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MORE GENERALIZED FUZZY SUBSEMIGROUPS/IDEALS IN SEMIGROUPS

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Abstract. The main motivation of this article is to generalized the concept of fuzzy ideals, (α,β)-fuzzy ideals, $(\in,\in\vee q_k)$-fuzzy ideals of semigroups. By using the concept of q_K^{δ}-quasi-coincident of a fuzzy point with a fuzzy set, we introduce the notions of $(\in,\in\vee q_k^{\delta})$-fuzzy left ideal, $(\in,\in\vee q_k^{\delta})$-fuzzy right ideal of a semigroup. Special sets, so called Q_k^{δ}-set and $\left[\lambda_k^{\delta}\right]_t$-set, condition for the Q_k^{δ}-set and $\left[\lambda_k^{\delta}\right]_t$-set to be left (resp. right) ideals are considered. We finally characterize different classes of semigroups (regular, left weakly regular, right weakly regular) in term of $(\in,\in\vee q_k^{\delta})$-fuzzy left ideal, $(\in,\in\vee q_k^{\delta})$-fuzzy right ideal and $(\in,\in\vee q_k^{\delta})$-fuzzy ideal of semigroup S.

1. Introduction

In 1965, Zadeh in his seminal paper [6], introduced the concept of a fuzzy set. This important paper has opened up new insights and application in a wide range of scientific fields. Further many researchers have done work on the generalization of fuzzy sets with vast applications in computer science, artificial intelligence, control engineering, expert, robotics, automata theory, finite state machine, graph theory logics and many branches of pure and applied mathematics. The concept of the fuzzy group was first given by Rosenfeld in [1]. Fuzzy semigroups have been first considered by Kuroki in his influential paper[12]. Kuroki in [13, 14], initiated fuzzy ideals, bi-ideals, quasi-ideals of semigroups. The monograph by Mordeson et al. [5], deals with the theory of fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines, and

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fuzzy languages. Fuzziness provides a natural place to the area of formal languages. The monograph by Mordeson and Malik [4], deals with the application of the fuzzy methodology to the automata and formal languages. In [7], Murali initiated the notion of belongingness of a fuzzy point to a fuzzy subset under an expected equality on a fuzzy subset. In [17] the idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined. These two concepts played a vital role in producing some different types of fuzzy subgroups. Using these ideas Bhakat and Das [17, 18, 19, 20], introduced the notion of (α, β)-fuzzy subgroups by using the "belong to" $(\in)$ relation and "quasi-coincident with" (q) relation between a fuzzy point and a fuzzy subgroup and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. The concept of an $(\in, \in \vee q)$-fuzzy subgroups is a possible generalization of Rosenfeld's fuzzy subgroups. The idea of an $(\in, \in \vee q)$-fuzzy subrings and ideals are initiated in [21]. Davvaz introduced the notion of an $(\in, \in \vee q)$-fuzzy sub nearrings and ideals of a near ring in [2]. Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup in [22]. In [9], Shabir et al, characterized regular semigroup by using the concept of (α, β)-fuzzy ideals. In [15], Kazanci and Yamak studied an $(\in, \in \vee q)$-fuzzy bi-ideals of a semigroup. In [10], regular semigroups are characterized by the properties of an $(\in, \in \vee q)$-fuzzy ideals. Generalizing the idea of the quasi-coincident of a fuzzy point with a fuzzy subset, Jun in [23], introduced $(\in, \in \vee q_k)$-fuzzy subgroups and an $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras, respectively. In [10], Shabir et al, discussed an $(\in, \in \vee q_k)$-fuzzy ideals in semigroups. In [11], Shabir et al, characterized the regular semigroups by an $(\in, \in \vee q_k)$-fuzzy ideals. In [3, 8]. the authors characterized semigroup in term of an $(\in, \in \vee q_k)$-fuzzy ideals, and apply the concept to left weakly regular and right weakly regular semigroups. The notion of general form of the quasi-coincidence of a fuzzy point with a fuzzy set is initiated by Jun in [25]. In [24], Kang generalized the concept of $(\in, \in \vee q_k)$-fuzzy subsemigroup and initiated the notion of $(\in, \in \vee q_k^\delta)$-fuzzy-fuzzy subsemigroup of semigroups.

The concept of an $(\in, \in \vee q_k^\delta)$-fuzzy-fuzzy subsemigroup, an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal, of a semigroup is a generalization of the concepts studied in [10, 11]. If we take $\delta = 1$, then we get, an $(\in, \in \vee q_k)$-fuzzy subsemigroup, an $(\in, \in \vee q_k)$-fuzzy left (right) ideal semigroup [11]. If we take $\delta = 1$ and $k = 0$, then we get, an $(\in, \in \vee q)$-subsemigroup, an $(\in, \in \vee q)$-fuzzy left (right) ideal, of a semigroup [10]. Which means that these fuzzy substructures become a special case of an $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup, an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of a semigroup. Due to the motivation and inspiration of the concept, we study the concept of an $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup, an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of a semigroup.

In this paper, we generalized the concept of an $(\in, \in \vee q_k)$-fuzzy left (right)ideal, an $(\in, \in \vee q_k)$-fuzzy ideals, and define q_k^δ-quasi-coincident, an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal, an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal. Given two special sets, so called Q_k^δ-set-set and $\left[\lambda_k^\delta\right]_t$-set, we discuss conditions for the Q_k^δ-set and $\left[\lambda_k^\delta\right]_t$-set to be left (resp. right) ideal. Finally we characterize different classes of semigroups (regular, left weakly regular, right weakly regular) in term of an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal, an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal of semigroup S.

2. Preliminaries

Throughout in this paper, S will denote a semigroup. A non-empty subset A of S is said to be a subsemigroup of S if $A^2 \subseteq A$. A non-empty subset I of a S is said to be a left(right) ideal of S if $SI \subseteq I(IS \subseteq I)$. A non-empty subset I of S is said to be an ideal if it is both left and right ideal of S.

An element a of S is said to be a regular element if there exists an element x in S such that $a = axa$. S is called regular if every element of S is regular. A fuzzy subset λ of a universe X is a function from X into the unit closed interval $[0,1]$, that is $\lambda : X \to [0,1]$. A fuzzy subset in a universe X of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is called a fuzzy point with support x and value t and is denoted by (x,t).

For a fuzzy subset λ in S, a fuzzy point (x,t) said to

- contained in λ denoted $(x,t) \in \lambda$, if $\lambda(x) \geq t$.
- be quasi-coincident with λ, denoted by $(x,t)\, q\lambda$, if $\lambda(x) + t > 1$.

For a fuzzy subset λ and fuzzy point (x,t) in a set S, we say that

- $(x,t) \in \wedge q\lambda$ if $(x,t) \in \lambda$ or $(x,t)\, q\lambda$.

Let λ be a fuzzy subset of S and $t \in [0,1]$. Then the set $C(\lambda; t) = \{x \in S : \lambda(x) \geq t\}$ is called the level subset of S.

Let A be a non-empty subset of S. We denote by λ_A, the characteristic function of A, that is the mapping of S into $[0,1]$ defined by

$$\lambda_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Obviously λ_A is a fuzzy subset of S.

Definition 2.1. [10] *Let λ be a fuzzy subset of S. Then, λ is said to be a $(\in, \in \vee q_k)$-fuzzy subsemigroup of S if for all $x, y \in S$ and $t_1, t_2 \in (0,1]$ the following condition hold:*

$$(x, t_1) \in \lambda, (y, t_2) \in \lambda \Rightarrow (xy, t_1 \wedge t_2) \in \vee q_k \lambda.$$

Definition 2.2. [10] *Let λ be a fuzzy subset of S. Then, λ is said to be an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S if for all $x, y \in S$ and $t \in (0,1]$ the following condition hold:*

$$(y, t) \in \lambda \Rightarrow (xy, t) \in \vee q_k \lambda \qquad (\text{resp. } (y, t) \in \lambda \Rightarrow (yx, t) \in \vee q_k \lambda).$$

Definition 2.3. [10] *Let λ be a fuzzy subset of S. Then, λ is said to be an $(\in, \in \vee q_k)$-fuzzy ideal of S if it is both an $(\in, \in \vee q_k)$-fuzzy left and an $(\in, \in \vee q_k)$-fuzzy right ideal of S.*

Theorem 2.4. [10] *A fuzzy subset λ of S is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S if and only if $C(\lambda; t)(\neq \emptyset)$, where $t \in (0,1]$, is a left (resp. right) ideal of S.*

3. General types of $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroups

In what follows, let S denote a semigroup unless otherwise specified. Generalizing the concept of $(x, t)\, q\lambda$, Jun [23], defined $(x, t)\, q_k\lambda$, where $k \in [0, 1)$ as $(x, t)\, q_k\lambda$ if $\lambda(x) + t + k > 1$ and $(x, t) \in \vee q_k \lambda$ if $(x, t) \in \lambda$ or$(x, t)\, q_k\lambda$. Jun et al in [25], considered the general form of the symbol $(x, t)\, q_k\lambda$ and $(x, t) \in \vee q_k\lambda$ as follows: For a fuzzy point (x, t) and fuzzy subset λ in a set X, we say that

i) $(x, t)\, q^\delta\lambda$ if $\lambda(x) + t > \delta$,

ii) $(x, t)\, q_k^\delta\lambda$ if $\lambda(x) + t + k > \delta$,

iii) $(x, t) \in \vee q_k^\delta\lambda$ if $(x, t) \in \lambda$ or $(x, t)\, q_k^\delta\lambda$;

iv) $(x, t)\, \overline{\alpha}\lambda$ if $(x, t)\, \alpha\lambda$ does not hold. For $\alpha \in \{\in, q, \in \vee q, \in \vee q_k, q_k^\delta, \in \vee q_k^\delta\}$.

where $k \in [0, 1)$ and $k < \delta$ in $[0, 1]$. Obviously, $(x, t)\, q^\delta\lambda$ implies $(x, t)\, q_k^\delta\lambda$.

By using the concept of $(\in, \in \vee q_k^\delta)$-fuzzy ideals of semigroup, we generalized [10], and introduce $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroups, $(\in, \in$

$\vee q_k^\delta$)-fuzzy left (right) ideal, $(\in, \in \vee q_k^\delta)$-fuzzy ideal of semigroup S and study some basic properties.

Definition 3.1. [24]*A fuzzy subset λ of S is called an $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup of S if it satisfies:*

$$(x, t_1)\, \alpha\lambda, (y, t_2)\, \alpha\lambda \Rightarrow (xy, t_1 \wedge t_2\}) \in \vee q_k^\delta \lambda$$

for all $x, y \in S$, and $t_1, t_2 \in (0, \delta]$ where $\alpha \in \left\{\in, q^\delta\right\}$

Theorem 3.2. [24]*A fuzzy subset λ in S ia an $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup of S if and only if it satisfies:*
$(\forall x, y \in S) \left(\lambda\,(xy) \geq \lambda\,(x) \wedge \lambda\,(y) \wedge \frac{\delta - k}{2}\right)$

Example 3.3. *Let $S = \{a, b, c\}$ and "$\cdot$" be a binary operation defined on S in the following table:*

$\cdot$	a	b	c
a	a	a	a
b	a	c	b
c	a	b	c

Then, $(S, \cdot)$ is a semigroup. Define a fuzzy subset $\lambda : S \to [0, 1]$ by

$$\lambda(a) = 0.6, \lambda(b) = 0.5 \text{ and } \lambda(c) = 0.2$$

for $k = 0.3$ and $\delta = 0.5$. Then by Theorem 3.2, λ is an $(\in, \in \vee q_{0.3}^{0.5})$-fuzzy subsemigroup of S.

(i) λ is not a fuzzy subsemigroup of S,

$$\lambda(b \cdot b) = \lambda(c) = 0.2 \ngeq 0.5 = \lambda(b) \wedge \lambda(b)$$

(ii) λ is not an $(\in, \in \vee q)$-fuzzy subsemigroup. Because,

$$\lambda(b \cdot b) = \lambda(c) = 0.2 \ngeq 0.5 = \lambda(b) \wedge \lambda(b) \wedge 0.5,$$

(iii) λ is not an $(\in, \in \vee q_{0.3})$-fuzzy subsemigroup. Because,

$$\lambda(b \cdot b) = \lambda(c) = 0.2 \ngeq 0.3 = \lambda(b) \wedge \lambda(b) \wedge 0.3,$$

From Example 3.3, we see that $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup is the generalization of fuzzy subsemigroups, $(\in, \in \vee q)$-fuzzy subsemigroups $(\in, \in \vee q_k)$-fuzzy subsemigroups. If we take, $\delta = 1$, then we get $(\in, \in \vee q_k)$-fuzzy subsemigroups of S. If we take, $\delta = 1$ and $k = 0$, then we get $(\in, \in \vee q)$-fuzzy subsemigroups of S. This means that these structures become special cases.

For a fuzzy set λ in S and $t \in (0, \delta]$, consider the Q_k^δ-set and $\left[\lambda_k^\delta\right]_t$-set with respect to t as follows:
$Q_k^\delta\,(\lambda; t) := \left\{x \in X |\, (x, t)\, q_k^\delta \lambda\right\}$ and $\left[\lambda_k^\delta\right]_t := \left\{x \in X |\, (x, t) \in \vee q_k^\delta \lambda\right\}.$

Remark 3.4. *For any $t, r \in (0, \delta]$, if $t \geq r$, then $Q_k^\delta(\lambda; r) \subseteq Q_k^\delta(\lambda; t)$ and $U(\lambda; r) \cup Q_k^\delta(\lambda; t) \subseteq U(\lambda; t) \cup Q_k^\delta(\lambda; t)$.*

Obviously, $\left[\lambda_k^\delta\right]_t = U(\lambda; t) \cup Q_k^\delta(\lambda; t)$.

Proposition 3.5. *Let λ, μ and η be any three fuzzy sets in a set S. Then,*

1) $[(\lambda \cup \mu)_k^\delta]_t = [\lambda_k^\delta]_t \cup [\mu_k^\delta]_t$.
2) $[(\lambda \cap \mu)_k^\delta]_t = [\lambda_k^\delta]_t \cap [\mu_k^\delta]_t$.
3) $[(\lambda \cup (\mu \cap \eta))_k^\delta]_t = [(\lambda \cup \mu)_k^\delta]_t \cap [(\lambda \cup \eta)_k^\delta]_t$.
4) $[(\lambda \cap (\mu \cup \eta))_k^\delta]_t = [(\lambda \cap \mu)_k^\delta]_t \cup [(\lambda \cap \eta)_k^\delta]_t$.

Proof. For any $x \in S$, we have

1)

$$\begin{aligned}
x &\in [(\lambda \cup \mu)_k^\delta]_t \Leftrightarrow (x, t) \in \vee q_k^\delta(\lambda \cup \mu) \\
&\Leftrightarrow (\lambda \cup \mu)(x) \geq t \text{ or } (\lambda \cup \mu)(x) + t + k > \delta \\
&\Leftrightarrow [\lambda(x) \geq t \text{ or } \mu(x) \geq t] \text{ or } [\lambda(x) + t + k > \delta \text{ or } \mu(x) + t + k > \delta] \\
&\Leftrightarrow [\lambda(x) \geq t \text{ or } \lambda(x) + t + k > \delta] \, or \, [\mu(x) \geq t \text{ or } \mu(x) + t + k > \delta] \\
&\Leftrightarrow (x, t) \in \vee q_k^\delta \lambda or \, (x, t) \in \vee q_k^\delta \mu \\
&\Leftrightarrow x \in [\lambda_k^\delta]_t \text{ or } x \in [\mu_k^\delta]_t \\
&\Leftrightarrow x \in [\lambda_k^\delta]_t \cup [\mu_k^\delta]_t
\end{aligned}$$

and

$$\begin{aligned}
x &\in [(\lambda \cap \mu)_k^\delta]_t \Leftrightarrow (x, t) \in \vee q_k^\delta(\lambda \cap \mu) \\
&\Leftrightarrow (\lambda \cap \mu)(x) \geq t \text{ or } (\lambda \cap \mu)(x) + t + k > \delta \\
&\Leftrightarrow [\lambda(x) \geq t \text{ and } \mu(x) \geq t] \text{ or } [\lambda(x) + t + k > \delta \text{ and } \mu(x) + t + k > \delta] \\
&\Leftrightarrow [\lambda(x) \geq t \text{ or } \lambda(x) + t + k > \delta] \text{ and } [\mu(x) \geq t \text{ or } \mu(x) + t + k > \delta] \\
&\Leftrightarrow (x, t) \in \vee q_k^\delta \lambda \text{ and } (x, t) \in \vee q_k^\delta \mu \\
&\Leftrightarrow x \in [\lambda_k^\delta]_t \text{ and } x \in [\mu_k^\delta]_t \\
&\Leftrightarrow x \in [\lambda_k^\delta]_t \cap [\mu_k^\delta]_t
\end{aligned}$$

This proves (1) and (2). Using (1) and (2), one can show that (3) and (4). □

Theorem 3.6. *If λ is an $\left(\in, \in \vee q_k^\delta\right)$-fuzzy subsemigroup of S, then $\left(\forall t \in (\frac{\delta - k}{2}, 1]\right) \left(Q_k^\delta(\lambda; t) \neq \emptyset \text{ is a subsemigroup of } S.\right)$*

Proof. Let $x, y \in Q_k^\delta(\lambda; t)$ for $x, y \in S$. Then $\lambda(x) + t + k > \delta$ and $\lambda(y) + t + k > \delta$. It follows from Theorem (3.2) that

$$\lambda(xy) \geq \lambda(x) \wedge \lambda(y) \wedge \frac{\delta - k}{2} \geq (\delta - t - k) \wedge \frac{\delta - k}{2} = \delta - t - k$$

and that $xy \in Q_k^\delta(\lambda; t)$. Hence, $Q_k^\delta(\lambda; t)$ is a subsemigroup of S. □

If we take, $\delta = 1$ in Theorem 3.6, then we have the following corollary.

Corollary 3.7. *If λ is an $\left(\in, \in \vee q_k^\delta\right)$-fuzzy subsemigroup of S, then $\left(\forall t \in (\frac{\delta-k}{2}, 1]\right) \left(Q_k^\delta(\lambda; t) \neq \emptyset \text{ is a subsemigroup of } S.\right)$*

Theorem 3.8. *Let λ be a fuzzy subset of S. Then λ is an $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup of S if and only if $[\lambda_k^\delta]_t$ is a subsemigroup of S for all $t \in (0, \delta]$.*

Proof. Assume that λ is an $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup of S, then

$$\lambda(xy) \geq \lambda(x) \wedge \lambda(y) \wedge \frac{\delta - k}{2}$$

by Theorem 3.2. Let $x, y \in [\lambda_k^\delta]_t$ for $t \in (0, \delta]$. Then, $(x, t) \in \vee q_k^\delta \lambda$ and $(y, t) \in \vee q_k^\delta \lambda$, that is, $\lambda(x) \geq t$ or $\lambda(x) + t > \delta - k$, and $\lambda(y) \geq t$ or $\lambda(y) + t > \delta - k$. Hence, we have the following four cases:

Case 1 : $\lambda(x) \geq t$ and $\lambda(y) \geq t$

Case 2 : $\lambda(x) \geq t$ and $\lambda(y) + t > \delta - k$

Case 3 : $\lambda(y) \geq t$ and $\lambda(y) + t > \delta - k$

Case 4 : $\lambda(x) + t > \delta - k$ and $\lambda(y) + t > \delta - k$

Case 1: Let $\lambda(x) \geq t$ and $\lambda(y) \geq t$. If $t > \frac{\delta-k}{2}$, then $\lambda(xy) \geq \lambda(x) \wedge \lambda(y) \wedge \frac{\delta-k}{2} = \frac{\delta-k}{2}$. Hence, $\lambda(xy) + t > \delta - k$ and so $(xy, t) q_k^\delta \lambda$. If $t \leq \frac{\delta-k}{2}$, then $\lambda(xy) \geq \lambda(a) \wedge \lambda(b) \wedge \frac{\delta-k}{2} \geq t$ and so $(xy, t) \in \lambda$. Therefore, $(xy, t) \in \vee q_k^\delta \lambda$, i.e. $xy \in [\lambda_k^\delta]_t$.

Case 2: Let $\lambda(x) \geq t$ and $\lambda(y) + t > \delta - k$. If $t > \frac{\delta-k}{2}$, then

$$\begin{aligned} \lambda(xy) &\geq \lambda(x) \wedge \lambda(y) \wedge \frac{\delta - k}{2} \\ &= \lambda(y) \wedge \frac{\delta - k}{2} \\ &> (\delta - t - k) \wedge \frac{\delta - k}{2} \\ &= \delta - t - k, \end{aligned}$$

that is $\lambda(y)+t>\delta-k$ and so $(xy,t)\in q_k^\delta\lambda$. If $t\leq\frac{\delta-k}{2}$, then

$$\begin{aligned}\lambda(ab) &\geq \lambda(a)\wedge\lambda(b)\wedge\frac{\delta-k}{2}\\ &\geq t\wedge\delta-t\wedge\frac{\delta-k}{2}=t\end{aligned}$$

Hence, $(xy,t)\in\lambda$. Therefore, $(xy,t)\in\vee q_k^\delta\lambda$, i.e. $xy\in[\lambda_k^\delta]_t$

Case 3: Let $\lambda(y)\geq t$ and $\lambda(y)+t>\delta-k$. If $t>\frac{\delta-k}{2}$, Then

$$\begin{aligned}\lambda(xy) &\geq \lambda(x)\wedge\lambda(y)\wedge\frac{\delta-k}{2}\\ &\geq \lambda(x)\wedge\frac{\delta-k}{2}\\ &> \delta-t-k\wedge\frac{\delta-k}{2}\\ &= \delta-t-k.\end{aligned}$$

Hence, $\lambda(y)+t>\delta-k$ and thus $(xy,t)q_k^\delta\lambda$. If $t\leq\frac{\delta-k}{2}$, then

$$\begin{aligned}\lambda(xy) &\geq \lambda(x)\wedge\lambda(y)\wedge\frac{\delta-k}{2}\\ &\geq \delta-t-k\wedge t\wedge\frac{\delta-k}{2}=t\end{aligned}$$

and so $(xy,t)\in\lambda$. Therefore, $(xy,t)\in\vee q_k^\delta\lambda$, i.e. $xy\in[\lambda_k^\delta]_t$

Case 4: Let $\lambda(x)+t>\delta-k$ and $\lambda(y)+t>\delta-k$. If $t>\frac{\delta-k}{2}$, then

$$\begin{aligned}\lambda(xy) &\geq \lambda(x)\wedge\lambda(y)\wedge\frac{\delta-k}{2}\\ &\geq \lambda(x)\wedge\frac{\delta-k}{2}\\ &> \delta-t-k\wedge\frac{\delta-k}{2}\\ &= \delta-t-k,\end{aligned}$$

that is $\lambda(xy)+t>\delta-k$ and so $(xy,t)\in\lambda$. If $t\leq\frac{\delta-k}{2}$, then

$$\begin{aligned}\lambda(xy) &\geq \lambda(x)\wedge\lambda(y)\wedge\frac{\delta-k}{2}\\ &\geq \delta-t-k\wedge t\wedge\frac{\delta-k}{2}\\ &= \frac{\delta-k}{2}\geq t\end{aligned}$$

Hence, $(xy, t) \in \lambda$ and thus $(xy, t) \in \vee q_k^{\delta}\lambda$. i.e. $xy \in [\lambda_k^{\delta}]_t$.Consequently, $[\lambda_k^{\delta}]_t$ is a subsemigroup of S.

Conversely, let λ be a fuzzy subset of S and $x, y \in S$ be such that $[\lambda_k^{\delta}]_t$ is a subsemigroup of S. Assume that $\lambda(xy) < t < \lambda(x) \wedge \lambda(y) \wedge \frac{\delta-k}{2}$ for some $t \in (0, \frac{\delta-k}{2}]$. Then, $x, y \in U(\lambda; t) \subseteq [\lambda_k^{\delta}]_t$ which implies that $xy \in [\lambda_k^{\delta}]_t$. Thus $\lambda(xy) \geq t$ or $\lambda(xy) + t > \delta - k$, which is a contradiction. Hence, $\lambda(xy) \geq \lambda(x) \wedge \lambda(y) \wedge \frac{\delta-k}{2}$ for all $x, y \in S$. Thus by Theorem 3.2, it follows that λ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subsemigroup of S. □

If we put $\delta = 1$ in Theorem 3.8, then we have the following corollary.

Corollary 3.9. *Let λ be a fuzzy subset of S. Then λ is an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S if and only if $[\lambda_k]_t$ is a subsemigroup of S for all $t \in (0, 1]$.*

Lemma 3.10. *For a fuzzy subset λ in S, the following are equivalent.*
1) *λ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subsemigroup of S.*
2) *The level subset $U(\lambda; t)$ of λ is a subsemigroup of S for all $t \in (0, \frac{\delta-k}{2}]$.*

Theorem 3.11. *Let $A_1 \subseteq A_2 \subseteq ... \subseteq A_r = S$ be a chain of subsemigroups of S. Then there exists an $(\in, \in \vee q_k^{\delta})$-fuzzy subsemigroup whose $(\in \vee q_k^{\delta})$-level subsemigroup are precisely the members of the chain.*

Proof. Let $\left\{t_i \in (0, \frac{\delta-k}{2}] | i = 1, 2, ..., r\right\}$ be such that $t_1 > t_2 > ... > t_r$. Define a fuzzy subset λ in S by

$$\lambda(x) = \begin{cases} t_1 & \text{if } x \in A_1, \\ t_2 & \text{if } x \in A_1 \backslash A_2, \\ & ... \\ t_r & \text{if } x \in A_1 \backslash A_r, \end{cases}$$

Then

$$U(\lambda; t) = \begin{cases} \emptyset & \text{if } t \in (t_1, \frac{\delta-k}{2}], \\ A_1 & \text{if } t \in (t_2, t_1], \\ & ... \\ A_r (= S) & \text{if } t \in (0, t_r], \end{cases}$$

and so λ is an $(\in, \in \vee q_k^{\delta})$-fuzzy subsemigroups of S by Lemma 3.10. Obviously, $(\in \vee q_k^{\delta})$-level subsemigroups of λ are precisely the member of the chain. □

Given a fuzzy subset λ in S, an $(\in, \in \vee q_k^{\delta})$-fuzzy subsemigroups μ of S is said to be the $(\in, \in \vee q_k^{\delta})$-fuzzy subsemigroups generated by λ in S

if $\lambda \subseteq \mu$ and for any $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroups η of S with $\lambda \subseteq \eta$ it must be $\mu \subseteq \eta$.

Theorem 3.12. *Let λ be a fuzzy subset in S with finite image. Define subsemigroups A_i of S as follows:*
$A_0 = \left\langle \left\{ x \in S | \lambda(x) \geq \frac{\delta - k}{2} \right\} \right\rangle,$
$A_i = \langle A_{i-1} \cup \{ x \in S | \lambda(x) = \sup \{ \lambda(y) | y \in S \backslash A_{i-1} \} \} \rangle$
for $i = 1, 2, ..., k$ where $k \leq \sharp\mathrm{Im}(\lambda)$, and $A_k = S$. Let λ^* be a fuzzy subset in S defined by

$$\lambda^*(x) = \begin{cases} \lambda(x) & \text{if } x \in A_0 \\ \sup \{ \lambda(y) | y \in S \backslash A_{i-1} \} & \text{if } x \in A_i \backslash A_{i-1} \end{cases}$$

Then, λ^* is the $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup of S generated by λ in S.

Proof. The construction of λ^* shows that $\lambda \subseteq \lambda^*$. Note that the $A_i's$ form a chain

$$A_0 \subseteq A_1 \subseteq ... \subseteq A_n = S$$

Let $x, y \in S$. If $x, y \in A_0$, then $xy \in A_0$ and so,

$$\begin{aligned} \lambda^*(xy) &= \lambda(xy) \geq \lambda(x) \wedge \lambda(y) \wedge \frac{\delta - k}{2} \\ &= \lambda^*(x) \wedge \lambda^*(y) \wedge \frac{\delta - k}{2}. \end{aligned}$$

If $x \in A_i \backslash A_{i-1}$ and $y \in A_i \backslash A_{i-1}$, then we may assume that $i < j$ without loss of generality. It follows that $xy \in A_j$ and so that,

$$\begin{aligned} \lambda^*(xy) &= \sup \{ \lambda(y) | y \in S \backslash A_{i-1} \} \\ &\geq \sup \{ \lambda(x) | x \in S \backslash A_{i-1} \} \wedge \sup \{ \lambda(y) | y \in S \backslash A_{j-1} \} \wedge \frac{\delta - k}{2} \\ &= \lambda^*(x) \wedge \lambda^*(y) \wedge \frac{\delta - k}{2}. \end{aligned}$$

Hence, λ^* is an $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup of S whose $(\in \vee q_k^\delta)$-level subsemigroup are precisely the members of the chain $A_0 \subseteq A_1 \subseteq ... \subseteq A_n = S$ by Theorem 3.11. Now let μ be any $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup of S which contain λ. If $x \in A_0$, then $\lambda^*(x) = \lambda(x) \leq \mu(x)$. Let $\{B_{t_i}\}$ be the class of $(\in \vee q_k^\delta)$-level subsemigroups of μ in S. let $x \in A_1 \backslash A_0$. Then,

$$\lambda^*(x) = \sup \{ \lambda(y) | y \in S \backslash A_0 \}$$

and $A_1 = \langle N_1 \rangle$ where,

$$N_1 = A_0 \cup \{ x \in S | \lambda(x) = \sup \{ \lambda(y) | y \in S \backslash A_0 \} \}.$$

Let $x \in N_1 \backslash A_0$. Then,

$$\begin{aligned} \lambda(x) &= \sup\{\lambda(y) \,|y \in S \backslash A_0\} \\ &\leq \inf\{\mu(x) \,|x \in N_1 \backslash A_0\} \\ &\leq \mu(x) \end{aligned}$$

since λ is contained in μ. Putting $t_{i1} = \inf\{\mu(x) \,|x \in N_1 \backslash A_0\}$ induces $x \in B_{t_{i1}}$ and so $N_1 \backslash A_0 \subseteq B_{t_{i1}}$. Since, $A_0 \subseteq B_{t_{i1}}$, we have $A_1 = \langle N_1 \rangle \subseteq B_{t_{i1}}$, and so $\mu(x) \geq t_{i1}$ for all $x \in A_1$. Therefore,

$$\lambda^*(x) = \sup\{\lambda(y) \,|y \in S \backslash A_0\} \leq t_{i1} \leq \mu(x)$$

for all $x \in A_1 \backslash A_0$. Similarly, we can prove that $\lambda^*(x) \leq \mu(x)$ for all $\in A_i \backslash A_{i-1}$ for $2 \leq i \leq k$. Therefore, λ^* is the $(\in, \in \vee q_k^\delta)$-fuzzy subsemigroup of S generated by λ in S. □

4. General types of $(\in, \in \vee q_k^\delta)$-fuzzy Ideals

In this section we define $(\in, \in \vee q_k^\delta)$-fuzzy left (resp. right) ideals, $(\in, \in \vee q_k^\delta)$-fuzzy ideals and discuss some of its properties.

Definition 4.1. *A fuzzy subset λ of S is said to be an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S if the following condition holds;*

$$(y,t) \in \lambda \Rightarrow (xy,t) \in \vee q_k^\delta \lambda \ ((y,t) \in \lambda \Rightarrow (yx,t) \in \vee q_k^\delta \lambda)$$

Definition 4.2. *A fuzzy subset λ of S is said to be an $(\in, \in \vee q_k^\delta)$-fuzzy ideal of S if it is both an $(\in, \in \vee q_k^\delta)$-fuzzy left and an $(\in, \in \vee q_k^\delta)$-fuzzy right ideals of S.*

Theorem 4.3. *Let I be a left (resp. right) ideal of S and λ a fuzzy subset of S defined by*

$$\lambda(x) = \begin{cases} \varepsilon & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon \geq \frac{\delta-k}{2}$. Then,

i) λ is $(q^\delta, \in \vee q_k^\delta)$-fuzzy left (resp. right) ideal of S.

ii) λ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp. right) ideal of S.

Proof. *i)* Let $x, y \in S$ and $t_y \in (0, \delta]$ be such that $(y, t_y)\, q^\delta \lambda$. Then, $\lambda(y) + t_y \geq \delta$. If $x \notin I$, then $\lambda(y) = 0$. Hence, $t_y > \delta$ which is a contradiction. Thus $y \in I$. Since, I is a left ideal of S, we have for $x \in S$, $xy \in I$. Thus, $\lambda(xy) = \varepsilon \geq \frac{\delta-k}{2}$. If $t_y \wedge t_y = t_y \leq \frac{\delta-k}{2}$, then $\lambda(xy) \geq t_y$ and so $(xy, t_y) \in \lambda$. If $t_y \wedge t_y = t_y > \frac{\delta-k}{2}$, then $\lambda(xy) + t_y + k >$

$\frac{\delta-k}{2}+\frac{\delta-k}{2}+k=\delta$, and so, $(xy,t_y)q_k^{\delta}\lambda$ .Therefore, $(xy,t)\in\vee q_k^{\delta}\lambda$. Hence, λ is $(q^{\delta},\in\vee q_k^{\delta})$-fuzzy left ideal of S

ii) Let $x,y\in S$ and $t\in(0,1]$ be such that $(y,t)\in\lambda$. Then, $\lambda(y)\geq t$, and so, $\lambda(y)=\varepsilon\geq\frac{\delta-k}{2}$, which implies that $y\in I$ and since I is a left(right) ideal of S, so we have $xy\in I$. Thus $\lambda(xy)=\varepsilon\geq\frac{\delta-k}{2}$. If $t\leq\frac{\delta-k}{2}$, then $\lambda(xy)=\varepsilon\geq\frac{\delta-k}{2}>t$, so $(xy,t)\in\lambda$. If $t>\frac{\delta-k}{2}$, then $\lambda(xy)+t+k>\frac{\delta-k}{2}+\frac{\delta-k}{2}+k=\delta$, and so, $(xy,t)q_k^{\delta}\lambda$. Therefore, $(xy,t)\in\vee q_k^{\delta}\lambda$. Hence, λ is an $(\in,\in\vee q_k^{\delta})$-fuzzy left deal of S. □

In the similar way we can prove the case of right ideal of S.

Corollary 4.4. *Let I be a ideal of S and λ a fuzzy subset of S defined by*

$$\lambda(x)=\begin{cases}\varepsilon & \text{if } x\in A,\\ 0 & \text{otherwise,}\end{cases}$$

where $\varepsilon\geq\frac{\delta-k}{2}$. Then,

i) λ is $(q^{\delta},\in\vee q_k^{\delta})$-fuzzy ideal of S.

ii) λ is an $(\in,\in\vee q_k^{\delta})$-fuzzy ideal of S.

If we put $\delta=1$ and $k=0$ in Theorem 4.3, then we have the following corollary.

Corollary 4.5. *[[10], Theorem 7] Let I be a left (resp. right) ideal of S and λ a fuzzy subset of S defined by*

$$\lambda(x)=\begin{cases}\varepsilon & \text{if } x\in A,\\ 0 & \text{otherwise,}\end{cases}$$

where $\varepsilon\geq\frac{\delta-k}{2}$. Then,

i) λ is $(q,\in\vee q_k)$-fuzzy (resp. right) ideal of S.

ii) λ is an $(\in,\in\vee q_k)$-fuzzy (resp. right) ideal of S.

In the following theorem we provide condition for an $(\in,\in\vee q_k^{\delta})$-fuzzy left (resp. right) ideal of S to be a $(q^{\delta},\in\vee q_k^{\delta})$-fuzzy left (resp. right) ideal.

Theorem 4.6. *Suppose that every fuzzy point has the value $t\in(0,1]$. Then every $(\in,\in\vee q_k^{\delta})$-fuzzy left (resp. right) ideal of S is a $(q^{\delta},\in\vee q_k^{\delta})$-fuzzy left (resp. right) ideal of S.*

Proof. Let λ be an $(\in,\in\vee q_k^{\delta})$-fuzzy left ideal of S and $x,y\in S, t\in(0,\frac{\delta-k}{2}]$ be such that $(y,t)\,q^{\delta}\lambda$. Then, $\lambda(y)+t>\delta$ it implies that $\lambda(y)>\delta-t\geq t$, i.e. $(y,t)\in\lambda$. By assumption it follows that $(xy,t)\in\vee q_k^{\delta}\lambda$. Hence, λ is a $(q^{\delta},\in\vee q_k^{\delta})$-fuzzy left ideal of S. □

Similarly we can prove the case of right ideal of S.

Corollary 4.7. *Let λ be an $(\in,\in)$, $(\in, q_k^\delta)$, $(\in,\in\vee q_k^\delta)$, $(\in,\in\wedge q_k^\delta)$, $(\in\vee q_k^\delta,\in)$, $(\in\vee q_k^\delta, q^\delta)$, $(\in\vee q_k^\delta,\in\wedge q_k^\delta)$, $(\in\vee q_k^\delta,\in\vee q_k^\delta)$-fuzzy left (resp. right) ideal of S. If every fuzzy point has the value $t\in(0,\frac{\delta-k}{2}]$, then λ is an $(q^\delta,\in\vee q_k^\delta)$-fuzzy left (resp. right) ideal of S.*

Theorem 4.8. *A fuzzy subset λ of S is an $(\in,\in\vee q_k^\delta)$-fuzzy left (resp. right) ideal of S if and only if $\lambda(xy)\geq\lambda(y)\wedge\frac{\delta-k}{2}(\lambda(xy)\geq\lambda(x)\wedge\frac{\delta-k}{2})$.*

Proof. Let λ be an $(\in,\in\vee q_k^\delta)$-fuzzy left ideal of S. Suppose that there exist $x,y\in S$ such that $\lambda(xy)\leq\lambda(y)\wedge\frac{\delta-k}{2}$. Choose $t\in(0,1]$ such that $\lambda(xy)<t<\lambda(y)\wedge\frac{\delta-k}{2}$. Then, $(y,t)\in\lambda$. But, $(xy,t)\notin\lambda$ and $\lambda(xy)+t+k<\frac{\delta-k}{2}+\frac{\delta-k}{2}+k=\delta$. Therefore, $(xy,t)\in\overline{\vee q_k^\delta}\lambda$. Which is a contradiction. Thus $\lambda(xy)\geq\lambda(y)\wedge\frac{\delta-k}{2}$.

Conversely, suppose that $\lambda(xy)\leq\lambda(y)\wedge\frac{\delta-k}{2}$. Let $x,y\in S$ and $t\in(0,1]$ be such that $(y,t)\in\lambda$. Then, $\lambda(y)\geq t$, thus $\lambda(xy)\geq\lambda(y)\wedge\frac{\delta-k}{2}\geq t\wedge\frac{\delta-k}{2}$. If $t\leq\frac{\delta-k}{2}$,then $\lambda(xy)\geq t$. So, $(xy,t)\in\lambda$. Thus $(xy,t)\in\vee q_k^\delta\lambda$. If $t>\frac{\delta-k}{2}$, then $\lambda(xy)\geq\frac{\delta-k}{2}$. So $\lambda(xy)+t+k>\frac{\delta-k}{2}+\frac{\delta-k}{2}+k=\delta$,which implies that $(xy,t)q_k^\delta\lambda$. Therefore, λ is an $(\in,\in\vee q_k^\delta)$-fuzzy left ideal of S.

In a similar fashion we can prove the case of right ideal of S. □

Example 4.9. *Let $S=\{1,2,3,4\}$ and a binary operation "$\cdot$"is defined in the following table:*

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	4	1
4	1	1	4	4

Then $(S,\cdot)$ is a semigroup. One can easily check that $\{1\},\{1,4\},\{1,2,4\},\{1,3,4\}$ and $\{1,2,3,4\}$ are ideals of S.

Define a fuzzy subset λ by $\lambda(1)=0.8,\lambda(2)=\lambda(3)=0.5$ and $\lambda(4)=0.2$. If we take, $k=0.3$ and $\delta=0.4$, then λ is an $(\in,\in\vee q_{0.3}^{0.4})$-fuzzy ideal of S.

(i) λ is not a fuzzy ideal of S. Because,

$$\lambda(3\cdot 3)=\lambda(4)=0.2\ngeq\lambda(3)=0.5,$$

(ii) λ is not an $(\in,\in\vee q)$-fuzzy ideal of S. Because,

$$\lambda(3\cdot 3)=\lambda(4)=0.2\ngeq\lambda(3)\wedge 0.5=0.5,$$

(iii) λ *is not an* $(\in, \in \vee q_{0.3})$*-fuzzy of* S*. Because,*

$$\lambda(3 \cdot 3) = \lambda(4) = 0.2 \ngeq \lambda(3) \wedge 0.3 = 0.3.$$

Clearly every fuzzy ideal, $(\in, \in \vee q)$*-fuzzy ideal,* $(\in, \in \vee q_k)$*-fuzzy ideal is an* $(\in, \in \vee q_k^\delta)$*-fuzzy ideal but the converse is not true.*

Corollary 4.10. *A fuzzy subset* λ *of* S *is an* $(\in, \in \vee q_k^\delta)$*-fuzzy ideal of* S *if and only if* $\lambda(xy) \geq \lambda(y) \wedge \frac{\delta-k}{2}$ *and* $\lambda(xy) \geq \lambda(x) \wedge \frac{\delta-k}{2}$.

If we put $\delta = 1$ in Theorem 4.8, then we have the following corollary.

Corollary 4.11. *[[10] Theorem 8] A fuzzy subset* λ *of* S *is an* $(\in, \in \vee q_k)$*-fuzzy left (resp. right) ideal of* S *if and only if* $\lambda(xy) \geq \lambda(y) \wedge \frac{1-k}{2}$ $(\lambda(xy) \geq \lambda(x) \wedge \frac{1-k}{2})$.

Theorem 4.12. *Let* λ *be a fuzzy subset of* S*. Then* λ *is an* $(\in, \in \vee q_k^\delta)$*-fuzzy left (resp. right) of* S *if and only if* $U(\lambda; t)(\neq 0)$ *is a left(right) of* S *for all* $t \in (0, \frac{\delta-k}{2}]$.

Proof. Assume that λ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. Let $t \in (0, \frac{\delta-k}{2}]$ and $x, y \in U(\lambda; t)$. Then $\lambda(y) \geq t$. It follows from Theorem 4.8 that

$$\lambda(xy) \geq \lambda(y) \wedge \frac{\delta-k}{2} \geq t \wedge \frac{\delta-k}{2} = t$$

so that $xy \in U(\lambda; t)$. Hence, $U(\lambda; t)$ is a left ideal of S.

Conversely, suppose that $U(\lambda; t)$ is a left ideal of S for all $t \in (0, \frac{\delta-k}{2}]$. If $\lambda(xy) < \lambda(y) \wedge \frac{\delta-k}{2}$ for $x, y \in S$. Choose $t \in (0,1)$ such that $\lambda(xy) < t \leq \lambda(y) \wedge \frac{\delta-k}{2}$. Then $t \in (0, \frac{\delta-k}{2}]$ and $x, y \in U(\lambda; t)$. Since, $U(\lambda; t)$ is a left ideal of S, it follows that $xy \in U(\lambda; t)$ so that $\lambda(xy) \geq t$. This is a contradiction. Therefore, $\lambda(xy) \geq \lambda(y) \wedge \frac{\delta-k}{2}$. Hence by Theorem 4.8, λ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S.

In similar way we can prove the case for $(\in, \in \vee q_k^\delta)$-fuzzy right ideal of S. □

Theorem 4.13. *Let* λ *be an* $(\in, \in \vee q_k^\delta)$*-fuzzy left ideal of* S *and* μ *be an* $(\in, \in \vee q_k^\delta)$*-fuzzy right ideal of* S*. Then* $\lambda \circ \eta$ *is an* $(\in, \in \vee q_k^\delta)$*-fuzzy ideal of* S*.*

Proof. Let $x, y \in S$. Then,

$$\begin{aligned}(\lambda \circ \mu)(y) \wedge \frac{\delta - k}{2} &= (\bigvee_{y=ab} \{\lambda(a) \wedge \mu(b)\}) \wedge \frac{\delta - k}{2} \\ &= \bigvee_{y=ab} \{\lambda(a) \wedge \mu(b) \wedge \frac{\delta - k}{2}\} \\ &= \bigvee_{y=ab} \{\lambda(a) \wedge \frac{\delta - k}{2} \wedge \mu(b)\}\end{aligned}$$

If $y = ab$; then $xy = x(ab) = (xa)b$. Since, λ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S, then by Theorem 4.8 we have $\lambda(xa) \geq \lambda(a) \wedge \frac{\delta-k}{2}$. Thus,

$$\begin{aligned}(\lambda \circ \mu)(y) \wedge \frac{\delta - k}{2} &= \left(\bigvee_{y=ab} \{\lambda(a) \wedge \mu(b)\}\right) \wedge \frac{\delta - k}{2} \\ &\leq \bigvee_{y=ab} \{\lambda(xa) \wedge \mu(b)\} \\ &\leq \bigvee_{y=ab} \{\lambda(a) \wedge \mu(b)\} \\ &= (\lambda \circ \mu)(xy)\end{aligned}$$

So, $(\lambda \circ \mu)(y) \wedge \frac{\delta-k}{2} \leq (\lambda \circ \mu)(xy)$. Similarly we can show that $(\lambda \circ \mu)(x) \wedge \frac{\delta-k}{2} \leq (\lambda \circ \mu)(xy)$. Thus, $\lambda \circ \mu$ is an $(\in, \in \vee q_k^\delta)$-fuzzy ideal of S. □

The following example show that $\lambda \circ \mu \nleq \lambda \wedge \mu$ in general.

Example 4.14. *Let $S = \{1, 2, 3, 4, 5\}$ and "$\cdot$" be a binary operation defined on S in the following table*

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	1	1	3	1
5	1	1	1	3	3

Then, $(S, \cdot)$ is a semigroup. One can easily check that, $\{1\}, \{1, 2\}$, $\{1, 2, 5\}$, $\{1, 2, 3, 5\}$, $\{1, 5\}$, $\{1, 2, 4, 5\}$ and $\{1, 2, 3, 4, 5\}$ are all ideals of S. Define fuzzy set λ and μ of S by

$$\lambda(1) = 0.7,\ \lambda(2) = 0.5,\ \lambda(3) = 0.3 = \lambda(4),\ \lambda(5) = 0.6$$

$$\mu(1) = 0.7,\ \mu(2) = 0.6,\ \mu(3) = 0.2 = \mu(5),\ \mu(4) = 0.3$$

Then we have,

$$U(\lambda;t) = \begin{cases} \{1,2,3,4,5\} & \text{if } t \in (0, 0.3] \\ \{1,2,4,5\} & \text{if } t \in (0.3, 0.5] \\ \{1,5\} & \text{if } t \in (0.5, 0.6] \\ \{1\} & \text{if } t \in (0.6, 0.7] \\ \phi & \text{if } t \in (0.7, 1] \end{cases}$$

$$U(\mu;r) = \begin{cases} \{1,2,3,4,5\} & \text{if } t \in (0, 0.2] \\ \{1,2,3,5\} & \text{if } t \in (0.2, 0.3] \\ \{1,2,5\} & \text{if } t \in (0.3, 0.6] \\ \{1,2\} & \text{if } t \in (0.6, 0.7] \\ \{1\} & \text{if } t \in (0.7, 1] \\ \phi & \text{if } t \in (0.7, 1] \end{cases}$$

Thus, by 4.12, λ and μ are $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S with $k = 0.2$ and $\delta = 0.8$. Now,

$$\begin{aligned} (\lambda \circ \mu)(3) &= \bigvee_{3=ab} \{\lambda(a), \mu(b)\} \\ &= \bigvee \{0.3, 0.3, 0.4\} \\ &= 0.4 \nleq (\lambda \wedge \mu)(3) = 0.2. \end{aligned}$$

Hence, $\lambda \circ \mu \nleq \lambda \wedge \mu$ in general.

Lemma 4.15. *If $\{I_n\}_{n\in\Lambda}$ is a family of $(\in, \in \vee q_k^\delta)$-fuzzy left (resp. right) ideal of S. Then $\bigcap_{n\in\Lambda} I_n$ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (resp. right) ideal of S.*

Proof. Let $\{I_n\}_{n\in\Lambda}$ is a family of an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S and $x, y \in S$. Then $\bigcap_{n\in\Lambda} I_n(xy) = \bigcap_{n\in\Lambda} (I_n(xy))$. (since each I_n is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S, so $I_n(xy) \geq I_n(y) \wedge \frac{\delta-k}{2}$, for all $n \in \Lambda$.)

Thus,

$$\begin{aligned}\left(\bigcap_{n\in\Lambda} I_n\right)(xy) &= \bigcap_{n\in\Lambda}\left(I_n\left(xy\right)\right)\\ &\geq \bigcap_{n\in\Lambda}\left(I_n\left(y\right)\wedge\frac{\delta-k}{2}\right)\\ &= \left(\bigcap_{n\in\Lambda} I_n\left(y\right)\right)\wedge\frac{\delta-k}{2}\\ &= \left(\bigcap_{n\in\Lambda} I_n\right)(y)\wedge\frac{\delta-k}{2}\end{aligned}$$

Hence, $\bigcap\limits_{n\in\Lambda} I_n$ is an $(\in,\in\vee q_k^\delta)$-fuzzy left ideal of S. □

In similar way we can prove the case of right ideal of S.

Corollary 4.16. *If $\{I_n\}_{n\in\Lambda}$ is a family of $(\in,\in\vee q_k^\delta)$-fuzzy ideal of S. Then, $\bigcap\limits_{n\in\Lambda} I_n$ is an $(\in,\in\vee q_k^\delta)$-fuzzy ideal of S.*

Remark 4.17. *The union of two $(\in,\in\vee q_k^\delta)$-fuzzy ideal of S need not be an $(\in,\in\vee q_k^\delta)$-fuzzy ideal of S.*

Example 4.18. *Consider a semigroup of example 4.14, if we define λ and μ by*

$$\lambda(1)=0.6,\lambda(2)=\lambda(3)=0.3,\lambda(4)=0.5,\lambda(5)=0.5$$

$$\mu(1)=0.7,\mu(2)=0.5,\mu(3)=0.2,\mu(4)=0.1,\mu(5)=0.4$$

and take, $k=0.2$ and $\delta=0.8$, then both λ and μ are $(\in,\in\vee q_{0.2}^{0.8})$-fuzzy ideals of S.

Now, $(\lambda\cup\mu)(4\cdot 4)=(\lambda\cup\mu)(3)=(\lambda\vee\mu)(3)\vee\frac{\delta-k}{2}=\lambda(3)\vee\mu(3)\vee\frac{\delta-k}{2}=0.3\vee 0.1\vee\frac{\delta-k}{2}=0.3$

But $(\lambda\vee\mu)(4)\vee\frac{\delta-k}{2}=\lambda(4)\vee\mu(4)\vee\frac{\delta-k}{2}=0.5\vee 0.1\vee\frac{\delta-k}{2}=0.5$. Which shows that, $(\lambda\vee\mu)(4\cdot 4)\vee\frac{\delta-k}{2}=(\lambda\vee\mu)(3)\vee\frac{\delta-k}{2}=0.3\ngeq 0.5=(\lambda\vee\mu)(4)\vee\frac{\delta-k}{2}$.

Theorem 4.19. *If λ is an $(\in,\in)$-fuzzy left (resp. right) ideal of S, then the set $Q_k^\delta(\lambda;t)\neq\emptyset$ is a left (resp. right) ideal of S.*

Proof. Let $a,b\in S$ and $t\in(0,1]$ be such that $(y,t)\in Q_k^\delta(\lambda;t)$. Then, $(y,t)\,q_k^\delta\lambda$, i.e. $\lambda(y)+t+k>\delta$. It implies that $\lambda(xy)+t+k\geq$

$\lambda(y) + t + k > \delta$ and so $(xy, t)\, q_k^{\delta} \lambda$. Hence, $xy \in Q_k^{\delta}(\lambda; t)$. Therefore, $Q_k^{\delta}(\lambda; t)$ is a left ideal of S. □

Similarly we can prove the case of right ideal of S.

Theorem 4.20. *Suppose λ is a fuzzy subset of S. If the set $Q_k^{\delta}(\lambda; t)$-set is a left (resp. right) ideal of S for all $t \in (\frac{\delta-k}{2}, 1]$, then λ is an $\left(\in, q_k^{\delta}\right)$-fuzzy left (resp. right) ideal of S.*

Proof. Let $x, y \in S$ and $t \in (\frac{\delta-k}{2}, 1]$ be such that $(y, t) \in \lambda$. Then, $\lambda(y) \geq t$. It implies that $\lambda(y) + t + k > \frac{\delta-k}{2} + \frac{\delta-k}{2} + k = \delta$, i.e. $(y, t)\, q_k^{\delta} \lambda$ and so, $y \in Q_k^{\delta}(\lambda; t)$. Thus by hypothesis we have, $(xy, t) \in Q_k^{\delta}(\lambda; t)$ and so $(xy, t)\, q_k^{\delta} \lambda$. Therefore, λ is an $\left(\in, q_k^{\delta}\right)$-fuzzy left ideal of S. □

In similar way we can prove the case of right ideal of S.

Theorem 4.21. *If λ is a $\left(q^{\delta}, \in \vee q_k^{\delta}\right)$-fuzzy left (resp. right) ideal of S, then the set $Q_k^{\delta}(\lambda; t)$ is a left (resp. right) ideal of S for all $t \in (\frac{\delta-k}{2}, 1]$.*

Proof. Let $x, y \in S$ and $t \in (\frac{\delta-k}{2}, 1]$ be such that $y \in Q_k^{\delta}(\lambda; t)$. Then, $(y, t)\, q_k^{\delta} \lambda$. Since, λ is a $\left(q^{\delta}, \in \vee q_k^{\delta}\right)$-fuzzy left ideal of S, we have $(xy, t) \in \vee q_k^{\delta} \lambda$, i.e. $(xy, t) \in \lambda$, then $\lambda(xy) \geq t > \delta - t$, since, $t > \frac{\delta-k}{2}$. Hence, $(xy, t)\, q_k^{\delta} \lambda$ and so $xy \in Q_k^{\delta}(\lambda; t)$. Therefore, $Q_k^{\delta}(\lambda; t)$ is a left ideal of S. □

In similar way we can prove the case for right ideal of S.
Using Theorem 4.3 and Theorem 4.21 we have the following theorem.

Theorem 4.22. *Let I be a left (resp. right) ideal of S and λ a fuzzy subset of S defined by*

$$\lambda(x) = \begin{cases} \varepsilon & \text{if } x \in A, \\ 0 & \text{otherwise}, \end{cases}$$

where $\varepsilon \geq \frac{\delta-k}{2}$. Then, the set $Q_k^{\delta}(\lambda; t) \neq \emptyset$ is a left (resp. right) ideal of S, for all $t \in (\frac{\delta-k}{2}, 1]$.

Proof. Straightforward □

Theorem 4.23. *Let λ be a fuzzy subset of S, if the set $\left[\lambda_k^{\delta}\right]_t \neq \emptyset$ is a left (resp. right) ideal of S for all $t \in (0, 1]$. Then, λ is a $\left(q^{\delta}, \in \vee q_k^{\delta}\right)$-fuzzy left (resp. right) ideal of S.*

Proof. Let $x, y \in S$ and $t \in (0, 1]$ be such that $(x, t)\, q_k^{\delta} \lambda$. Then, $y \in Q_k^{\delta}(\lambda; t) \subseteq \left[\lambda_k^{\delta}\right]_t$. From the hypothesis it implies that $xy \in \left[\lambda_k^{\delta}\right]_t$. Hence, $(xy, t) \in \vee q_k^{\delta} \lambda$. Therefore, λ is a $\left(q^{\delta}, \in \vee q_k^{\delta}\right)$-fuzzy left ideal of S. □

In the similar way we can prove the case for right ideal of S.

Here is arise a question that, if λ is an $\left(\in, \in \vee q_k^{\delta}\right)$-fuzzy left (resp. right) ideal of S, then is the set $Q_k^{\delta}(\lambda; t)$ is a left (resp. right) ideal of S?

Answer of the question is negative for $t \leq \frac{\delta - k}{2}$ as shown in the following example:

Example 4.24. *Consider the semigroup of Example 4.14. Define a fuzzy subset λ by; $\lambda(1) = 0.6, \lambda(2) = 0.5 = \lambda(5), \lambda(3) = 0.3 = \lambda(4)$. Then, λ is an $\left(\in, \in \vee q_{0.2}^{0.8}\right)$-fuzzy left ideal of S, where $k = 0.2$ and $\delta = 0.8$. But the set*

$$Q_{0.8}^{0.2}(\lambda; t) = \{1, 2, 5\}$$

is not a left ideal of S. Because $5 \cdot 5 = 3 \notin Q_{0.8}^{0.2}(\lambda; t)$.

But the following theorem answers the above question.

Theorem 4.25. *Let λ be an $(\in, \in \vee q_k^{\delta})$-fuzzy left (resp. right) ideal of S. Then the set $Q_k^{\delta}(\lambda; t) \neq \emptyset$ is a left (resp. right) ideal of S for all $t \in (\frac{\delta - k}{2}, 1]$.*

Proof. Suppose, $Q_k^{\delta}(\lambda; t) \neq \emptyset$ for $t \in (\frac{\delta - k}{2}, 1]$. Let $x \in S$ and $y \in Q_k^{\delta}(\lambda; t)$. Then, $(y, t)\, q_k^{\delta} \lambda$ i.e. $\lambda(y) + t + k > q_k^{\delta}$. It implies that

$$\begin{aligned} \lambda(xy) + t + k &\geq \left\{ \lambda(y) \wedge \frac{\delta - k}{2} \right\} + t + k \\ &= \lambda(y) + t + k \wedge \frac{\delta - k}{2} + t + k \\ &> \delta. \end{aligned}$$

Thus, $(xy, t)\, q_k^{\delta} \lambda$. Hence, $xy \in Q_k^{\delta}(\lambda; t)$ and therefore, $Q_k^{\delta}(\lambda; t)$ is a left ideal of S. □

Similarly we can show the case of right ideal of S.

Theorem 4.26. *A fuzzy subset λ in S is an $(\in, \in \vee q_k^{\delta})$-fuzzy left (resp. right) ideal of S. Then the set $\left[\lambda_k^{\delta}\right]_t \neq \emptyset$ is a left (resp. right) ideal of S for all $t \in (0, 1]$.*

Proof. Suppose that λ is an $(\in, \in \vee q_k^{\delta})$-fuzzy left ideal of S and $y \in \left[\lambda_k^{\delta}\right]_t$, then $(y, t) \in \vee q_k^{\delta} \lambda$, i.e. $\lambda(y) > t$ or $\lambda(y) + t + k > \delta$. Since, λ is an $(\in, \in \vee q_k^{\delta})$-fuzzy left ideal of S, we have $\lambda(xy) \geq \lambda(y) \wedge \frac{\delta - k}{2}$. Thus we have two cases:

Case (1) : $\lambda(y) \geq t$. If $t > \frac{\delta-k}{2}$, then $\lambda(xy) \geq \lambda(y) \wedge \frac{\delta-k}{2} = \frac{\delta-k}{2}$ and hence $(xy,t)\, q_k^{\delta} \lambda$. If $t \leq \frac{\delta-k}{2}$, then $\lambda(xy) \geq \lambda(y) \wedge \frac{\delta-k}{2} \geq t$ and so $(xy,t) \in \lambda$. Hence, $(xy,t) \in \vee q_k^{\delta} \lambda$.

Case (2) : Let $\lambda(y) + t + k > \delta$. If $t > \frac{\delta-k}{2}$, then $1 - t < \frac{\delta-k}{2} < t$ and

$$\begin{aligned} \lambda(xy) &\geq \lambda(y) \wedge \frac{\delta-k}{2} \\ &= \{.\lambda(y) \text{ if } \lambda(y) < \frac{\delta-k}{2} \frac{\delta-k}{2} \text{ if } \lambda(y) \geq \frac{\delta-k}{2} \\ &> 1-t. \end{aligned}$$

Hence, $(xy,t) \in \vee q_k^{\delta} \lambda$. If $t \leq \frac{\delta-k}{2}$, then $\lambda(xy) \geq \lambda(y) \wedge \frac{\delta-k}{2} \geq 1 - t \wedge \frac{\delta-k}{2} = \frac{\delta-k}{2} \geq t$, and so $(xy.t) \in \lambda$. Thus $(xy,t) \in \vee q_k^{\delta} \lambda$. Hence, in any case $\left[\lambda_k^{\delta}\right]_t$ is a left ideal of S.

Conversely, let λ that is a fuzzy subset of S and $t \in (0,1]$ such that is a left ideal of S. Assume that $\lambda(xy) < t \leq \lambda(y) \wedge \frac{\delta-k}{2}$ for some $t \in (0, \frac{\delta-k}{2}]$ and $x, y \in S$. Then, $y \in U(\lambda; t) \subseteq \left[\lambda_k^{\delta}\right]_t$, it follows that $xy \in \left[\lambda_k^{\delta}\right]_t$. Hence, $\lambda(xy) \geq t$ or $\lambda(xy) + t + k > \delta$, which is a contradiction. It implies that $\lambda(xy) \geq \lambda(y) \wedge \frac{\delta-k}{2}$ for all $x, y \in S$. Therefore, λ is an $(\in, \in \vee q_k^{\delta})$-fuzzy left ideal of S. □

In similar way we can prove the case of right ideal of S.

5. Regular semigroups

In this section, we define the upper/lower parts of an $(\in, \in \vee q_k^{\delta})$-fuzzy left (resp. right) ideal and characterize regular, left weakly regular and right weakly regular semigroups in terms of lower parts of $(\in, \in \vee q_k^{\delta})$-fuzzy left (resp. right) ideals.

Definition 5.1. *Let λ be a fuzzy subset of S. Then, the (δ,k)-upper part of λ is denoted by $\left(\lambda_k^{\delta}\right)^+$ and is defined by:*

$$\left(\lambda_k^{\delta}\right)^+(x) = \lambda(x) \vee \frac{\delta-k}{2}$$

for all $x \in S$, where $\delta \in (0,1], k \in [0,1)$ and $k < \delta$ in $[0,1]$.Clearly, $\left(\lambda_k^{\delta}\right)^+$ is a fuzzy subset of S. Let A be a non-empty subset of S and λ a fuzzy subset of S. Then the (δ,k)-upper part of the characteristic function λ_A, is denoted by $\left(\lambda_k^{\delta}\right)_A^+$ and is defined by:

$$\left(\lambda_k^{\delta}\right)_A^+ : S \to [0,1] \quad x \mapsto \left(\lambda_k^{\delta}\right)_A^+(x) = \begin{cases} \delta & \text{if } x \in A \\ \frac{\delta-k}{2} & \text{otherwise} \end{cases}$$

Definition 5.2. *Let λ be a fuzzy subset of S. Then the (δ,k)-lower part of λ is denoted by $\left(\lambda_k^\delta\right)^-$ and is defined by:*

$$\left(\lambda_k^\delta\right)^-(x)=\lambda(x)\wedge\frac{\delta-k}{2}$$

for all $x\in S$ where $\delta\in(0,1], k\in[0,1)$ and $k<\delta$ in $[0,1]$. Clearly, $\left(\lambda_k^\delta\right)^-$is a fuzzy subset of S. Let A be a non-empty subset of S and λ a fuzzy subset of S. Then the (δ,k)-lower part of the characteristic function λ_A, is denoted by $\left(\lambda_k^\delta\right)_A^-$ and is defined by:

$$\left(\lambda_k^\delta\right)_A^-:S\to[0,1]\quad x\mapsto\left(\lambda_k^\delta\right)_A^-(x)=\begin{cases}\frac{\delta-k}{2} & \text{if } x\in A\\ 0 & \text{otherwise}\end{cases}$$

Definition 5.3. *Let λ and μ be two fuzzy subset of S. We define the fuzzy subsets of $\left(\lambda_k^\delta\right)^+$, $\left(\lambda\wedge_k^\delta\mu\right)^+$, $\left(\lambda\vee_k^\delta\mu\right)^+$, $\left(\lambda\circ_k^\delta\mu\right)^+$, $\left(\lambda_k^\delta\right)^-$, $\left(\lambda\wedge_k^\delta\mu\right)^-$, $\left(\lambda\vee_k^\delta\mu\right)^-$ and $\left(\lambda\circ_k^\delta\mu\right)^-$ of S as follows;*

i) $\left(\lambda_k^\delta\right)^+(x)=\lambda(x)\vee\frac{\delta-k}{2}$

ii) $\left(\lambda\wedge_k^\delta\mu\right)^+(x)=(\lambda\wedge\mu)(x)\vee\frac{\delta-k}{2}$

iii) $\left(\lambda\vee_k^\delta\mu\right)^+(x)=(\lambda\vee\mu)(x)\vee\frac{\delta-k}{2}$

iv) $\left(\mu\circ_k^\delta\mu\right)^+(x)=(\lambda\circ\eta)(x)\vee\frac{\delta-k}{2}$

v) $\left(\lambda_k^\delta\right)^-(x)=\lambda(x)\wedge\frac{\delta-k}{2}$

vi) $\left(\lambda\wedge_k^\delta\mu\right)^-(x)=(\lambda\wedge\mu)(x)\wedge\frac{\delta-k}{2}$

vii) $\left(\lambda\vee_k^\delta\mu\right)^-(x)=(\lambda\vee\mu)(x)\wedge\frac{\delta-k}{2}$

viii) $\left(\lambda\circ_k^\delta\mu\right)^-(x)=(\lambda\circ\mu)(x)\wedge\frac{\delta-k}{2}$

Lemma 5.4. *Let λ and μ be two fuzzy subset of S. Then the following hold.*

i) $\left(\lambda\wedge_k^\delta\mu\right)^+=\left(\lambda_k^\delta\right)^+\wedge\left(\mu_k^\delta\right)^+$

ii) $\left(\lambda\vee_k^\delta\mu\right)^+=\left(\lambda_k^\delta\right)^+\vee\left(\mu_k^\delta\right)^+$

iii) $\left(\lambda\circ_k^\delta\mu\right)^+=\left(\lambda_k^\delta\right)^+\circ\left(\mu_k^\delta\right)^+$

Proof. Let $x\in S$.

i)

$$\begin{aligned}\left(\lambda \wedge_k^\delta \mu\right)^+(x) &= (\lambda \wedge \mu)(x) \vee \frac{\delta - k}{2} \\ &= \lambda(x) \wedge \mu(x) \vee \frac{\delta - k}{2} \\ &= \left(\lambda(x) \vee \frac{\delta - k}{2}\right) \wedge \left(\mu(x) \vee \frac{\delta - k}{2}\right) \\ &= \left(\lambda_k^\delta\right)^+(x) \wedge \left(\mu_k^\delta\right)^+(x) \\ &= \left(\left(\lambda_k^\delta\right)^+ \wedge \left(\mu_k^\delta\right)^+\right)(x).\end{aligned}$$

ii)

$$\begin{aligned}\left(\lambda \vee_k^\delta \mu\right)^+(x) &= (\lambda \vee \mu)(x) \vee \frac{\delta - k}{2} \\ &= \lambda(x) \vee \mu(x) \vee \frac{\delta - k}{2} \\ &= \left(\lambda(x) \vee \frac{\delta - k}{2}\right) \vee \left(\mu(x) \vee \frac{\delta - k}{2}\right) \\ &= \left(\lambda_k^\delta\right)^+(x) \vee \left(\mu_k^\delta\right)^+(x) \\ &= \left(\left(\lambda_k^\delta\right)^+ \vee \left(\mu_k^\delta\right)^+\right)(x).\end{aligned}$$

iii) If it is not possible to express x as $x = ab$ for all $a, b \in S$, then $\left(\left(\lambda \circ_k^\delta \mu\right)^+\right)(x) = 0$. Thus, $\left(\left(\lambda \circ_k^\delta \mu\right)^+\right)(x) = (\lambda \circ \mu)(x) \vee \frac{\delta-k}{2} = 0$. As x is not express in the form $x = ab$, so $\left(\left(\lambda_k^\delta\right)^+ \circ \left(\mu_k^\delta\right)^+\right)(x) = 0$. Thus, in this case $\left(\lambda \circ_k^\delta \mu\right)^+ = \left(\lambda_k^\delta\right)^+ \circ \left(\mu_k^\delta\right)^+$. If it is possible to express $x = ab$

for $a, b \in S$. Then,

$$\begin{aligned}
\left(\lambda \circ_k^\delta \mu\right)^+(x) &= (\lambda \circ \mu)(x) \vee \frac{\delta - k}{2} \\
&= \left[\bigvee_{x=ab} \{\lambda(a) \wedge \mu(b)\}\right] \vee \frac{\delta - k}{2} \\
&= \bigvee_{x=ab} \left(\{\lambda(a) \wedge \mu(b)\} \vee \frac{\delta - k}{2}\right) \\
&= \bigvee_{x=ab} \left(\lambda(a) \wedge \mu(b) \vee \frac{\delta - k}{2}\right) \\
&= \bigvee_{x=ab} \left\{\left(\lambda(a) \vee \frac{\delta - k}{2}\right) \wedge \left(\mu(b) \vee \frac{\delta - k}{2}\right)\right\} \\
&= \bigvee_{x=ab} \left\{\left(\lambda_k^\delta\right)^+ \wedge \left(\mu_k^\delta\right)^+\right\} \\
&= \left(\left(\lambda_k^\delta\right)^+ \circ \left(\mu_k^\delta\right)^+\right)(x).
\end{aligned}$$

□

Lemma 5.5. *Let λ and μ be two fuzzy subset of S. Then the following hold.*

i) $\left(\lambda \wedge_k^\delta \mu\right)^- = \left(\lambda_k^\delta\right)^- \wedge \left(\mu_k^\delta\right)^-$

ii) $\left(\lambda \vee_k^\delta \mu\right)^- = \left(\lambda_k^\delta\right)^- \vee \left(\mu_k^\delta\right)^-$

iii) $\left(\mu \circ_k^\delta \mu\right)^- = \left(\lambda_k^\delta\right)^- \circ \left(\mu_k^\delta\right)^-$

Proof. The proof follows from [10]. □

Lemma 5.6. *Let A and B be two non-empty subset of S. Then the following hold.*

i) $\left(\lambda_A \wedge_k^\delta \lambda_B\right)^- = \left(\lambda_k^\delta\right)^-_{A\cap B}$

ii) $\left(\lambda_A \vee_k^\delta \lambda_B\right)^- = \left(\lambda_k^\delta\right)^-_{A\cup B}$

iii) $\left(\lambda_A \circ_k^\delta \lambda_B\right)^- = \left(\lambda_k^\delta\right)^-_{AB}$

Proof. The proof follows from [10]. □

Lemma 5.7. *Let I be a non-empty subset of S. Then, I is a left (right) ideal of S if and only if $(C_k^\delta)_I^-$ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S.*

Proof. Let I be a left (right) ideal of S. Then by Theorem 4.3, $(C_k^\delta)_I^-$ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S.

Conversely, suppose that $(C_k^\delta)_I^-$ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S. Let $y \in I$. Then, $(C_k^\delta)_I^+(y) = \frac{\delta-k}{2}$. So $\left(y, \frac{\delta-k}{2}\right) \in (C_k^\delta)_I^-$. Since, $(C_k^\delta)_I^-$ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S, so $\left(xy\,, \frac{\delta-k}{2}\right) \in \vee q_k^\delta (C_k^\delta)_I^-$ which implies that $\left(xy, \frac{\delta-k}{2}\right) \in (C_k^\delta)_I^-$ or $\left(xy, \frac{\delta-k}{2}\right) \in q_k^\delta (C_k^\delta)_I^-$. Thus, $(C_k^\delta)_I^-(xy) \geq \frac{\delta-k}{2}$ or $(C_k^\delta)_I^-(xy) + \frac{\delta-k}{2} + k > \delta$. If $(C_k^\delta)_I^-(xy) + \frac{\delta-k}{2} + k > \delta$, then $(C_k^\delta)_I^-(xy) > \frac{\delta-k}{2}$. Thus, $(C_k^\delta)_I^-(xy) \geq \frac{\delta-k}{2}$, which implies that $(C_k^\delta)_I^-(xy) = \frac{\delta-k}{2}$. Hence, $xy \in I$. Therefore, I is a left ideal of S. □

From the above Lemma we deduce the following corollary.

Corollary 5.8. *Let I be a non-empty subset of S. Then, I is a ideal of S if and only if $(C_k^\delta)_I^-$ is an $(\in, \in \vee q_k^\delta)$-fuzzy ideal of S.*

From Example 4.9 we see that every fuzzy ideal, an $(\in, \in \vee q)$-fuzzy ideal and an $(\in, \in \vee q_k)$-fuzzy ideal are $(\in, \in \vee q_k^\delta)$-fuzzy ideal but the converse is not true. In the following theorem we show that if λ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S then $\left(\lambda_k^\delta\right)^-$is fuzzy left (right) ideal of S.

Theorem 5.9. *If λ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S. Then, $\left(\lambda_k^\delta\right)^-$ is fuzzy left(right) ideal of S.*

Proof. Suppose λ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S. Then for all $x, y \in S$, we have $\lambda(xy) \geq \lambda(y) \wedge \frac{\delta-k}{2}$, which implies that $\lambda(xy) \wedge \frac{\delta-k}{2} \geq \lambda(y) \wedge \frac{\delta-k}{2}$, so $\left(\lambda_k^\delta\right)^-(xy) \geq \left(\lambda_k^\delta\right)^-(y)$. Hence, $\left(\lambda_k^\delta\right)^-$ is an $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal of S. □

From the above theorem we have the following corollary.

Corollary 5.10. *If λ is an $(\in, \in \vee q_k^\delta)$-fuzzy ideal of S, then $\left(\lambda_k^\delta\right)^-$ is fuzzy ideal of S.*

Next we characterize regular semigroup in term of $(\in, \in \vee q_k^\delta)$-fuzzy ideals.

Theorem 5.11. [11] *For a semigroup S the following conditions are equivalent.*

(1) S is regular.
(2) $R \cap L = RL$ for every right ideal R and every left ideal L of S.

Theorem 5.12. *For a semigroup S the following conditions are equivalent:*

i) S is regular

ii) $\left(\lambda \wedge_k^\delta \mu\right)^- = \left(\lambda \circ_k^\delta \mu\right)^-$ for every $(\in, \in \vee q_k^\delta)$-fuzzy right ideal λ and every $(\in, \in \vee q_k^\delta)$-fuzzy left ideal μ of S.

Proof. *i)* $\Rightarrow$ *ii)* Let λ be an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal and μ an $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. For $x \in S$, we have

$$\begin{aligned}
\left(\lambda \circ_k^\delta \mu\right)^-(x) &= (\lambda \circ \mu)(x) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=ab} \{\lambda(a) \wedge \mu(b)\}\right) \wedge \frac{\delta - k}{2} \\
&= \bigvee_{x=ab} \left\{\lambda(a) \wedge \mu(b) \wedge \frac{\delta - k}{2}\right\} \\
&= \bigvee_{x=ab} \left\{\left(\lambda(a) \wedge \frac{\delta - k}{2}\right) \wedge \left(\mu(b) \wedge \frac{\delta - k}{2}\right) \wedge \frac{\delta - k}{2}\right\} \\
&\leq \bigvee_{x=ab} \left\{(\lambda(ab) \wedge \mu(ab)) \wedge \frac{\delta - k}{2}\right\} \\
&= \lambda(a) \wedge \mu(b) \wedge \frac{\delta - k}{2} \\
&= \left(\lambda \wedge_k^\delta \mu\right)^-(x)
\end{aligned}$$

So, $\left(\lambda \circ_k^\delta \mu\right)^- \leq \left(\lambda \wedge_k^\delta \mu\right)^-$. Since, S is regular, so there exists an element $x \in S$ such that $x = xax$. Thus,

$$\begin{aligned}
\left(\lambda \circ_k^\delta \mu\right)^-(x) &= (\lambda \circ \mu)(x) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=ab} \{\lambda(a) \wedge \mu(b)\}\right) \wedge \frac{\delta - k}{2} \\
&\geq \lambda(xa) \wedge \mu(x) \wedge \frac{\delta - k}{2} \\
&\geq \left(\lambda(x) \wedge \frac{\delta - k}{2} \wedge \mu(x)\right) \wedge \frac{\delta - k}{2} \\
&= \lambda(x) \wedge \mu(x) \wedge \frac{\delta - k}{2} \\
&= \left(\lambda \wedge_k^\delta \mu\right)^-(x)
\end{aligned}$$

Hence, $\left(\lambda \circ_k^\delta \mu\right)^- \geq \left(\lambda \wedge_k^\delta \mu\right)^-$. Thus, $\left(\lambda \circ_k^\delta \mu\right)^- = \left(\lambda \wedge_k^\delta \mu\right)^-$.

ii) $\Rightarrow$ *i)*: Let R and L be left and right ideal of S. Then by Lemma 5.7, $(C_k^\delta)_R^-$ and $(C_k^\delta)_L^-$ are $(\in, \in \vee q_k^\delta)$-fuzzy right ideal and $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S, respectively. Thus we have

$$\begin{aligned}(C_k^\delta)_{RL}^- &= \left(C_R \circ_k^\delta C_L\right)^- \\ &= \left(C_R \wedge_k^\delta C_L\right)^- \text{ (since by } (i)) \\ &= (C_k^\delta)_{R\cap L}.\end{aligned}$$

Thus, $R \cap L = RL$. Hence by Theorem 5.11 it follows that S is regular.

□

Definition 5.13. [3, 8] *An element x of a semigroup S is said to be right weakly regular if there exists $a, b \in S$ such that $x = xaxb$ and left weakly regular if $x = axbx$. A semigroup is said to be right (left) weakly regular if all its elements are right (left) weakly regular.*

Example 5.14. *Let $S = \{1, 2, 3, \}$ be a semigroup with a binary operation "$\cdot$" defined on S in the following Cayley table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	c	d
c	a	b	c	d
d	a	b	c	d

Then, S is a left weakly regular semigroup. Define a fuzzy subset λ in S by

$$\lambda(a) = 0.3, \lambda(b) = 0.5, \lambda(c) = 0.4, \lambda(d) = 0.5.$$

Then, λ is an $(\in, \in \vee \mu)$-fuzzy ideal of S, where $k = 0.4$ and $\delta = 0.8$.

Lemma 5.15. [8]*For a semigroup S the following are equivalent:*
i) S is a left (right) weakly regular.
ii) $L_1 \cap L_2 \subseteq L_1 L_2$ where L_1 and L_2 are left (right) ideals of S.
iii) $L_1[x] \cap L_2[x] \subseteq L_1[x] L_2[x]$ for some $x \in S$.

Theorem 5.16. *For a semigroup S the following are equivalent:*
i) S is left(right) weakly regular,
ii) $\left(\lambda \wedge_k^\delta \mu\right)^- \leq \left(\lambda \circ_k^\delta \mu\right)^-$ for all $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal λ and μ of S.

Proof. $i) \Rightarrow ii)$: Let λ and μ be $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. Since, S is left regular, then for each $x \in S$, there exists $a, b \in S$, such that $x = axbx$, so we have,

$$\begin{aligned}
\left(\lambda \circ_k^\delta \mu\right)^- (x) &= (\lambda \circ \mu)(x) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=pq} \{\lambda(p) \wedge \mu(q)\}\right) \wedge \frac{\delta - k}{2} \\
&\geq \lambda(ax) \wedge \mu(bx) \wedge \frac{\delta - k}{2} \\
&\geq \lambda(x) \wedge \mu(x) \wedge \frac{\delta - k}{2} \\
&= \left(\lambda \wedge_k^\delta \mu\right)^- (x)
\end{aligned}$$

Therefore, $\left(\lambda \wedge_k^\delta \mu\right)^- \leq \left(\lambda \circ_k^\delta \mu\right)^-$.

$ii) \Rightarrow i)$: Let $L_1$and L_2be two left ideals of S. Then by Lemma 5.7 $(C_k^\delta)_{L_1}^-$ and $(C_k^\delta)_{L_2}^-$ are $(\in, \in \vee q_k^\delta)$-fuzzy left ideal of S. Let, $x \in L_1 \cap L_2$, then $(C_k^\delta)_{L_1 L_2}^- = (C)_{L_1} \circ_k^\delta (C)_{L_2} \geq (C)_{L_1} \wedge_k^\delta (C)_{L_2} = (C_k^\delta)_{L_1 \cap L_2}^- \geq \frac{\delta - k}{2}$. Thus, $x \in L_1 L_2$.Therefore, $L_1 \cap L_2 \subseteq L_1 L_2$. Hence it follows from 5.15 that S is left regular semigroup. □

Theorem 5.17. *For a semigroup S the following are equivalent:*

i) S is left (right) weakly regular;

ii) $\left(\lambda \circ_k^\delta \mu\right)^- \wedge \left(\mu \circ_k^\delta \lambda\right)^- \geq \left(\lambda \wedge_k^\delta \mu\right)^-$for every $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal λ and η of S.

Proof. $i) \Rightarrow ii)$: Let λ and μ be two $(\in, \in \vee q_k^\delta)$-fuzzy left ideals of S. Since, S is left weakly regular, then for all $x \in S$ there exists $a, b \in S$ such that $x = axbx$. So we have,

$$\begin{aligned}
\left(\lambda \circ_k^\delta \mu\right)^- (x) &= (\lambda \circ \mu)(x) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=pq} \{\lambda(p) \wedge \mu(q)\}\right) \wedge \frac{\delta - k}{2} \\
&\geq \lambda(ax) \wedge \mu(bx) \wedge \frac{\delta - k}{2} \\
&\geq \lambda(x) \wedge \mu(x) \wedge \frac{\delta - k}{2} \\
&= \left(\lambda \wedge_k^\delta \mu\right)^- (x)
\end{aligned}$$

Also,

$$\begin{aligned}
\left(\eta \circ_k^\delta \lambda\right)^-(x) &= (\eta \circ \lambda)(x) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=pq} \{\eta(p) \wedge \lambda(q)\}\right) \wedge \frac{\delta - k}{2} \\
&\geq \eta(ax) \wedge \lambda(bx) \wedge \frac{\delta - k}{2} \\
&\geq \eta(x) \wedge \lambda(x) \wedge \frac{\delta - k}{2} \\
&= \left(\lambda \wedge_k^\delta \mu\right)^-(x)
\end{aligned}$$

Therefore, $\left(\lambda \circ_k^\delta \mu\right)^- \wedge \left(\mu \circ_k^\delta \lambda\right)^- \geq \left(\lambda \wedge_k^\delta \mu\right)^-$.

ii) $\Rightarrow$ *i)*: Let λ and μ be any two $(\in, \in \vee q_k^\delta)$-fuzzy left ideals of S. Then by assumption $\left(\lambda \circ_k^\delta \mu\right)^- \wedge \left(\mu \circ_k^\delta \lambda\right)^- \geq \left(\lambda \wedge_k^\delta \mu\right)^-$, so by Theorem 5.16, S is left weakly regular semigroup. □

Theorem 5.18. *For a semigroup S the following are equivalent:*

i) S is left (right) weakly regular

ii) $\left(\lambda \wedge_k^\delta \mu \wedge_k^\delta \eta\right)^- \leq \left(\lambda \circ_k^\delta \mu \circ_k^\delta \eta\right)^-$ for every $(\in, \in \vee q_k^\delta)$-fuzzy left (right) ideal λ, μ and η of S.

Proof. *i)* $\Rightarrow$ *ii)*: Let λ, μ and η be three $(\in, \in \vee q_k^\delta)$-fuzzy left ideals of S. Since S is left weakly regular. Therefore, for $x \in S$ there exists $a, b \in S$,such that $x = axbx$. Then $x = axbx = a(axbx)bx = aaxb(axba)bx =$

$(aaxbax)(bx)(bx)$. Thus

$$\begin{aligned}
\left(\lambda \circ_k^\delta \mu \circ_k^\delta \eta\right)^- (x) &= (\lambda \circ \mu \circ \eta)(x) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=pq} \{\lambda(p) \wedge (\mu \circ \eta)(q)\}\right) \wedge \frac{\delta - k}{2} \\
&\geq \lambda(aaxbax) \wedge (\mu \circ \eta)((bx)(bx)) \wedge \frac{\delta - k}{2} \\
&\geq \lambda(x) \wedge \mu(bx) \wedge \eta(bx) \wedge \frac{\delta - k}{2} \\
&\geq \lambda(x) \wedge \mu(x) \wedge \eta(x) \wedge \frac{\delta - k}{2} \\
&= \left(\lambda \wedge_k^\delta \mu \wedge_k^\delta \eta\right)^- (x)
\end{aligned}$$

Therefore, $\left(\lambda \wedge_k^\delta \mu \wedge_k^\delta \eta\right)^- \leq \left(\lambda \circ_k^\delta \mu \circ_k^\delta \eta\right)^-$.

$ii) \Rightarrow i)$: Let λ and μ be two $(\in, \in \vee q_k^\delta)$-fuzzy left ideals of S. Since, S itself is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideals of S. So by Lemma 5.7 $\left(C_k^\delta\right)_S^-$ is an $(\in, \in \vee q_k^\delta)$-fuzzy left ideals of S. Thus by (ii), we have

$$\begin{aligned}
\left(\lambda \wedge_k^\delta \eta\right)^- (x) &= (\lambda \wedge \eta)(x) \wedge \frac{\delta - k}{2} \\
&= \left(\lambda \wedge \left(C_k^\delta\right)_S^- \wedge \eta\right)(x) \wedge \frac{\delta - k}{2} \\
&= \left(\lambda \wedge_k^\delta \left(C_k^\delta\right)_S^- \wedge_k^\delta \eta\right)^- (x) \\
&\leq \left(\lambda \circ_k^\delta \left(C_k^\delta\right)_S^- \circ_k^\delta \eta\right)^- (x) \\
&= (\lambda \circ C_S \circ \eta)(x) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=pq} \{\lambda(p) \wedge (S \circ \eta)(q)\}\right) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=pq} \left\{\lambda(p) \wedge \left(\bigvee_{q=st} \{S(s) \wedge \eta(t)\}\right)\right\}\right) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=pq} \left\{\lambda(p) \wedge \left(\bigvee_{q=st} \{1 \wedge \eta(t)\}\right)\right\}\right) \wedge \frac{\delta - k}{2} \\
&\leq \left(\bigvee_{x=pq} \left\{\lambda(p) \wedge \left(\bigvee_{q=st} \{\eta(st)\}\right)\right\}\right) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=pq} \{\lambda(p) \wedge \eta(q)\}\right) \wedge \frac{\delta - k}{2} \\
&= \left(\bigvee_{x=pq} \left\{\lambda(p) \wedge \eta(q) \wedge \frac{\delta - k}{2}\right\}\right) \\
&= \left(\lambda \circ_k^\delta \eta\right)^- (x)
\end{aligned}$$

Therefore, $\left(\lambda \wedge_k^\delta \eta\right)^- (x) \leq \left(\lambda \circ_k^\delta \eta\right)^- (x)$ for every $(\in, \in \vee q_k^\delta)$-fuzzy left ideals λ and η of S. Hence by Theorem 5.16, S is left weakly regular semigroup. □

Theorem 5.19. *For a semigroup S the following are equivalent.*

i) S is a right (left) weakly regular;

ii) $\left(\lambda_k^\delta\right)^- = \left(\lambda \circ_k^\delta \lambda\right)^-$ for every $(\in, \in \vee q_k^\delta)$-fuzzy right (left) ideal μ of S.

Proof. $i) \Rightarrow ii)$: Let S be a right weakly regular semigroup and μ an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal of S. Then for each $x \in S$, there exists $a, b \in S$ such that $x = xaxb$, so we have

$$\begin{aligned}
\left(\lambda \circ_k^\delta \lambda\right)^- (x) &= \left(\bigvee_{x=pq} \{\lambda(p) \wedge \lambda(q)\}\right) \wedge \frac{\delta - k}{2} \\
&= \bigvee_{x=pq} \left\{\lambda(p) \wedge \lambda(q) \wedge \frac{\delta - k}{2}\right\} \\
&= \lambda(xa) \wedge \lambda(xb) \wedge \frac{\delta - k}{2} \\
&\geq \lambda(x) \wedge \lambda(x) \wedge \frac{\delta - k}{2} \\
&= \lambda(x) \wedge \frac{\delta - k}{2}
\end{aligned}$$

Therefore, $\left(\lambda_k^\delta\right)^- \leq \left(\lambda \circ_k^\delta \lambda\right)^-$ but $\left(\lambda_k^\delta\right)^- \geq \left(\lambda \circ_k^\delta \lambda\right)^-$. Hence, $\left(\lambda_k^\delta\right)^- = \left(\lambda \circ_k^\delta \lambda\right)^-$.

$ii) \Rightarrow i)$: Let $x \in S$ and $R[x]$ a right ideal of S generated by x. Then by Lemma 5.7 $\left(C_k^\delta\right)^-_{R[x]}$ is an $(\in, \in \vee q_k^\delta)$-fuzzy right ideal of S. Now, $\left(C_k^\delta\right)^-_{R[x]R[x]}(x) = (C)_{R[x]} \circ_k^\delta (C)_{R[x]}(x) = \left(C_k^\delta\right)^-_{R[x]}(x)$. Which implies that, $x \in R[x]R[x] = (x \cup xS)(x \cup xS) = x^2 \cup xxS \cup xSx \cup xSxS$. Thus, $x = x^2$ or $x = xax$ where, $a = x$ or $x = xbx$ where, $x = b$ or $x = xuxv$ where $u = v = a$. When $x = x^2$, then $x = x^2x^2 = xxxx = xpxq$ where $p = q = x$. When $x = xxa = (xxa)ax = alxa$ where $l = xa$. Then, $x = xbx = (xbx)bx = xbxm$ where $m = bx$. Therefore, S is a right weakly regular semigroup. □

6. CONCLUSION

The aim of our thus study is to support research and the improvement of fuzzy technology by studying the generalized fuzzy semigroups. The objective is to describe new operational improvements in fuzzy semigroups which will also be of developing importance in the upcoming. Since the concept of fuzzy ideal of a semigroup play a vital role in the study of semigroup structure, by using the idea of a quasi-coincidence of a fuzzy point with a fuzzy set, we use the idea of Davvaz et al. to semigroup and defined a new generalization of ideals of semigroups. A generalization of fuzzy ideals in semigroups of type $(\in, \in \vee q_k^\delta)$-fuzzy ideals are defined, where $k \in [0,1)$ and $k < \delta$ in $[0,1]$. We have proved some basic results that define the relation between these notions and ideals of semigroups. We define fuzzy finite state machine and $(\in, \in \vee q_k^\delta)$-fuzzy transformation semigroup. Moreover, we give some applications of the present concept in fuzzy finite state machine and $(\in, \in \vee q_k^\delta)$-fuzzy transformation semigroup. We hope that the research through this way can be continued and indeed, this work would attend as a font for further study of the theory of semigroups.

Future aspect of the study is that, we can apply the present concept to other algebraic structures, i.e Ring, Hemiring, Nearring etc. We will define $(\in, \in \vee q_k^\delta)$-fuzzy soft ideals in semigroups.

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