Honam Mathematical J. **39** (2017), No. 4, pp. 515–525
https://doi.org/10.5831/HMJ.2017.39.4.515

NORMAL SECTION CURVES AND CURVATURES

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Abstract. In this paper, we study planar normal section curves. We have interpreted curvatures of normal section curves. On the other hand we have investigated sufficient and necessary conditions for a normal section curve to be biharmonic.

1. Introduction

Surfaces and submanifolds with planar normal sections and their classifications were first studied by Chen [4]. Chen described the normal section curve as follows: Let M be an n-dimensional submanifold of m-dimensional Euclidean space E^m. For any non-zero tangent vector t to M at p, the vector t and normal space $T_p^\perp M$ determine an $(m-n+1)$-dimensional vector space $E(p,t)$ in E^m. The intersection of M and $E(p,t)$ is a curve γ which is called a normal section curve. In general the normal section curve γ is a twisted space curve in $E(p,t)$. A submanifold M is said to have pointwise k-planar normal sections, $(2 \leq k \leq m-n)$, if each normal section γ at p satisfies

$$\gamma' \wedge \gamma'' \wedge ... \wedge \gamma^{(k+1)} = 0. \tag{1}$$

Chen [5, 6, 8, 9], Kim [16, 17], Li [7] classified submanifolds by using curves of normal section. Also, Kim [17] investigated semi-Riemannian case of such studies in 1980s by assuming γ is being timelike or spacelike.

In [12], the first author, Şahin and Güneş studied lightlike surfaces of Minkowski 3-space having degenerate or non-degenerate planar normal sections and proved that every lightlike surface of Minkowski 3-space has degenerate planar normal sections. In [13] the necessary and sufficient conditions for half-lightlike submanifolds of R_2^4 to have degenerate or

Received May 24, 2017. Accepted September 25, 2017.
2010 Mathematics Subject Classification. 53A04, 53C40, 58E20.
Key words and phrases. Normal section curve, planar normal section, biharmonic map, biharmonic curve.
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non degenerate planar normal sections are obtained.

On the other hand biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. Recently, biharmonic functions on Riemannian manifolds were studied by Caddeo [2], Caddeo and Vanhecke [3], and Sario, Nakai and Wang [20].

Let $C^\infty(M,N)$ denotes the space of smooth maps $\Psi:(M,g)\to(N,h)$ between two Riemannian manifolds. A map $\Psi\in C^\infty(M,N)$ is called harmonic if it is a critical point of the energy functional

$$E:C^\infty(M,N)\to R,\quad E(\Psi)=\frac{1}{2}\int_M |d\Psi|^2 v_g$$

and harmonicity of Ψ is characterized by the vanishing of the tension field $\tau(\Psi)=trace\nabla d\Psi$, where ∇ is a connection induced from the Levi-Civita connection ∇^M of M and the pull-back connection ∇^Ψ. As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by Eells and Sampson in [11]. Biharmonic maps $\Psi:(M,g)\to(N,h)$ between Riemannian manifolds are the critical points of the bienergy functional

$$E_2(\Psi)=\frac{1}{2}\int_M |\tau(\Psi)|^2 v_g.$$

The first variation formula for the bienergy which is derived in [14, 15] shows that the Euler-Lagrange equation for the bienergy is

$$\tau_2(\Psi)=-J(\tau(\Psi))=-\Delta\tau(\Psi)-traceR^N(d\Psi,\tau(\Psi))d\Psi=0,$$

where $\Delta=-trace(\nabla^\Psi\nabla^\Psi-\nabla^\Psi_\nabla)$ is the rough Laplacian on the sections of $\Psi^{-1}TN$ and $R^N(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$ is the curvature operator on N. From the expression of the bitension field τ_2, it is clear that a harmonic map is automatically a biharmonic map. So non-harmonic biharmonic maps which are called proper biharmonic maps are more interesting.

In a different setting, B. Y. Chen [10] defined biharmonic submanifolds $M\subset E^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H=0$, where Δ is the rough Laplacian, and stated the following

- Conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, that is minimal.

If the definition of biharmonic maps is applied to Riemannian immersions into Euclidean space, the notion of Chen's biharmonic submanifold is obtained, so the two definitions agree.

The non-existence theorems for the case of non-positive sectional curvature codomains, as well as the

- Generalized Chen's conjecture: Biharmonic submanifolds of a manifold N with $Riem^N \leq 0$ are minimal,

encouraged the study of proper biharmonic submanifolds, that is submanifolds such that the inclusion map is a biharmonic map in spheres or another non-negatively curved spaces.

Of course, the first and easiest examples can be found by looking at differentiable curves in a Riemannian manifold. Obviously geodesics are biharmonic. Non-geodesic biharmonic curves are called proper biharmonic curves.

Until recently while studying submanifolds with planar normal sections the authors mentioned above used Gauss and Weingarten formulas to characterize such submanifolds. Motivated by these studies, in this paper with another aspect we characterize normal section curves by using the intrinsic properties, namely curvatures. Also we find necessary and sufficient conditions for a normal section curve to be biharmonic.

2. Preliminaries

2.1. Submanifolds with Planar Normal Section

Let M be an n-dimensional submanifold in an m-dimensional E^m Euclidean space. Let ∇ and $\bar{\nabla}$ be the covariant differentiations of M and E^m, respectively. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\bar{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where $X, Y \in \chi(M)$, $\xi \in \chi^{\perp}(M)$. It is well known that $h(X,Y)$ is a normal-bundle-valued symmetric tensor of type $(0,2)$ and $-A_\xi X$ and $D_X \xi$ denote the tangential and normal components of $\bar{\nabla}_X \xi$, respectively, as well as the second fundamental form related to the shape operator by

$$\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle\,,$$

where $\langle , \rangle$ denotes the scalar product in E^m.

For the second fundamental form h, the covariant derivative of h is defined by

$$(2)\qquad \left(\bar{\nabla}_X h\right)(Y,Z) = D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

$\bar{\nabla}h$ is a normal-bundle-valued tensor of type $(0,3)$ satisfying the following equation of Codazzi given by

$$\left(\bar{\nabla}_X h\right)(Y,Z) = \left(\bar{\nabla}_Y h\right)(X,Z).$$

Moreover, we also have from (2) that

$$\left(\bar{\nabla}_X h\right)(Y,Z) = \left(\bar{\nabla}_X h\right)(Z,Y).$$

Theorem 2.1. [5] *Let M be an n-dimensional $(n \geq 2)$ submanifold in an E^m Euclidean m-space. Then, M has pointwise planar normal sections if and only if the following relation is satisfied for any vector t tangent to M:*

$$\left(\bar{\nabla}_t h\right)(t,t) \wedge h(t,t) = 0.$$

Theorem 2.2. [5] *Let M be an n-dimensional $(n \geq 2)$ submanifold in an Euclidean m-space E^m. Then the following statements are equivalent.*

(a) The second fundamental form satisfy $\left(\bar{\nabla}_t h\right)(t,t) = 0$ for t tangent to M,

(b) The second fundamental form is parallel, i.e. $\bar{\nabla}h = 0$,

(c) M has pointwise planar normal sections and each normal section at p has one of its vertices at p.

Theorem 2.3. [5] *An n-dimensional $(n \geq 2)$ submanifold in an Euclidean m-space E^m has planar geodesic if and only if it has planar normal sections of the same constant curvatures.*

2.2. Biharmonic maps between Riemannian manifolds

Let (M,g) and (N,h) be Riemannian manifolds and $\Psi : (M,g) \to (N,h)$ be a smooth map. The tension field of Ψ is given by $\tau(\Psi) = trace \nabla d\Psi$, where $\nabla d\Psi$ is the second fundamental form of Ψ defined by $\nabla d\Psi(X,Y) = \nabla^{\Psi}_X d\Psi(Y) - d\Psi(\nabla^M_X Y)$, $X, Y \in \Gamma(TM)$. For any compact domain $\Omega \subseteq M$, the bienergy is defined by

$$E_2(\Psi) = \frac{1}{2}\int_{\Omega} |\tau(\Psi)|^2 v_g.$$

Then a smooth map Ψ is called biharmonic map if it is a critical point of the bienergy functional for any compact domain $\Omega \subseteq M$. For the bienergy we have the following first variation formula:

$$(3)\qquad \frac{d}{dt} E_2(\Psi_t;\Omega)|_{t=0} = \int_{\Omega} < \tau_2(\Psi), w > v_g,$$

where v_g is the volume element, w is the variational vector field associated to the variation $\{\Psi_t\}$ of Ψ and

$$\tau_2(\Psi) = -J(\tau_2(\Psi)) = -\Delta^{\Psi}\tau(\Psi) - traceR^N(d\Psi, \tau(\Psi))d\Psi. \tag{4}$$

$\tau_2(\Psi)$ is called bitension field of Ψ. Here Δ^{Ψ} is the rough Laplacian on the sections of the pull-back bundle $\Psi^{-1}TN$ which is defined by

$$\Delta^{\Psi}V = -\sum_{i=1}^{m}\{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}V - \nabla^{\Psi}_{\nabla^M_{e_i}e_i}V\}, \quad V \in \Gamma(\Psi^{-1}TN), \tag{5}$$

where ∇^{Ψ} is the pull-back connection on the pull-back bundle $\Psi^{-1}TN$ and $\{e_i\}_{i=1}^m$ is an orthonormal frame on M.

Let M be a Riemannian manifold and $\gamma : I \to M$ be a differentiable curve parameterized by arc length. By using the definition of the tension field we have

$$\tau(\gamma) = \nabla^{\gamma}_{\frac{\partial}{\partial s}} d\gamma(\frac{\partial}{\partial s}) = \nabla_T T, \tag{6}$$

where $T = \gamma'$. Consider a smooth variation of γ, that is a smooth map $\beta : I \times (-\delta, \delta) \to M$, $\beta(s,t) = \gamma_t(s)$, such that $\gamma_0 = \gamma$ (see [19]). Then from (3)-(5) we can write the first variation formula for the bienergy functional of γ:

$$\begin{aligned}\frac{d}{dt}E_2(\gamma_t; I)|_{t=0} &= \int_I < \nabla^{\beta}_{\frac{\partial}{\partial s}}\nabla^{\beta}_{\frac{\partial}{\partial s}} d\beta(\frac{\partial}{\partial t}) - \nabla^{\beta}_{\nabla^I_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}}\frac{\partial \beta}{\partial t}, \tau(\gamma_t) > \quad |_{t=0}ds \\ &+ \int_I < R^M(\frac{\partial \beta}{\partial t}, d\beta(\frac{\partial}{\partial s}))d\beta(\frac{\partial}{\partial s}), \tau(\gamma_t) > \quad |_{t=0}ds,\end{aligned}$$

where ∇^I denotes the connection on I. Since $\nabla^I_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} = 0$ and the Laplace operator is self-adjoint, then we have

$$\frac{d}{dt}E_2(\gamma_t; D)|_{t=0} = \int_I < \nabla^3_T T - R^M(T, \nabla_T T)T, w > \quad ds.$$

Here w is the variation vector field of γ and ∇ denotes the connection on M. In this case biharmonic equation for the curve γ reduces to

$$\nabla^3_T T - R(T, \nabla_T T)T = 0, \tag{7}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (7) (see also [18]).

3. Normal Section Curves and Curvatures on Riemannian Manifolds

Let $\gamma : I \subset R \to E^m$ be a unit speed curve in E^m. The curve is called Frenet curve of osculating order r if its higher order derivatives $\gamma'(s)$, $\gamma''(s)$, $\gamma'''(s)$, ..., $\gamma^{(r)}(s)$ are linearly independent and $\gamma'(s)$, $\gamma''(s)$, $\gamma'''(s)$, ..., $\gamma^{(r+1)}(s)$ are linearly dependent, for all $s \in I$. For each Frenet curve of osculating order r one can associate an orthonormal r-frame $v_1, v_2, ..., v_r$ along γ which is called the Frenet frame and $(r-1)$ functions $\kappa_1, \kappa_2, ..., \kappa_{r-1} : I \to R$, namely Frenet curvatures, such that the Frenet formulas defined by in the usual way

$$\begin{aligned} T'(s) &= v_1'(s) = \kappa_1(s)v_2(s), \\ v_2'(s) &= -\kappa_1(s)T(s) + \kappa_2(s)v_3(s), \\ &\vdots \\ v_i'(s) &= -\kappa_{i-1}(s)v_{i-1}(s) + \kappa_i(s)v_{i+1}(s), \\ v_{i+1}'(s) &= -\kappa_i(s)v_i(s). \end{aligned} \tag{8}$$

Let M be a differentiable n-dimensional submanifold in $(n + r)$-dimensional Euclidean space E^{n+r}. If each normal section γ of M is a Frenet curve of osculating order r then M is said to have r-planar normal sections. For every normal sections γ of M if it is a W-curve of rank r in M then M is called weak helical submanifold of order r. If each r-planar normal section is a geodesic then the submanifold M is said to have geodesic r-planar normal sections. For every geodesic normal sections of M if it is a W-curve of rank r in M then M is called weak geodesic helical submanifold of order r [1].

Assume that γ is a normal section curve of a differentiable n-dimensional submanifold M in E^{n+2}. Then by using (8) we write

$$\begin{aligned} \gamma'(s) &= T(s) = v_1(s), \\ \gamma''(s) &= \kappa_1(s)v_2(s), \\ \gamma'''(s) &= \kappa_1^2(s)T(s) - \kappa_1'(s)v_2(s). \end{aligned} \tag{9}$$

A submanifold M is said to have 2-planar normal sections if each normal section γ at p satisfies

$$\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s) = 0. \tag{10}$$

Hence, from (9) and (10) we have

Theorem 3.1. *Let M be an n-dimensional submanifold in E^{n+2}. Then M has 2-planar normal sections if and only if the normal section curve γ of M is a geodesic.*

Corollary 3.2. *An n-dimensional submanifold M in E^{n+2} has geodesic 2-planar normal sections.*

Next, we suppose that M has 3-planar normal sections. Then by using Frenet formulas given by (8), we get

$$\begin{aligned}
\gamma'(s) &= T(s) = v_1(s), \\
\gamma''(s) &= \kappa_1(s)v_2(s), \\
\gamma'''(s) &= -\kappa_1^2(s)T(s) + \kappa_1'(s)v_2(s) + \kappa_1(s)\kappa_2(s)v_3(s), \\
\gamma^{(iv)}(s) &= \left(-3\kappa_1(s)\kappa_1'(s)\right)T(s) + (\kappa_1''(s) - \kappa_1^3(s) - \kappa_1(s)\kappa_2^2(s))v_2(s) \\
&\quad +(2\kappa_1'(s)\kappa_2(s) + \kappa_1(s)\kappa_2'(s))v_3(s).
\end{aligned}$$

Then we obtain

$$\left\{\begin{array}{c}
T(s) \wedge (\kappa_1(s)v_2(s)) \wedge \begin{pmatrix} -\kappa_1^2(s)T(s) + \kappa_1'(s)v_2(s) \\ +\kappa_1(s)\kappa_2(s)v_3(s) \end{pmatrix} \\
\wedge \begin{pmatrix} (-3\kappa_1(s)\kappa_1'(s))\,T(s) \\ +(\kappa_1''(s) - \kappa_1^3(s) - \kappa_1(s)\kappa_2^2(s))v_2(s) \\ +(2\kappa_1'(s)\kappa_2(s) + \kappa_1(s)\kappa_2'(s))v_3(s) \end{pmatrix}
\end{array}\right. = 0,$$

which implies

$$\begin{aligned}
0 = &-\kappa_1^3(s)\left(2\kappa_1'(s)\kappa_2(s) + \kappa_1(s)\kappa_2'(s)\right)T(s) \\
&+\kappa_1(s)\kappa_1'(s)\left(2\kappa_1'(s)\kappa_2(s) + \kappa_1(s)\kappa_2'(s)\right)v_2(s) \\
&-\left(\kappa_1(s)\kappa_1'(s)\left(\kappa_1''(s) - \kappa_1^3(s) - \kappa_1(s)\kappa_2^2(s)\right) + 3\kappa_1^4(s)\kappa_1'(s)\right)v_3(s).
\end{aligned}$$

Therefore, we have

$$(11)\quad \left\{\begin{array}{c}
\kappa_1^3(s)\left(2\kappa_1'(s)\kappa_2(s) + \kappa_1(s)\kappa_2'(s)\right) = 0, \\
\kappa_1(s)\kappa_1'(s)\left(2\kappa_1'(s)\kappa_2(s) + \kappa_1(s)\kappa_2'(s)\right) = 0, \\
\kappa_1(s)\kappa_1'(s)\left(\kappa_1''(s) - \kappa_1^3(s) - \kappa_1(s)\kappa_2^2(s)\right) + 3\kappa_1^4(s)\kappa_1'(s) = 0,
\end{array}\right.$$

for all $s \in I$.

Now, we investigate following cases.

CASE 1. $\kappa_1 = 0$. Then it is obvious that above equations are provided. That is, 3-planar normal section is a geodesic curve.

So we have

Corollary 3.3. *Let M be an n-dimensional submanifold of E^{n+3}. Then M has 3-planar normal sections with $\kappa_1 = 0$ if and only if normal section curve γ of M is a geodesic.*

CASE 2. κ_1 =constant$\neq 0$. From the first equation in (11), we get κ_2 =constant. In particular, if $\kappa_2 = 0$, then the normal section curve is a circle. On the other hand, if κ_2 is a nonzero constant then the normal section curve is a helix.

So we have

Corollary 3.4. *Let M be an n-dimensional submanifold of E^{n+3} and the curvature of the normal section curve be a nonzero constant. Then M has 3-planar normal sections if and only if normal section curve γ of M is either a circle or a helix.*

CASE 3. $\kappa_1 \neq$ constant. From the first and the second equations of (11), we have

$$\kappa_1^2 \kappa_2 = c, \tag{12}$$

where c is a constant. Then, if (12) is used in the last equation of (11), we get the following second order nonlinear ordinary differential equation

$$\kappa_1^3 \kappa_1'' + 2\kappa_1^6 - c^2 = 0. \tag{13}$$

Corollary 3.5. *Let M be an n-dimensional submanifold of E^{n+3} and γ be the normal section curve of M with $\kappa_1 \neq$ constant and $\kappa_2 = 0$. Then M has 3-planar normal sections if and only if κ_1 is a solution of*

$$\kappa_1'' + 2\kappa_1^3 = 0.$$

Corollary 3.6. *Let M be an n-dimensional submanifold of E^{n+3} and γ be the normal section curve of M with $\kappa_1 \neq$ constant and κ_2 =constant$\neq$ 0. Then M can not have 3-planar normal sections.*

Theorem 3.7. *Let M be an n-dimensional submanifold of E^{n+3} and γ be the normal section curve of M with $\kappa_1 \neq$ constant and $\kappa_1^2 \kappa_2 =$ c =constant. Then M has 3-planar normal sections if and only if κ_1 is a solution of second order nonlinear ordinary differential equation given by*

$$\kappa_1^3 \kappa_1'' + 2\kappa_1^6 - c^2 = 0.$$

Finally, we examine conditions for a normal section curve γ being biharmonic in a Riemannian space form $M^n(k)$. It is well-known that in a Riemannian space form, the Riemannian curvature tensor is of the form

$$R(X,Y)Z = k\{< Y,Z > X - < X,Z > Y\},$$

for any $X,Y,Z \in \chi(M)$. Then we get

$$R(T,\nabla_T T)T = -k\kappa_1 v_2.$$

From (7) and the last equation above, the biharmonic equation of γ is given by

$$\begin{aligned} 0 &= \nabla_T^3 T - R(T,\nabla_T T)T \\ &= -3\kappa_1\kappa_1' T + (-\kappa_1^3 + \kappa_1'' - \kappa_1\kappa_2^2 + k\,\kappa_1)v_2 \\ &\quad +(2\kappa_1'\kappa_2 + \kappa_1\kappa_2')v_3 + (\kappa_1\kappa_2\kappa_3)\,v_4. \end{aligned}$$

Hence we find γ is biharmonic if and only if

$$\left\{ \begin{array}{c} \kappa_1\kappa_1' = 0, \\ -\kappa_1^3 + \kappa_1'' - \kappa_1\kappa_2^2 + k\,\kappa_1 = 0, \\ 2\kappa_1'\kappa_2 + \kappa_1\kappa_2' = 0, \\ \kappa_1\kappa_2\kappa_3 = 0. \end{array} \right. \tag{14}$$

If we search non-geodesic solitions, namely κ_1 =constant$\neq 0$, from (14) we find either $\kappa_1 = \pm\sqrt{k}$, $\kappa_2 = 0$ (circle), or $\kappa_1^2 + \kappa_2^2 = k$ (helix), (see also, [18]).

Then we give

Theorem 3.8. *Let N be a submanifold of $M^n(k)$. Then N has 3-planar normal sections if and only if normal section curves satisfying $\kappa_1^2 + \kappa_2^2 = k$, $k \geq 0$, are biharmonic.*

Acknowledgement. This paper was supported by Adıyaman University, under Scientific Research Project No. EFMAP/2016-0002. The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.

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