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SOME RELATIONS BETWEEN $\zeta(2n+1)$ AND $\zeta(2n+1,\alpha)$ FOR SPECIAL VALUES OF α

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Abstract. Hurwitz zeta function occurs in various parts of mathematics. In particular, it plays an important role in some area of number theory. In this paper, using a certain transformation formula, we find some identities of relations between $\zeta(2n+1)$ and $\zeta(2n+1,\alpha)$ for special values of α.

1. Introduction and preliminaries

The Hurwitz zeta function is defined for complex s with $\mathrm{Re}(s) > 1$ and α with $\mathrm{Re}(\alpha) > 0$ by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}.$$

Let $\zeta(s)$ be the Riemann zeta function. For $\alpha = \frac{1}{2}$, it is easy to see that

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s),$$

which gives a relation between $\zeta(s)$ and $\zeta(s, \frac{1}{2})$. In this paper, we find this kind of relations between $\zeta(2n+1)$ and $\zeta(2n+1,\alpha)$ for special values of α and positive integer n.

Let N be a positive integer and let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. For every $\tau \in \mathbb{H}$,

$$V\tau = V(\tau) = \frac{a\tau + b}{c\tau + d}$$

denotes a modular transformation with $c > 0$ and $c \equiv 0 \pmod N$. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors, and define the

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associated vectors R and H by

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

For a complex number w, let $e(w) = e^{2\pi iw}$ and let the branch of the argument be defined by $-\pi \leq \arg w < \pi$. For $\tau \in \mathbb{H}$ and any complex number s, define

$$A_N(\tau, s; r, h) = \sum_{Nm+r_1>0} \sum_{n-h_2>0} \frac{e\left(Nmh_1 + ((Nm + r_1)\tau + r_2)(n - h_2)\right)}{(n - h_2)^{1-s}}$$

and

$$H_N(\tau, s; r, h) = A_N(\tau, s; r, h) + e\left(\frac{s}{2}\right) A_N(\tau, s; -r, -h).$$

For real x, β and complex s with $\mathrm{Re}(s) > 1$, let

$$\psi(x, \beta, s) := \sum_{n+\beta>0} \frac{e(nx)}{(n + \beta)^s}.$$

Let λ_N denote the characteristic function of the integers modulo N. For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x - [x]$. Then the modular transformation formula for $H_N(\tau, s; r, h)$ is given by the following theorem which plays a principal role to obtain our results.

Theorem 1.1. ([2]) *Let* $Q = \left\{\tau \in \mathbb{C} \mid \mathrm{Re}(\tau) > -\frac{d}{c}\right\}$ *and* $\varrho_N = c\{R_2\} - Nd\{R_1/N\}$, $c = c'N$. *Then for* $\tau \in Q$ *and all* s,

$$\begin{aligned}
&(c\tau + d)^{-s} H_N(V\tau, s; r, h) \\
&= H_N(\tau, s; R, H) \\
&\quad -\lambda_N(r_1) \frac{e(-r_1h_1)\Gamma(s)}{(-2\pi i)^s (c\tau + d)^s} \left(\psi(h_2, r_2, s) + e\left(\frac{s}{2}\right) \psi(-h_2, -r_2, s)\right) \\
&\quad +\lambda_N(R_1) \frac{e(-R_1H_1)\Gamma(s)}{(-2\pi i)^s} \left(\psi(H_2, R_2, s) + e\left(-\frac{s}{2}\right) \psi(-H_2, -R_2, s)\right) \\
&\quad +(2\pi i)^{-s} L_N(\tau, s; R, H),
\end{aligned}$$

where

$$\begin{aligned}
&L_N(\tau, s; R, H) \\
&= \sum_{j=1}^{c'} e(-H_1(Nj + N[R_1/N] - c) - H_2([R_2] + 1 + [(Njd + \varrho_N)/c] - d)) \\
&\quad \cdot \int_C u^{s-1} \frac{e^{-(c\tau+d)(Nj - N\{R_1/N\})u/c}}{e^{-(c\tau+d)u} - e(cH_1 + dH_2)} \frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u - e(-H_2)} du,
\end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u=0$ is the only zero of

$$\left(e^{-(c\tau+d)u} - e(cH_1+dH_2)\right)(e^u - e(-H_2))$$

lying inside the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0<\arg u<2\pi$.

In fact, Theorem 1.1 is another version of Theorem 2.1 in [1]. If $N=1$ in Theorem 1.1, then we have Theorem 2.1 in [1]. Also, if $r'=(\frac{r_1}{N},r_2)$ and $h'=(Nh_1,h_2)$, then we see that $H_N(\tau,s;r,h)=H(N\tau,s;r',h')$.

2. Main results

Let $B_n(x)$, $n\ge 0$ be the Bernoulli polynomials which come from the generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}\ (|t|<2\pi).$$

The n-th Bernoulli number B_n, $n\ge 0$, is defined by $B_n=B_n(0)$. Put $\bar{B}_n(x)=B_n(\{x\})$, $n\ge 0$. Let $N=1$, $r=(r_1,r_2)$, $h=(0,0)$, $V\tau = 1-\frac{1}{\tau}$ and $s=-2n$, where $n,r_1\in\mathbb{Z}$, $r_2\notin\mathbb{Z}$. By Theorem 1.1, we see that

$$\begin{aligned}\tau^{2n}H_1(V\tau,-2n;r,0) &= H_1(\tau,-2n;R,0)+(2\pi i)^{2n}L_1(\tau,-2n;R,0)\\ &\quad - \lim_{s\to -2n}\frac{\Gamma(s)}{(-2\pi i\tau)^s}\left(\psi(0,r_2,s)+e\left(\frac{s}{2}\right)\psi(0,-r_2,s)\right).\end{aligned}$$

It is easy to see that

$$\begin{aligned}H_1(V\tau,-2n;r,0) &= \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{e(((m+1)V\tau+r_2)k)+e(((m+1)V\tau-r_2)k)}{k^{2n+1}}\\ &= \sum_{k=1}^{\infty}\frac{\cosh(2\pi ikr_2)}{k^{2n+1}\sinh(\pi ik/\tau)}e^{-\pi ik/\tau}\\ &= \sum_{k=1}^{\infty}\frac{2\cos(2\pi r_2k)}{k^{2n+1}(e^{2\pi ik/\tau}-1)}\end{aligned}$$

and

$$H_1(\tau,-2n;R,0) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{e(((m+\{r_2\})\tau-r_2)k)+e(((m-\{r_2\})\tau+r_2)k)}{k^{2n+1}}$$

$$= \sum_{k=1}^{\infty} \frac{\cosh(\pi ik(-2r_1 + (2\{r_2\} - 1)\tau))}{k^{2n+1}\sinh(-\pi ik\tau)}.$$

Next, using the residue theorem, we have

$$\begin{aligned} L_1(\tau, -2n; R, 0) &= \int_C u^{-2n-1} \frac{e^{-\tau(1-\{r_2\})u}}{e^{-\tau u} - 1} \frac{1}{e^u - 1}\, du \\ &= (-\tau)^{-1} \int_C u^{-2n-3} \sum_{\ell=0}^{\infty} \frac{B_\ell(1-\{r_2\})}{\ell!}(-\tau u)^\ell \sum_{m=0}^{\infty} \frac{B_m}{m!} u^m\, du \\ &= -2\pi i \sum_{k=0}^{2n+2} \frac{\bar{B}_k(r_2) B_{2n+2-k}}{k!(2n+2-k)!} \tau^{k-1}. \end{aligned}$$

For $\mathrm{Re}(s) < 0$, applying the formula in [3], p. 37, we obtain that

$$\begin{aligned} &\frac{\Gamma(s)}{(-2\pi i\tau)^s} \left(\psi(0, r_2, s) + e\left(\frac{s}{2}\right)\psi(0, -r_2, s)\right) \\ &= \frac{\Gamma(s)}{(-2\pi i\tau)^s} \left(\sum_{m+r_2>0} \frac{1}{(m+r_2)^s} + e\left(\frac{s}{2}\right) \sum_{m-r_2>0} \frac{1}{(m-r_2)^s}\right) \\ &= \frac{\Gamma(s)}{(-2\pi i\tau)^s} \left(\zeta(s, \{r_2\}) + e^{\pi is}\zeta(s, 1-\{r_2\})\right) \\ &= \frac{(-i\tau)^{-s}}{\sin(\pi s)} \left(\sum_{k=1}^{\infty} \frac{\sin(2k\pi\{r_2\} + \pi s/2)}{k^{1-s}} + e^{\pi is} \sum_{k=1}^{\infty} \frac{\sin(-2k\pi\{r_2\} + \pi s/2)}{k^{1-s}}\right) \\ &= \frac{(-i\tau)^{-s}}{\sin(\pi s)} \sum_{k=0}^{\infty} \frac{e^{-2k\pi i\{r_2\}}(e^{3\pi is/2} - e^{-2\pi is/2})}{k^{1-s}} \\ &= \tau^{-s} \sum_{k=0}^{\infty} \frac{e^{-2k\pi i\{r_2\}}}{k^{1-s}}. \end{aligned}$$

Hence it follows that, for $n > 0$,

$$\begin{aligned} &\lim_{s \to -2n} \frac{\Gamma(s)}{(-2\pi i\tau)^s} \left(\psi(0, r_2, s) + e\left(\frac{s}{2}\right)\psi(0, -r_2, s)\right) \\ &= \tau^{2n}\psi(-r_2, 0, 2n+1). \end{aligned}$$

Putting $\tau = i$, we now obtain that, for $n > 0$,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{2\cos(2\pi r_2 k)}{k^{2n+1}(e^{2\pi k} - 1)} &= (-1)^n \sum_{k=1}^{\infty} \frac{\cosh(\pi k(2r_1 - 2\{r_2\} + 1))}{k^{2n+1}\sinh(\pi k)} \\ &\quad -(2\pi)^{2n+1} \sum_{k=0}^{2n+2} \frac{\bar{B}_k(r_2) B_{2n+2-k}}{k!(2n+2-k)!} i^k - \psi(-r_2, 0, 2n+1). \end{aligned} \tag{2.1}$$

Theorem 2.1. *For any integer $n > 0$,*

$$\zeta\left(2n+1,\frac{1}{3}\right) = \frac{3^{2n+1}-1}{2}\zeta(2n+1) + \frac{(-1)^{n+1}(6\pi)^{2n+1}B_{2n+1}(1/3)}{2\sqrt{3}(2n+1)!}$$

and

$$\zeta\left(2n+1,\frac{2}{3}\right) = \frac{3^{2n+1}-1}{2}\zeta(2n+1) + \frac{(-1)^{n+1}(6\pi)^{2n+1}B_{2n+1}(2/3)}{2\sqrt{3}(2n+1)!}.$$

Proof. Let $r_2 = \frac{1}{3}$ and see the imaginary parts in (2.1). Recalling that $B_{2k+1} = 0,\ k \geq 1$, we have

$$\begin{aligned}\operatorname{Im}\left(\sum_{k=0}^{2n+2}\frac{\bar{B}_k(1/3)B_{2n+2-k}}{k!(2n+2-k)!}i^k\right) &= \sum_{k=0}^{n}(-1)^k\frac{\bar{B}_{2k+1}(1/3)B_{2n-2k+1}}{(2k+1)!(2n-2k+1)!}\\ &= \frac{(-1)^n(2\pi)^{2n+1}B_{2n+1}(1/3)}{2(2n+1)!}.\end{aligned}$$

Since

$$\begin{aligned}\psi\left(-\frac{1}{3},0,2n+1\right) &= \sum_{k=1}^{\infty}\frac{e^{-2k\pi i/3}}{k^{2n+1}}\\ &= \sum_{k=1}^{\infty}\frac{\cos(2k\pi/3)}{k^{2n+1}} - i\sum_{k=1}^{\infty}\frac{\sin(2k\pi/3)}{k^{2n+1}},\end{aligned}$$

we see that

$$\begin{aligned}\operatorname{Im}\left(\psi\left(-\frac{1}{3},0,2n+1\right)\right) &= -\sum_{k=1}^{\infty}\frac{\sin(2k\pi/3)}{k^{2n+1}}\\ &= -\frac{\sqrt{3}}{2}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^{2n+1}} + \frac{\sqrt{3}}{2}\sum_{k=0}^{\infty}\frac{1}{(3k+2)^{2n+1}}\\ &= -\frac{\sqrt{3}}{2}\cdot 3^{-2n-1}\left(\zeta\left(2n+1,\frac{1}{3}\right) - \zeta\left(2n+1,\frac{2}{3}\right)\right).\end{aligned}$$

Thus, equating the imaginary parts in (2.1), we obtain that

$$\zeta\left(2n+1,\frac{1}{3}\right) - \zeta\left(2n+1,\frac{2}{3}\right) = \frac{(-1)^{n+1}(6\pi)^{2n+1}B_{2n+1}(1/3)}{\sqrt{3}(2n+1)!}. \tag{2.2}$$

An elementary calculation shows that

$$\zeta\left(2n+1,\frac{1}{3}\right) + \zeta\left(2n+1,\frac{2}{3}\right) = (3^{2n+1}-1)\zeta(2n+1). \tag{2.3}$$

Adding and subtracting (2.2) and (2.3), we complete the proof of the theorem. □

Theorem 2.2. *For any integer $n>0$,*

$$\zeta\left(2n+1,\frac{1}{4}\right)=2^{2n}(2^{2n+1}-1)\zeta(2n+1)+\frac{(-1)^{n+1}2^{6n+1}\pi^{2n+1}B_{2n+1}(1/4)}{(2n+1)!}$$

and

$$\zeta\left(2n+1,\frac{3}{4}\right)=2^{2n}(2^{2n+1}-1)\zeta(2n+1)+\frac{(-1)^{n+1}2^{6n+1}\pi^{2n+1}B_{2n+1}(3/4)}{(2n+1)!}.$$

Proof. Let $r_2=\frac{1}{4}$ and equate the imaginary parts in (2.1). By the similar way in the proof of Theorem 2.1, we have

$$\operatorname{Im}\left(\sum_{k=0}^{2n+2}\frac{\bar{B}_k(1/4)B_{2n+2-k}}{k!(2n+2-k)!}i^k\right)=\frac{(-1)^n(2\pi)^{2n+1}B_{2n+1}(1/4)}{2(2n+1)!}$$

and

$$\begin{aligned}\operatorname{Im}\left(\psi\left(-\frac{1}{4},0,2n+1\right)\right)&=-\sum_{k=1}^{\infty}\frac{\sin(k\pi/2)}{k^{2n+1}}\\&=-\sum_{k=0}^{\infty}\frac{1}{(4k+1)^{2n+1}}+\sum_{k=0}^{\infty}\frac{1}{(4k+3)^{2n+1}}\\&=-4^{-2n-1}\left(\zeta\left(2n+1,\frac{1}{4}\right)-\zeta\left(2n+1,\frac{3}{4}\right)\right).\end{aligned}$$

Thus we obtain that

$$\zeta\left(2n+1,\frac{1}{4}\right)-\zeta\left(2n+1,\frac{3}{4}\right)=\frac{(-1)^{n+1}(8\pi)^{2n+1}B_{2n+1}(1/4)}{2(2n+1)!}.\tag{2.4}$$

It is easy to see that

$$\begin{aligned}&\zeta\left(2n+1,\frac{1}{4}\right)+\zeta\left(2n+1,\frac{3}{4}\right)\\&=4^{2n+1}\left(\zeta(2n+1)-\sum_{k=0}^{\infty}\frac{1}{(4k)^{2n+1}}-\sum_{k=0}^{\infty}\frac{1}{(4k+2)^{2n+1}}\right)\\&=(2^{4n+2}-2^{2n+1})\zeta(2n+1).\end{aligned}\tag{2.5}$$

Combining equations (2.4) and (2.5), the desired results follow. □

Theorem 2.3. *For any integer $n>0$,*

$$\begin{aligned}\zeta\left(2n+1,\frac{1}{6}\right)&=(2^{2n}-2^{-1})(3^{2n+1}-1)\zeta(2n+1)\\&\quad+(2^{2n}+2^{-1})\frac{(-1)^{n+1}(6\pi)^{2n+1}B_{2n+1}(1/3)}{\sqrt{3}(2n+1)!}\end{aligned}$$

and

$$\zeta\left(2n+1,\frac{5}{6}\right)=(2^{2n}-2^{-1})(3^{2n+1}-1)\zeta(2n+1)$$
$$+(2^{2n}+2^{-1})\frac{(-1)^{n+1}(6\pi)^{2n+1}B_{2n+1}(2/3)}{\sqrt{3}(2n+1)!}.$$

Proof. For any integer $n>0$,

$$\zeta\left(2n+1,\frac{1}{3}\right)=\sum_{k=0}^{\infty}\left(2k+\frac{1}{3}\right)^{-2n-1}+\sum_{k=0}^{\infty}\left(2k+1+\frac{1}{3}\right)^{-2n-1}$$
$$=2^{-2n-1}\zeta\left(2n+1,\frac{1}{6}\right)+2^{-2n-1}\zeta\left(2n+1,\frac{2}{3}\right)$$

and

$$\zeta\left(2n+1,\frac{2}{3}\right)=\sum_{k=0}^{\infty}\left(2k+\frac{2}{3}\right)^{-2n-1}+\sum_{k=0}^{\infty}\left(2k+1+\frac{2}{3}\right)^{-2n-1}$$
$$=2^{-2n-1}\zeta\left(2n+1,\frac{1}{3}\right)+2^{-2n-1}\zeta\left(2n+1,\frac{5}{6}\right).$$

Hence we have that, for $n>0$,

$$\zeta\left(2n+1,\frac{1}{6}\right)=2^{2n+1}\zeta\left(2n+1,\frac{1}{3}\right)-\zeta\left(2n+1,\frac{2}{3}\right) \tag{2.6}$$

and

$$\zeta\left(2n+1,\frac{5}{6}\right)=2^{2n+1}\zeta\left(2n+1,\frac{2}{3}\right)-\zeta\left(2n+1,\frac{1}{3}\right). \tag{2.7}$$

Apply Theorem 2.1 to (2.6) and (2.7). Then the theorem follows. □

Unfortunately, for other values of α, Theorem 1.1 does not provide useful equations like (2.2) or (2.4). For example, we obtain that, for $r_2=\frac{1}{8}$ in (2.1),

$$\zeta\left(2n+1,\frac{1}{8}\right)+\zeta\left(2n+1,\frac{3}{8}\right)-\zeta\left(2n+1,\frac{5}{8}\right)-\zeta\left(2n+1,\frac{7}{8}\right)$$
$$=\frac{(-1)^n(8\pi)^{2n+1}}{\sqrt{2}(2n+1)!}\left(8^{2n+1}B_{2n+1}\left(\frac{3}{8}\right)+B_{2n+1}\left(\frac{3}{4}\right)\right)$$

This is why our results are very resrictive.

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