$. It is well-known that $f \prec F$ is

---

Received February 13, 2017. Accepted October 25, 2017.
2010 Mathematics Subject Classification. 30C20, 30C45, 30C80
Key words and phrases. starlike functions, vertical strip domain, integral operator, Briot-Bouquet differential subordination.
This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP; Ministry of Science, ICT & Future Planning) (No. NRF-2017R1C1B5076778)
*Corresponding author

equivalent to $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$, if F is univalent in $\mathbb{D}$ (See [9, p. 36]).

For real numbers α and β such that $\alpha < 1 < \beta$, let us consider a function $p_{\alpha,\beta} : \mathbb{D} \to \mathbb{C}$ defined by

$$p_{\alpha,\beta}(z) = 1 + \mathrm{i}\frac{\beta - \alpha}{\pi} \log\left(\frac{1 - e^{2\pi \mathrm{i}\frac{1-\alpha}{\beta-\alpha}} z}{1 - z}\right). \tag{1}$$

Kuroki and Owa [5] showed that the function $p_{\alpha,\beta}$ maps $\mathbb{D}$ onto the vertical strip domain $\Omega_{\alpha,\beta} := \{w : \alpha < \mathfrak{R}\{w\} < \beta\}$ conformally. Therefore by the notion of subordination, $p \in \mathcal{P}(\alpha, \beta)$ can be represented by $p \prec p_{\alpha,\beta}$ in $\mathbb{D}$. Recently, Sim and Kwon [10] found some sufficient conditions on p so that p belongs to the class $\mathcal{P}(\alpha, \beta)$ which generalize some parts of the results due to Marx [6] and Strohhäcker [11] by using the mapping properties of the function $p_{\alpha,\beta}$.

Now, we introduce an equivalent condition for $p \in \mathcal{P}(\alpha, \beta)$.

Lemma 1.1. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Then, $p \in \mathcal{P}(\alpha, \beta)$ if and only if $\mathfrak{R}\left\{\mathrm{e}^{\mathrm{i}\varphi} q(z)\right\} > 0$ in $\mathbb{D}$, where $q : \mathbb{D} \to \mathbb{C}$ and $\varphi \in \mathbb{R}$ is defined by*

$$q(z) = \exp\left\{\frac{\pi \mathrm{i}}{\beta - \alpha}(p(z) - 1)\right\} \tag{2}$$

and

$$\varphi = -\frac{\alpha + \beta - 2}{2(\beta - \alpha)}\pi, \tag{3}$$

respectively.

Proof. Let $\theta = ((1-\alpha)\pi)/(\beta-\alpha)$ and consider a function $\Lambda_\theta : \mathbb{D} \to \mathbb{C}$ defined by $\Lambda_\theta = (1 - z)/(1 - \mathrm{e}^{2\mathrm{i}\theta} z)$. Since

$$\Lambda_\theta(\mathbb{D}) = \{w \in \mathbb{C} : -\theta < \arg(w) < -\theta + \pi\}$$

and $\mathfrak{R}\{w\mathrm{e}^{\mathrm{i}(\theta - \frac{\pi}{2})}\} > 0$ for $w \in \Lambda_\theta(\mathbb{D})$, we have

$$\mathrm{e}^{\mathrm{i}\varphi}\Lambda_\theta(\mathbb{D}) = \mathrm{e}^{\mathrm{i}(\theta-\pi/2)}\Lambda_\theta(\mathbb{D}) = \{w \in \mathbb{C} : \mathfrak{R}(w) > 0\}. \tag{4}$$

Let $p \in \mathcal{P}(\alpha, \beta)$. Then we have $p \prec p_{\alpha,\beta}$ in $\mathbb{D}$, where $p_{\alpha,\beta}$ is the function given by (1). By the definition of subordination, there exists a Schwarz function $w : \mathbb{D} \to \mathbb{D}$ such that $q(z) = \Lambda_\theta(w(z))$. Therefore $\mathfrak{R}\left\{\mathrm{e}^{\mathrm{i}\varphi} q(z)\right\} > 0$ in $\mathbb{D}$ follows from (4). Conversely, if $\mathfrak{R}\left\{\mathrm{e}^{\mathrm{i}\varphi} q(z)\right\} > 0$ in $\mathbb{D}$, then

$$-\theta = -\frac{\pi}{2} - \varphi < \arg\{q(z)\} < \frac{\pi}{2} - \varphi = -\theta + \pi, \quad z \in \mathbb{D}.$$

That is, $q \prec \Lambda_\theta$ in $\mathbb{D}$ and this is equivalent to $p \in \mathcal{P}(\alpha, \beta)$. □

Let $\mathcal{A}$ be the class of functions $f \in \mathcal{H}$ normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}^*(\alpha, \beta)$ denote the class of functions in $f \in \mathcal{A}$ satisfying $zf'/f \in \mathcal{P}(\alpha, \beta)$. It is trivial that if $\alpha \geq 0$, the functions in $\mathcal{S}^*(\alpha, \beta)$ are starlike.

For $f \in \mathcal{A}$, μ and $\nu \in \mathbb{C}$ with $\mu \neq 0$, define an integral operator $I_{\mu,\nu} : \mathcal{A} \to \mathcal{A}$ defined by

$$F(z) := I_{\mu,\nu}[f](z) = \left(\frac{\mu + \nu}{z^\nu} \int_0^z f^\mu(t) t^{\nu - 1} \mathrm{d}t \right)^{1/\mu}. \tag{5}$$

If we put $\mu = 1$ in (5), then the operator $I_{\mu,\nu}$ reduces the well known one called Bernardi's integral operator [2]. Let us put $p = zF'/F$. Then simple calculations leads us to the Briot-Bouquet differential equation $p(z) + zp'(z)/(\mu p(z) + \nu) = h(z)$, where h is defined by $h = zf'/f$.

Now, for given α and β such that $\alpha < 1 < \beta$, $\mu \in \mathbb{C} \setminus \{0\}$ and $\nu \in \mathbb{C}$, let us define η_* and η^* by

$$\eta_* = \eta_*(\alpha, \beta, \mu, \nu) := \inf \left\{ \tilde{\eta} < 1 : f \in \mathcal{S}^*(\tilde{\eta}, \hat{\eta}) \Rightarrow I_{\mu,\nu}[f] \in \mathcal{S}^*(\alpha, \beta) \right\}$$

and

$$\eta^* = \eta^*(\alpha, \beta, \mu, \nu) := \sup \left\{ \hat{\eta} > 1 : f \in \mathcal{S}^*(\tilde{\eta}, \hat{\eta}) \Rightarrow I_{\mu,\nu}[f] \in \mathcal{S}^*(\alpha, \beta) \right\}.$$

Since the function $p_{\alpha,\beta}$ is a convex univalent function in $\mathbb{D}$, by applying the results on the Briot-Bouquet differential subordinations due to Eenigenburg *et al.* [3] (See also [8]) with $h = p_{\alpha,\beta}$, we can obtain the following implication:

$$\alpha < \Re \left\{ p(z) + \frac{zp'(z)}{\mu p(z) + \nu} \right\} < \beta \Longrightarrow \alpha < \Re \{p(z)\} < \beta, \quad z \in \mathbb{D},$$

whenever $\mu \geq 0$ and $\Re\{\mu\alpha + \nu\} > 0$. Namely, $\eta_* \leq \alpha$ and $\eta^* \geq \beta$. And, in [1], Attiya and Bulboacă generalized this implication by using the Janowski's type.

Motivated by the above, the purpose of this paper is to find the bounds of η_* and η^*. In Section 2, we find some several sufficient conditions on functions to be in the class $\mathcal{P}(\alpha, \beta)$ by using the equivalent condition given in Lemma 1.1 and the methods used in [4]. We inform that some of them are generalizations of the results given in [10]. And, in Section 3, we deal with the Briot-Bouquet differential subordinations associated with the vertical strip domain. The bounds η_* and η^* are obtained by applying this result.

To investigate the results in this paper, we need the following lemma.

Lemma 1.2. [7, p. 24] *Let q be analytic and injective on $\overline{\mathbb{D}} \setminus \mathbf{E}(q)$, where*

$$\mathbf{E}(q) := \left\{ \zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty \right\},$$

and $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus \mathbf{E}(q)$. And let

$$p(z) = a + a_n z^n + \cdots \quad (n \geq 1)$$

be an analytic function in $\mathbb{D}$ with $p(0) = a$. If p is not subordinate to q, then there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D} \setminus \mathbf{E}(q)$ for which

(i) *$p(z_0) = q(\zeta_0)$ and*
(ii) *$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ for some m with $m \geq n \geq 1$.*

2. Some sufficient conditions to be in $\mathcal{P}(\alpha, \beta)$

Theorem 2.1. *Let α and β be real numbers such that $\alpha < 1 < \beta$ and $\alpha + \beta \geq 2$. Let $P : \mathbb{D} \to \mathbb{C}$ be a function such that $\mathfrak{R}\{P(z)\} \geq 0$ for all $z \in \mathbb{D}$. If p is an analytic function in $\mathbb{D}$ with $p(0) = 1$ and*

(6)
$$\alpha - \frac{\beta-\alpha}{\pi}\frac{1+\sin\varphi}{\cos\varphi}\mathfrak{R}\{P(z)\} < \mathfrak{R}\left\{p(z) + P(z)zp'(z)\right\} < \beta + \frac{\beta-\alpha}{\pi}\frac{1-\sin\varphi}{\cos\varphi}\mathfrak{R}\{P(z)\}, \quad z \in \mathbb{D},$$

where φ is given by (3). Then $p \in \mathcal{P}(\alpha, \beta)$.

Proof. Let us consider a function $q : \mathbb{D} \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ defined by (2). Then we have

$$zq'(z) = \frac{\pi \mathrm{i}}{\beta - \alpha} zp'(z)q(z), \quad z \in \mathbb{D}. \tag{7}$$

Let us define a function $k : \mathbb{D} \to \mathbb{C}$ by

$$k(z) = \frac{\mathrm{e}^{\mathrm{i}\varphi} + \overline{\mathrm{e}^{\mathrm{i}\varphi}} z}{1 - z}. \tag{8}$$

We see that q and k are analytic in $\mathbb{D}$ with

$$q(0) = 1, \quad k(0) = \mathrm{e}^{\mathrm{i}\varphi} \quad \text{and} \quad k(\mathbb{D}) = \{w : \mathfrak{R}(w) > 0\}.$$

Now we suppose that $\mathrm{e}^{\mathrm{i}\varphi}q$ is not subordinate to k. Then by Lemma 1.2, there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D} \setminus \{1\}$ such that

$$\mathrm{e}^{\mathrm{i}\varphi} q(z_0) = k(\zeta_0) = \mathrm{i}\rho, \quad \rho \in \mathbb{R} \tag{9}$$

and

$$\mathrm{e}^{\mathrm{i}\varphi} z_0 q'(z_0) = m\zeta_0 k'(\zeta_0), \quad m \geq 1. \tag{10}$$

Indeed we have $\rho \in \mathbb{R}\setminus\{0\}$ since the function q cannot vanish in $\mathbb{D}$. Also we note that

$$\zeta_0 = k^{-1}(\mathrm{e}^{\mathrm{i}\varphi} q(z_0)) = \frac{\mathrm{e}^{\mathrm{i}\varphi} q(z_0) - \mathrm{e}^{\mathrm{i}\varphi}}{\mathrm{e}^{\mathrm{i}\varphi} q(z_0) + \overline{\mathrm{e}^{\mathrm{i}\varphi}}}$$

and

$$\zeta_0 k'(\zeta_0) = -\frac{\rho^2 - 2\rho\sin\varphi + 1}{2\cos\varphi} =: \sigma. \tag{11}$$

From (7), (9) and (10), we have

$$p(z_0) = 1 + \frac{\beta-\alpha}{\pi}(-\varphi + \arg(\mathrm{i}\rho)) - \frac{\beta-\alpha}{\pi}\mathrm{i}\log|\rho| \tag{12}$$

and

$$z_0 p'(z_0) = -\frac{\beta-\alpha}{\pi\rho} m\sigma. \tag{13}$$

Here note that $\mathfrak{R}\{p(z_0)\} = \beta$ when $\rho > 0$ and $\mathfrak{R}\{p(z_0)\} = \alpha$ when $\rho < 0$. For the case $\rho > 0$, we get

$$\begin{aligned}
&\mathfrak{R}\left\{p(z_0) + P(z_0) z_0 p'(z_0)\right\} \\
&= \beta + m\frac{\beta-\alpha}{2\pi\cos\varphi}\frac{\rho^2 - 2\rho\sin\varphi + 1}{\rho}\mathfrak{R}\{P(z_0)\} \\
&\geq \beta + \frac{\beta-\alpha}{2\pi\cos\varphi}\frac{\rho^2 - 2\rho\sin\varphi + 1}{\rho}\mathfrak{R}\{P(z_0)\} \\
&\geq \beta + \frac{\beta-\alpha}{\pi}\frac{1-\sin\varphi}{\cos\varphi}\mathfrak{R}\{P(z_0)\}.
\end{aligned}$$

This is a contradiction to (6). For the case $\rho < 0$, we get

$$\mathfrak{R}\left\{p(z_0) + P(z_0) z_0 p'(z_0)\right\} \leq \alpha - \frac{\beta-\alpha}{\pi}\frac{1+\sin\varphi}{\cos\varphi}\mathfrak{R}\{P(z_0)\}.$$

This also contradicts to (6). Therefore we obtain

$$\mathfrak{R}\left\{\mathrm{e}^{\mathrm{i}\varphi} q(z)\right\} > 0, \quad z \in \mathbb{D},$$

and by Lemma 1.1, we have $p \in \mathcal{P}(\alpha,\beta)$. □

Remark 2.2. *If we put* $P(z) \equiv 1$ *in Theorem 2.1, we can obtain the result in* [10, Theorem 2.2].

Theorem 2.3. *Let α, β be real numbers such that $\alpha < 1 < \beta$ and $\alpha + \beta \geq 2$. If p is an analytic function in $\mathbb{D}$ with $p(0) = 1$ and*

$$
\begin{aligned}
(14)\qquad & -\frac{\beta-\alpha}{2\pi\cos\varphi}(\Psi(z)+2\sin\varphi) < \Re\left\{zp'(z)\right\} \\
& \qquad\qquad < \frac{\beta-\alpha}{2\pi\cos\varphi}(\Psi(z)-2\sin\varphi), \quad z \in \mathbb{D},
\end{aligned}
$$

where

$$\Psi(z) = \exp\left[\frac{\pi}{\beta-\alpha}\Im\left\{p(z)\right\}\right] + \exp\left[-\frac{\pi}{\beta-\alpha}\Im\left\{p(z)\right\}\right]$$

and φ is given by (3)*. Then $p \in \mathcal{P}(\alpha, \beta)$.*

Proof. Suppose that $e^{i\varphi}q$ is not subordinate to k, where q and k are defined by (2) and (8), respectively. Then, as the proof of Theorem 2.1, there exists a $z_0 \in \mathbb{D}$ for which satisfies (12) and (13) with $\rho \in \mathbb{R} \setminus \{0\}$ and σ given by (11).

For the case $\rho > 0$, from (12), we have

$$\rho = \exp\left[-\frac{\pi}{\beta-\alpha}\Im\left\{p(z_0)\right\}\right].$$

So we have

$$
\begin{aligned}
\Re\left\{z_0p'(z_0)\right\} &\geq \frac{\beta-\alpha}{2\pi\cos\varphi}\left(\rho+\frac{1}{\rho}-2\sin\varphi\right) \\
&= \frac{\beta-\alpha}{2\pi\cos\varphi}\left(\Psi(z_0)-2\sin\varphi\right),
\end{aligned}
$$

which is a contradiction to (14). For the case $\rho < 0$, we have

$$\rho = -\exp\left[-\frac{\pi}{\beta-\alpha}\Im\left\{p(z_0)\right\}\right].$$

And this leads us to get

$$
\begin{aligned}
\Re\left\{z_0p'(z_0)\right\} &\leq \frac{\beta-\alpha}{2\pi\cos\varphi}\left(\rho+\frac{1}{\rho}-2\sin\varphi\right) \\
&= -\frac{\beta-\alpha}{2\pi\cos\varphi}\left(\Psi(z_0)+2\sin\varphi\right),
\end{aligned}
$$

which is a contradiction to (14). Hence $e^{i\varphi}q$ is subordinate to k in $\mathbb{D}$ and Lemma 1.1 yields that the function p belongs to the class $\mathcal{P}(\alpha, \beta)$. □

Theorem 2.4. *Let α, β be real numbers such that $\alpha < 1 < \beta$, $\alpha+\beta \geq 2$ and $2\alpha+\beta < 3$. And let $c \in [0,1]$ be given. If p is an analytic function in $\mathbb{D}$ with $p(0)=1$ and*

$$\begin{aligned}
& -\frac{c}{2\cos\varphi}\left|\Im\left\{p(z)\right\}\right| - \frac{(\beta-\alpha)(1+2\sin\varphi)}{2\pi\cos\varphi} \\
& < \Re\left\{zp'(z)\right\} \\
& < \frac{c}{2\cos\varphi}\left|\Im\left\{p(z)\right\}\right| + \frac{(\beta-\alpha)(1-2\sin\varphi)}{2\pi\cos\varphi}, \quad z \in \mathbb{D},
\end{aligned} \tag{15}$$

where φ is given by (3)*. Then*

$$\alpha < \Re\left\{p(z)\right\} < \beta, \quad z \in \mathbb{D}.$$

Proof. First of all, we note that $1+2\sin\varphi > 0$, since $2\alpha+\beta < 3$. Also, we have $1-2\sin\varphi > 0$, since $\varphi < 0$. Therefore the left-side and the right-side in the inequality (15) is negative and positive, respectively, at $z=0$. This means that the inequality (15) is well-defined.

For given $c \in [0,1]$, let us define a function $g : [1,\infty) \to \mathbb{R}$ by

$$g(x) = x^2 - cx\log x - x + 1.$$

Differentiating the function g, we have

$$g'(x) = 2x - c\log x - c - 1 \quad \text{and} \quad g''(x) = 2 - \frac{c}{x}.$$

Since $g'(1) = 1-c \geq 0$ and $g''(x) > 0$ on $[1,\infty)$, $g'(x) \geq 0$ on there. This with $g(1) > 0$ leads us to $g(x) \geq 0$ on $[1,\infty)$. Thus we have

$$x + \frac{1}{x} \geq c\log x + 1, \quad x \in [1,\infty).$$

Also, we have

$$x + \frac{1}{x} \geq -c\log x + 1, \quad x \in (0,1].$$

Therefore we obtain

$$x + \frac{1}{x} \geq c\left|\log x\right| + 1, \quad x \in \mathbb{R}_+ := \left\{x \in \mathbb{R} : x > 0\right\}. \tag{16}$$

Now, suppose that $e^{i\varphi}q$ is not subordinate to k, where q and k are defined by (2) and (8), respectively. Then there exists a $z_0 \in \mathbb{D}$ for which satisfies (12) and (13) with $\rho \in \mathbb{R}\setminus\{0\}$ and σ given by (11).

For the case $\rho > 0$, using the inequality (16), we have

$$\begin{aligned}\Re\left\{z_0 p'(z_0)\right\} &= m\frac{\beta-\alpha}{2\pi\cos\varphi}\frac{\rho^2-2\rho\sin\varphi+1}{\rho} \\ &\geq \frac{\beta-\alpha}{2\pi\cos\varphi}\left(\rho+\frac{1}{\rho}-2\sin\varphi\right) \\ &\geq \frac{\beta-\alpha}{2\pi\cos\varphi}\left(c\left|\log\rho\right|+1-2\sin\varphi\right) \\ &= \frac{\beta-\alpha}{2\pi\cos\varphi}\left(\frac{c\pi}{\beta-\alpha}\left|\Im\left\{p(z_0)\right\}\right|+1-2\sin\varphi\right),\end{aligned}$$

which is a contradiction to (15). Similar calculations leads us to get

$$\Re\left\{z_0 p'(z_0)\right\} \leq -\frac{\beta-\alpha}{2\pi\cos\varphi}\left(\frac{c\pi}{\beta-\alpha}\left|\Im\left\{p(z_0)\right\}\right|+1+2\sin\varphi\right),$$

when $\rho < 0$. This is also a contradiction to (15). Therefore we obtain $e^{i\varphi}q \prec k$ in $\mathbb{D}$ and $p \in \mathcal{P}(\alpha,\beta)$. □

Theorem 2.5. *Let α, β be real numbers such that $\alpha < 1 < \beta$ and $\alpha+\beta \geq 2$. If p is an analytic function in $\mathbb{D}$ with $p(0) = 1$, $p'(z) \neq 0$ in $\mathbb{D}$ and*

(17)
$$-\frac{(\beta-\alpha)(1+\sin\varphi)}{\alpha\pi\cos\varphi} < \left[\Re\left\{\frac{p(z)}{zp'(z)}\right\}\right]^{-1} < \frac{(\beta-\alpha)(1-\sin\varphi)}{\beta\pi\cos\varphi}, \quad z\in\mathbb{D},$$

where φ is given by (3)*. Then*

$$\alpha < \Re\left\{p(z)\right\} < \beta, \quad z \in \mathbb{D}.$$

Proof. Suppose that $e^{i\varphi}q$ is not subordinate to k, where q and k are defined by (2) and (8), respectively. Then, as the proof of Theorem 2.1, there exists a $z_0 \in \mathbb{D}$ for which satisfies (12) and (13) with $\rho \in \mathbb{R}\setminus\{0\}$ and σ given by (11).

For the case $\rho > 0$, we have

$$\Re\left\{\frac{p(z_0)}{z_0p'(z_0)}\right\} = \frac{\Re\left\{p(z_0)\right\}}{z_0p'(z_0)} = \frac{2\beta\pi\rho\cos\varphi}{m(\beta-\alpha)(\rho^2-2\rho\sin\varphi+1)}. \tag{18}$$

Applying the inequalities $m \geq 1$ and $(\rho^2-2\rho\sin\varphi+1)/\rho \geq 2(1-\sin\varphi)$ to the equation (18), we obtain

$$\Re\left\{\frac{p(z_0)}{z_0p'(z_0)}\right\} \leq \frac{\beta\pi\cos\varphi}{(\beta-\alpha)(1-\sin\varphi)},$$

which is a contradiction to (17). For the case $\rho < 0$, similar calculations leads us to get

$$\Re\left\{\frac{p(z_0)}{z_0 p'(z_0)}\right\} \geq -\frac{\alpha\pi\cos\varphi}{(\beta-\alpha)(1+\sin\varphi)},$$

which is a contradiction to (17). And this completes the proof of Theorem 2.5. □

3. The Briot-Bouquet differential subordinations associated with the class $\mathcal{P}(\alpha,\beta)$

Theorem 3.1. *Let α, β, μ and ν be real numbers such that $\alpha < 1 < \beta$, $\mu \geq 0$ and $\mu\alpha + \nu > 0$. Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Let γ and δ be real constants such that*

$$\gamma = \min_{\rho>0} \frac{\rho^2 + 2\rho\sin\varphi + 1}{\rho\left((\mu\alpha+\nu)^2 + \left(\frac{(\beta-\alpha)\mu}{\pi}\right)^2 (\log\rho)^2\right)} \tag{19}$$

and

$$\delta = \min_{\rho>0} \frac{\rho^2 - 2\rho\sin\varphi + 1}{\rho\left((\mu\beta+\nu)^2 + \left(\frac{(\beta-\alpha)\mu}{\pi}\right)^2 (\log\rho)^2\right)}, \tag{20}$$

where φ is given by (3)*. If*

$$\alpha - \frac{\gamma(\beta-\alpha)(\mu\alpha+\nu)}{2\pi\cos\varphi} < \Re\left\{p(z) + \frac{zp'(z)}{\mu p(z)+\nu}\right\} < \beta + \frac{\delta(\beta-\alpha)(\mu\beta+\nu)}{2\pi\cos\varphi}, \quad z\in\mathbb{D},$$

then $p \in \mathcal{P}(\alpha,\beta)$.

Proof. First of all, we show the existence of values γ and δ. For these, let us define two functions l_1 and $l_2 : (0,\infty) \to \mathbb{R}$ by

$$l_1(\rho) = \frac{\rho^2 + 2\rho\sin\varphi + 1}{\rho\left((\mu\alpha+\nu)^2 + \left(\frac{(\beta-\alpha)\mu}{\pi}\right)^2 (\log\rho)^2\right)}$$

and

$$l_2(\rho) = \min_{\rho>0} \frac{\rho^2 - 2\rho\sin\varphi + 1}{\rho\left((\mu\beta+\nu)^2 + \left(\frac{(\beta-\alpha)\mu}{\pi}\right)^2 (\log\rho)^2\right)},$$

respectively. Since $(\rho^2+2\rho\sin\varphi+1)/\rho \geq 2(1+\sin\varphi)$ for all $\rho>0$, we have $l_1(\rho)>0$ for all $\rho>0$. Applying L'Hospital's rule twice, we get

$$\lim_{\rho\to\infty} l_1(\rho) = \frac{\pi^2}{(\beta-\alpha)^2\mu^2}\lim_{\rho\to\infty}\frac{\rho}{1+\log\rho} = \infty. \tag{21}$$

And from the equation

$$\begin{aligned}
&\rho\left\{(\mu\alpha+\nu)^2+\left(\frac{(\beta-\alpha)\mu}{\pi}\right)^2(\log\rho)^2\right\}\\
&= -2\left(\frac{(\beta-\alpha)\mu}{\pi}\right)^2(\rho\log\rho) \to 0^+, \quad \text{as } \rho\to 0^+,
\end{aligned}$$

we have

$$\lim_{\rho\to 0^+} l_1(\rho) = \infty. \tag{22}$$

Since the function l_1 is continuous on $(0,\infty)$, it follows from (21) and (22) that the constant $\gamma=\min_{\rho>0} l_1(\rho)$ exists. Similarly, the function l_2 also has a minimum δ on $(0,\infty)$.

Now, suppose that $e^{i\varphi}q$ is not subordinate to k, where q and k are defined by (2) and (8), respectively. Then, as the proof of Theorem 2.1, there exists a $z_0\in\mathbb{D}$ for which satisfies (12) and (13) with $\rho\in\mathbb{R}\setminus\{0\}$ and σ given by (11).

For the case $\rho>0$, we have

$$\begin{aligned}
\Re\left\{p(z_0)+\frac{z_0p'(z_0)}{\mu p(z_0)+\nu}\right\} &= \beta + m\frac{(\beta-\alpha)(\mu\beta+\nu)}{2\pi\cos\varphi} l_2(\rho)\\
&\geq \beta + \frac{\delta(\beta-\alpha)(\mu\beta+\nu)}{2\pi\cos\varphi},
\end{aligned}$$

which is a contradiction to the hypothesis. For the case $\rho<0$, put $\tilde{\rho}=-\rho>0$. Then,

$$\begin{aligned}
\Re\left\{p(z_0)+\frac{z_0p'(z_0)}{\mu p(z_0)+\nu}\right\} &= \alpha - m\frac{(\beta-\alpha)(\mu\alpha+\nu)}{2\pi\cos\varphi} l_1(\tilde{\rho})\\
&\leq \alpha - \frac{\gamma(\beta-\alpha)(\mu\alpha+\nu)}{2\pi\cos\varphi},
\end{aligned}$$

which is a contradiction to the hypothesis and this completes the proof of Theorem 3.1. □

Remark 3.2. *If we put $\mu=1$ and $\nu=0$ in Theorem 3.1, we can obtain the result in* [10, Theorem 2.4].

Corollary 3.3. *Let α, β, μ and ν be real numbers such that $\alpha < 1 < \beta$, $\mu \geq 0$ and $\mu\alpha + \nu > 0$. And let γ and δ be real constants given by* (19) *and* (20)*, respectively. If $f \in \mathcal{S}^*(A, B)$, where*

$$A = \alpha - \frac{\gamma(\beta - \alpha)(\mu\alpha + \nu)}{2\pi \cos\varphi} \quad \text{and} \quad B = \beta + \frac{\delta(\beta - \alpha)(\mu\beta + \nu)}{2\pi \cos\varphi}$$

with φ given by (3)*, then the function $I_{\mu,\nu}[f]$ given by* (5) *is in the class $\mathcal{S}^*(\alpha, \beta)$. That is, $\eta_* \leq A$ and $\eta^* \geq B$.*

References

[1] A. A. Attiya and T. Bulboacă, *A general theorem associated with the Briot-Bouquet differential subordination*, J. Comput. Anal. Appl. **16(1)** (2014), 722-730.

[2] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135** (1969), 429-446.

[3] P. Eenigenburg, P. T. Mocanu, S. S. Miller and M. O. Reade, *On a Briot-Bouquet differential subordination*, General Inequalities 3, International Series of Numerical Mathematics **64** (1983), 339-348.

[4] I. H. Kim and N. E. Cho, *Sufficient conditions for Carathéodory functions*, Computers and Mathematics with Applications **59** (2010), 2067-2073.

[5] K. Kuroki and S. Owa, *Notes on new class for certain analytic functions*, Adv. Math. Sci. J. **1(2)** (2012), 127-131.

[6] A. Marx, *Untersuchungen über schlichte Abildung*, Math. Ann. **107** (1932/33), 40-67.

[7] S. S. Miller and P. T. Mocanu, *Differential Subordination, Theory and Application*, Marcel Dekker, Inc., New York, Bassel, 2000.

[8] S. S. Miller and P. T. Mocanu, *Univalent solutions of Briot-Bouquet differential equations*, J. Differential Equations **56** (1985), 297-309.

[9] Ch. Pommerenke, *Univalent functions*, Vandenhoeck und Ruprecht, 1975.

[10] Y. J. Sim and O. S. Kwon, *Some results on analytic functions associated with vertical strip domain*, Proc. Jangjeon Math. Soc. **19(4)** (2016), 653-661.

[11] E. Strohhäcker, *Beiträge zur Theorie der schlichten Funktionen*, Math. Z. **37** (1933), 356-380.

Young Jae Sim
Department of Mathematics, Kyungsung University,
309, Suyeong-ro, Nam-gu, Busan, Korea.
E-mail: yjsim@ks.ac.kr

Oh Sang Kwon
Department of Mathematics, Kyungsung University,

309, Suyeong-ro, Nam-gu, Busan, Korea.
E-mail: oskwon@ks.ac.kr