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A CLASS OF MAPPINGS BETWEEN R_z-SUPERCONTINUOUS FUNCTIONS AND R_δ-SUPERCONTINUOUS FUNCTIONS

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Abstract. A new class of functions called R_θ-supercontinuous functions is introduced. Their basic properties are studied and their place in the hierarchy of strong variants of continuity, which already exist in the literature, is elaborated. The class of R_θ-supercontinuous functions properly contains the class of R_z-supercontinuous functions [39] which in turn properly contains the class of R_{cl}-supercontinuous functions [43] and so includes all *cl*-supercontinuous (clopen continuous) functions ([38], [34]) and is properly contained in the class of R_δ-supercontinuous functions [24].

1. Introduction

From early days of mathematics the notion of continuity is of fundamental importance in almost all subdisciplines of mathematics. In many circumstances in geometry, analysis, topology and topologico-analytic situations continuity is not sufficient and a condition stronger than continuity is required to meet the demand of a particular situation. Several strong forms of continuity have been defined and studied by host of authors ([1], [19], [20], [22], [23], [28], [29], [31], [33], [34], [37], [38]). Moreover compact maps (operators) which arise in functional analysis also represent a strong variant of continuity. Hence it is of considerable significance both from intrinsic interest as well as from the applications view point to formulate and study new strong variants of continuity. In this paper, we introduce a new class of functions called 'R_θ-supercontinuous functions', study their basic properties and discuss their place in the hierarchy of strong variants of continuity that already exist in the mathematical literature. It turns out that the class of R_θ-supercontinuous functions properly contains the class of R_z-supercontinuous functions [39] which in turn properly contains the class of R_{cl}-supercontinuous functions [43] and so includes all *cl*-supercontinuous [38] (*clopen* continuous [34]) functions and is properly contained in the class of R_δ-supercontinuous functions [24].

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The organization of the paper is as follows: Section 2 is devoted to basic definitions and preliminaries. In Section 3 we introduce the notion of 'R_θ-supercontinuous function' and discuss its place in the hierarchy of strong variants of continuity that already exist in the literature. Examples are included to reflect upon the distinctiveness of notions so introduced from the existing ones. Basic properties of R_θ-supercontinuous functions are studied wherein it is shown that (i) R_θ-supercontinuity is stable under the restrictions, shrinking and expansion of range and composition of functions; (ii) a function into a product space is R_θ-supercontinuous if and only if its composition with each projection map is R_θ-supercontinuous; and (iii) if X is an R_1-space, then f is R_θ-supercontinuous if and only if its graph function is R_θ-supercontinuous. The interplay between topological properties and R_θ-supercontinuous functions is investigated in Section 4. In Section 5, properties of graphs of R_θ-supercontinuous functions are studied. The notion of r_θ-quotient topology is introduced in Section 6. In Section 7, we retopologize the domain of an R_θ-supercontinuous function in such a way that it is simply a continuous function and conclude with alternative proofs of certain results of the preceding sections.

2. Basic definitions and preliminaries

A subset H of a space X is called a **regular G_δ-set** [30] if H is the intersection of a sequence of closed sets whose interiors contain H, i.e. $H = \bigcap\limits_{n=1}^{\infty} F_n = \bigcap\limits_{n=1}^{\infty} F_n^o$, where each F_n is a closed subset of X. The complement of a regular G_δ-set is called a **regular F_σ-set**. A subset A of a space X is said to be **regular open** if it is the interior of its closure, i.e. $A = \overline{A}^o$. The complement of a regular open set is referred to as **regular closed**. Any intersection of regular closed (respectively zero, respectively clopen) sets is called **δ-closed** [45] (respectively **z-closed** [19], respectively ***cl*-closed** [38]) set. A point $x \in X$ is called **θ-limit point** [45] of $A \subset X$ if every closed neighbourhood of x intersects A. Let A_θ denote the set of all θ-limit points of A. The set A is called **θ-closed** if $A = A_\theta$. The complement of θ-closed set is referred to as **θ-open set**. An open set U of a space X is said to be **F-open** [22] (**r-open** [23]) if for each $x \in U$ there exists a zero(closed) set Z in X such that $x \in Z \subset U$; equivalently if U is expressible as a union of zero (closed) sets in X. An open set U in X is said to be **r_z-open** [39] (respectively **r_{cl}-open** [43]) if for each $x \in U$ there exists a z-closed (respectively cl-closed) set A containing x such that $A \subset U$; equivalently U is expressible as a union of z-closed (respectively cl-closed) sets.

Lemma 2.1 ([15], [18]). *A set U in a space X is θ-open if and only if for each $x \in U$ there exists an open set V containing x such that $\overline{V} \subset U$.*

Next, we include definitions of those variants of continuity which exist in the literature and are related to the theme of the present paper.

Definitions 2.2. *A function $f : X \to Y$ from a topological space X into a topological space Y is said to be*

(a) ***strongly continuous*** [29] *if $f(\overline{A}) \subset \overline{f(A)}$ for each subset A of X.*
(b) ***perfectly continuous*** *(*[25], [33]*) if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$.*
(c) cl-***supercontinuous*** [38] *(≡* ***clopen continuous*** [34]*) if for each $x \in X$ and each open set V containing $f(x)$, there is a clopen set U containing x such that $f(U) \subset V$.*
(d) z-***supercontinuous*** [19] *(*$\mathbf{D}_\delta$-***supercontinuous*** [20]*) if for each $x \in X$ and for each open set V containing $f(x)$, there exists a cozero (regular F_σ) set U containing x such that $f(U) \subset V$.*
(e) ***strongly θ-continuous*** *(*[28], [32]*) if for each $x \in X$ and for each open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U}) \subset V$.*
(f) F-***supercontinuous*** [22] *(respectively* R-***supercontinuous*** [23], *respectively* $\mathbf{R}_{cl}$-***supercontinuous*** [43]*) if for each $x \in X$ and each open set V containing $f(x)$, there exists an F-open (respectively r-open, respectively r_{cl}-open) set U containing x such that $f(U) \subset V$.*
(g) ***supercontinuous*** [31] *if for each $x \in X$ and for each open set V containing $f(x)$, there exists a regular open set U containing x such that $f(U) \subset V$.*
(h) $\mathbf{R_z}$-***supercontinuous*** [39] *if for each $x \in X$ and for each open set V containing $f(x)$, there exists a r_z-open set U containing x such that $f(U) \subset V$.*
(i) $\mathbf{R}_\delta$-***supercontinuous*** [24] *if for each $x \in X$ and for each open set V containing $f(x)$, there exists an r_δ-open set U containing x such that $f(U) \subset V$.*
(j) ***quasi perfectly continuous*** [27] *if $f^{-1}(V)$ is clopen in X for every θ-open set V in Y.*

Definitions 2.3. *A topological space X is said to be*

(a) $\mathbf{R_0}$-***space*** [5] *if for each open set U in X, $x \in U$ implies $\overline{\{x\}} \subset U$.*
(b) $\mathbf{R_1}$-***space*** [5] *if $x, y \in X$, $x \notin \overline{\{y\}}$, then x and y are contained in disjoint open sets.*
(c) $\mathbf{R}_\theta$-***space*** [21] *if for each $x \in X$ and each open set U containing x there exists a θ-closed set F such that $x \in F \subset U$.*

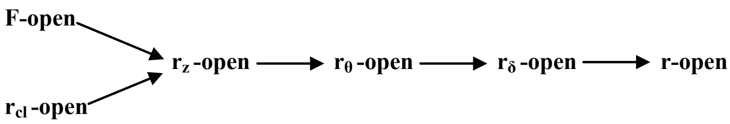
Theorem 2.4. [21, Theorem 7.5] *A topological space X is an R_1-space if and only if it is an R_θ-space.*

Earlier R_0-spaces were defined by Shanin [35] and rediscovered by Vaidyanathswamy who called them π_1-spaces (see [44], p.98) and later on by Davis [5].

In his studies on paracompactness Yang [46] defined R_1-spaces who called them T_2'-axiom and were rediscovered by Davis [5].

3. R_θ-supercontinuous functions

Let X be a topological space. An open subset U of a space X is called $\mathbf{r_\theta}$**-open** if for each $x \in U$, there exists a θ-closed set C_x such that $x \in C_x \subseteq U$, equivalently U is expressible as a union of θ-closed sets. The complement of an r_θ-open set will be referred to as $\mathbf{r_\theta}$**-closed**.
The following implications are immediate from definitions.



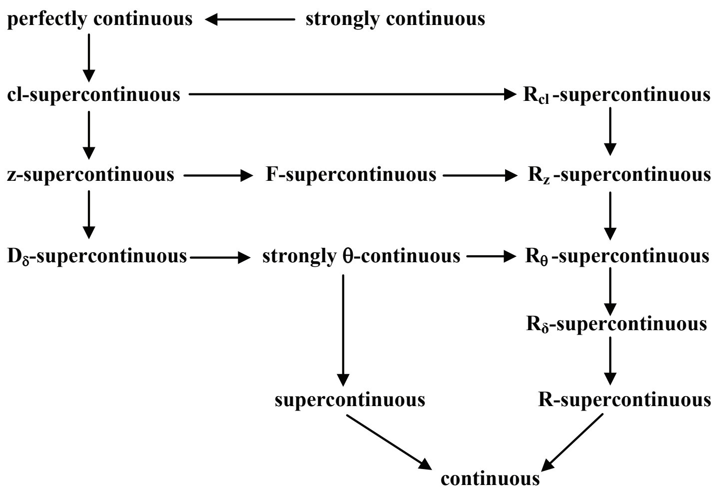
Simple examples can be given to show that none of the above implications are reversible in general (see [42]).

Remark 3.1. *In the light of Theorem 2.4, a topological space (X, τ) is an R_1-space if and only if each open set in X is r_θ-open.*

Definition 3.2. *A function $f : X \to Y$ from a topological space X into topological space Y is said to be* R_θ***-supercontinuous*** *at a point $x \in X$, if for each open set V containing $f(x)$ there exists an r_θ-open set U containing x such that $f(U) \subset V$. The function f is said to be R_θ-supercontinuous on X, if it is R_θ-supercontinuous at each $x \in X$.*

Remark 3.3. *In view of Remark 3.1, we observe that the notion of R_θ-supercontinuity coincides with continuity if domain is an R_1-space.*

The following diagram well illustrates the place of *R_θ-supercontinuity* in the hierarchy of other strong variants of continuity which already exist in the literature and are related to the theme of the present paper.



However, none of the above implications in general is reversible, which is either well known (see [9], [19], [24], [39]) or follows from the following observations and examples.

Example 3.4. *The function f in* [24, Example 3.4] *is an R_δ-supercontinuous function but it is not R_θ-supercontinuous.*

Example 3.5. *Let $X = Y$ be the regular space due to Hewitt* [8] *on which every continuous real valued function is constant and let f denote the identity map defined on X. Since the space is not completely regular, f is an R_θ-supercontinuous function which is not R_z-supercontinuous.*

Example 3.6. *Let $X = \mathbb{R}$ be the real line equipped with Smirnov topology* [42] *and $1_X : (X, \tau) \to (X, \tau)$. Then 1_X is an R_θ-supercontinuous function which is not strongly θ-continuous. As (X, τ) is R_1-space, hence every open set is r_θ-open set. Now take $U = (-1, 1) \setminus K$, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then U is an r_θ-open set containing 0. Since $1_X(\mathrm{cl}(U)) \nsubseteq U$, 1_X is not strongly θ-continuous.*

Definition 3.7. *Let X be a topological space and let $A \subset X$. A point $x \in X$ is said to be an* ***r_θ-adherent point*** *of A if every r_θ-open set containing x intersects A. Let A_{r_θ} denote the set of all r_θ-adherent points of the set A. It follows that $\bar{A} \subset A_{r_\theta}$ and A is r_θ-closed if and only if $A = A_{r_\theta}$.*

Example 3.8. *Let $X = \mathbb{R}$ be the regular space due to Hewitt* [8] *on which every continuous real valued function is constant. The only r_z-closed sets in X are $\emptyset$ and X itself. Let H be any non-empty proper closed set in X. Then $H_{r_z} = X$. Moreover, in view of Remark 3.1 $H_{r_\theta} = H$.*

The space X in the following example is due to T. Soundararajan (see [41], Proposition 2.1).

Example 3.9. *Let $X = [0, 1] \cup [2, 3] \cup A$, where A is a countably infinite set disjoint from $[0, 1]$ and $[2, 3]$. Every element of A is isolated. Each element of $[0, 1) \cup (2, 3]$ has a base of usual euclidean neighbourhoods. A set G containing 1 is a neighbourhood of 1 provided it contains some $(\epsilon, 1]$, $0 < \epsilon < 1$, and all but finitely many elements of A. Similarly a set containing 2 is a neighbourhood of 2 if it contains some $[2, \epsilon')$, $2 < \epsilon' < 3$ and all but finitely many elements of A. The space X is a non-Hausdorff, weakly Hausdorff space and hence an R_δ-space which is not an R_1-space. Let $B = [0, 1]$, then $B_{r_\theta} = B \cup A$ but $B_{r_\delta} = B$.*

Remark 3.10. *For any set B in (X, τ), it is easy to observe that $\overline{B} \subseteq B_{r_\delta} \subseteq B_{r_\theta} \subseteq B_{r_z}$ due to* [24, 39].

Theorem 3.11. *For a function $f : X \to Y$ from a topological space X into a topological space Y, the following statements are equivalent:*

(i) f is R_θ-supercontinuous.
(ii) $f^{-1}(V)$ is r_θ-open for every open set $V \subset Y$.
(iii) $f^{-1}(B)$ is r_θ-closed for every closed set $B \subset Y$.

(iv) $f^{-1}(S)$ *is* r_θ*-open for every subbasic open set* $S\subset Y$.
(v) $f(A_{r_\theta})\subset\overline{f(A)}$ *for every set* $A\subset X$.
(vi) $(f^{-1}(B))_{r_\theta}\subset f^{-1}(\overline{B})$ *for every set* $B\subset Y$.

Proof. Easy. □

Definition 3.12. *A filter base* $\mathcal{B}$ *is said to* r_θ***-converge to a point*** $x\in X$ *(written as* $\mathcal{B}\xrightarrow{r_\theta}x$*) if every* r_θ*-open set containing* x *contains a member of* $\mathcal{B}$*.*

Theorem 3.13. *A function* $f: X\to Y$ *from a topological space* X *into a topological space* Y *is* R_θ*-supercontinuous if and only if* $f(\mathcal{B})\to f(x)$ *for every filter base* $\mathcal{B}$ *in* X *that* r_θ*-converges to* x*.*

Proof. Suppose that f is R_θ-supercontinuous and $\mathcal{B}$ is a filter base in X that r_θ-converges to $x\in X$. Let U be any open set in Y containing $f(x)$. By Theorem 3.11 (ii), $f^{-1}(U)$ is an r_θ-open set containing x. Since the filter base $\mathcal{B}$ converges to x, there exists a $B\in\mathcal{B}$ such that $B\subset f^{-1}(U)$. Then $f(B)\subset f(f^{-1}(U))\subset U$ and so $f(\mathcal{B})\to f(x)$.

Conversely, let V be an open subset of Y containing $f(x)$. Now, the filter $\mathcal{B}$ generated by the filterbase $\mathcal{B}_x$ consisting of r_θ-open subsets of X containing x, r_θ-converges to x. Since by hypothesis $f(\mathcal{B})\to f(x)$, there exists a member $f(H)$ of $f(\mathcal{B})$ such that $f(H)\subset V$. Choose $B\in\mathcal{B}_x$ such that $B\subset H$. Since B is an r_θ-open set containing x and since $f(B)\subset f(H)\subset V$, f is R_θ-supercontinuous. □

Theorem 3.14. *If* $f: X\to Y$ *is quasi perfectly continuous and* $g: Y\to Z$ *is* R_θ*-supercontinuous, then* $g\circ f: X\to Z$ *is cl-supercontinuous.*

Proof. Let W be an open set in Z. Since g is R_θ-supercontinuous, $g^{-1}(W)$ is an r_θ-open set in Y. Let $g^{-1}(W)=\bigcup V_\alpha$, where each V_α is a θ-closed set in Y. Since f is quasi perfectly continuous and since a clopen set is both a zero set and a cozero set, in view of [36, Theorem 2.2] each $f^{-1}(V_\alpha)$ is a clopen set. Thus $(g\circ f)^{-1}(W)=f^{-1}(g^{-1}(W))=\bigcup f^{-1}(V_\alpha)$ is cl-open. □

Theorem 3.15. *If* $f: X\to Y$ *is an* R_θ*-supercontinuous function and* $g: Y\to Z$ *is a continuous function, then their composition* $g\circ f$ *is* R_θ*-supercontinuous. In particular, the composition of two* R_θ*-supercontinuous functions is* R_θ*-supercontinuous.*

Definition 3.16. *A function* $f: X\to Y$ *is said to be* $\mathbf{R}_\theta$***-open (*** $\mathbf{R}_\theta$***-closed)*** *if* $f(A)$ *is open (closed) in* Y *for every* r_θ*-open (*r_θ*-closed) set* A *in* X*.*

Every open (closed) function is R_θ-open (R_θ-closed). However, reverse implication is not true in general as is well exhibited by the following example.

Example 3.17. *Let* $X=Y=\{a,b\}$*. Let* X *be endowed with Sierpinski topology and* Y *be equipped with indiscrete topology. Then the identity function* f *from* X *onto* Y *is* R_θ*-open as well as* R_θ*-closed function but is neither open nor closed.*

Theorem 3.18. *If* $f: X\to Y$ *is an* R_θ*-open (*R_θ*-closed),* R_θ*-supercontinuous surjection and* $g: Y\to Z$ *is any function, then* $g\circ f$ *is* R_θ*-supercontinuous if and only if g is continuous. Moreover, if in addition f maps* r_θ*-open (*r_θ*-closed) sets to* r_θ*-open (*r_θ*-closed) sets, then g is* R_θ*-supercontinuous.*

Proof. Sufficiency is immediate in view of Theorem 3.15. To prove necessity, suppose that $g \circ f$ is R_θ-supercontinuous and let W be an open (closed) subset of Z. Then $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is r_θ-open (r_θ-closed) in X. Since f is an R_θ-open (R_θ-closed) surjection $f(f^{-1}(g^{-1}(W))) = g^{-1}(W)$ is an open (closed) set in Y and so g is continuous. The last assertion is immediate in view of Theorem 3.15. □

Definition 3.19 ([16], [17])**.** *A subset S of a space X is said to be θ-**embedded** in X if every θ-open (θ-closed) set in S is the intersection of a θ-open (θ-closed) set in X with S.*

Theorem 3.20. *For a function $f : X \to Y$, the following statements are true.*

(i) If f is R_θ-supercontinuous and if B is a subspace of X which is θ-embedded in X, then the restriction function $f|B : B \to Y$ is R_θ-supercontinuous.
(ii) Let $\{U_\alpha : \alpha \in \Lambda\}$ be an r_θ-open cover of X such that each U_α is θ-embedded in X. If $f_\alpha = f|U_\alpha$ is R_θ-supercontinuous for each $\alpha \in \Lambda$, then f is R_θ-supercontinuous.
(iii) Let $X = \bigcup_{i=1}^{n} H_i$, where each H_i is an r_θ-closed θ-embedded set in X. If for each i, $f_i = f|H_i$ (i=1,..., n) is R_θ-supercontinuous, then f is R_θ-supercontinuous.

Proof. (i) Let V be any open set in Y. Since f is R_θ-supercontinuous, $f^{-1}(V)$ is an r_θ-open set in X. Then $f^{-1}(V) = \bigcup\{B_\alpha : \alpha \in \Lambda\}$, where B_α is a θ-closed set in X. Now $(f|B)^{-1}(V) = f^{-1}(V) \cap B = \bigcup\{B_\alpha \cap B : \alpha \in \Lambda\}$. Since B is θ-embedded in X, $B_\alpha \cap B$ is a θ-closed set in B for each $\alpha \in \Lambda$. Thus $(f|B)^{-1}(V)$ is an r_θ-open set being an open set which is the union of θ-closed sets. Hence $f|B$ is R_θ-supercontinuous.

(ii) Let V be an open subset of Y. Then $f^{-1}(V) = \bigcup\{f_\alpha^{-1}(V) : \alpha \in \Lambda\}$. Now, since each f_α is R_θ-supercontinuous, each $f_\alpha^{-1}(V)$ is r_θ-open in U_α. Since each U_α is r_θ-open and θ-embedded in X, each $f_\alpha^{-1}(V)$ is r_θ-open in X. Furthermore, since any union of r_θ-open sets is r_θ-open, $f^{-1}(V)$ is an r_θ-open set in X and so f is R_θ-supercontinuous.

(iii) Let H be any closed subset of Y. Then $f^{-1}(H) = \bigcup_{i=1}^{n} f_i^{-1}(H)$. Since each f_i is R_θ-supercontinuous, each $f_i^{-1}(H)$ is r_θ-closed in H_i. Since each H_i is θ-embedded in X, it can be easily verified that $f_i^{-1}(H)$ is r_θ-closed in X. Since a finite union of r_θ-closed sets is r_θ-closed, $f^{-1}(H)$ is r_θ-closed in X and hence f is R_θ-supercontinuous. □

The following result shows that R_θ-supercontinuity is preserved under the shrinking of its range.

Theorem 3.21. *Let $f : X \to Y$ be R_θ-supercontinuous and $f(X)$ be endowed with the subspace topology. Then $f : X \to f(X)$ is R_θ-supercontinuous.*

It is easily verified that R_θ-supercontinuity is stable under expansion of its range.

Theorem 3.22. *A function into a product space is R_θ-supercontinuous if and only if its composition with each projection map is R_θ-supercontinuous.*

Proof. Suppose that the function $f : X \to \prod_{\alpha\in\Lambda} X_\alpha$ is R_θ-supercontinuous. Let $f_\alpha = \pi_\alpha \circ f$, where $\pi_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \to X_\alpha$ denotes the projection onto the α-coordinate space X_α. Since projection maps are continuous, in view of Theorem 3.15, each f_α is R_θ-supercontinuous.

Conversely, suppose that each $\pi_\alpha \circ f = f_\alpha : X \to X_\alpha$ is R_θ-supercontinuous. Since arbitrary unions and finite intersections of r_θ-open sets is r_θ-open, in order to prove the R_θ-supercontinuity of f, it is sufficient to show that inverse image under f of every subbasic open set in $\prod_{\alpha\in\Lambda} X_\alpha$ is r_θ-open in X. Let $V_\beta \times \prod_{\alpha\neq\beta} Y_\alpha$ be a subbasic open set in $\prod_{\alpha\in\Lambda} X_\alpha$. Then $f^{-1}(V_\beta \times \prod_{\alpha\neq\beta} Y_\alpha) = f^{-1}(\pi_\beta^{-1}(V_\beta)) = f_\beta^{-1}(V_\beta)$ is r_θ-open in X. So f is R_θ-supercontinuous. □

Theorem 3.23. *Let $f : X \to Y$ be any function and $g : X \to X \times Y$ be the graph function defined by $g(x) = (x, f(x))$ for each $x \in X$. Then g is R_θ-supercontinuous if and only if f is R_θ-supercontinuous and X is an R_1-space.*

Proof. It is easy to observe that $g = 1_X \times f$, where 1_X denotes the identity function defined on X. By Theorem 3.22, g is R_θ-supercontinuous if and only if 1_X and f both are R_θ-supercontinuous. Moreover, R_θ-*supercontinuity* of 1_X implies that every open set in X is r_θ-open and so in view of Remark 3.1, X is an R_1-space. □

Remark 3.24. *The hypothesis of R_1-space in Theorem 3.23 cannot be omitted. For let $X = \mathbb{R}$ be the real line with the $cofinite$ topology* [42] *and Y be the real line with indiscrete topology. Let $f : X \to Y$ be any function. Clearly, f is R_θ-supercontinuous but the graph function $g : X \to X \times Y$ is not R_θ-supercontinuous.*

Theorem 3.25. *Let $f : \prod_{\alpha\in\Lambda} X_\alpha \to \prod_{\alpha\in\Lambda} Y_\alpha$ be a mapping defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$, where $f_\alpha : X_\alpha \to Y_\alpha$ for each $\alpha \in \Lambda$. Then f is R_θ-supercontinuous if and only if each f_α is R_θ-supercontinuous.*

Proof. In order to prove necessity, let V_β be any open set in Y_β. Then $\pi_\beta^{-1}(V_\beta) = V_\beta \times \prod_{\alpha\neq\beta} Y_\alpha$ is a subbasic open set in $\prod_{\alpha\in\Lambda} Y_\alpha$. Now since f is R_θ-supercontinuous, $f^{-1}(\pi_\beta^{-1}(V_\beta)) = f_\beta^{-1}(V_\beta) \times (\prod_{\alpha\neq\beta} X_\alpha)$ is an r_θ-open set in $\prod_{\alpha\in\Lambda} X_\alpha$. Thus $f_\beta^{-1}(V_\beta)$ is an r_θ-open set in X_β and hence f_β is R_θ- supercontinuous.

Conversely, let $V = V_\beta \times \prod_{\alpha\neq\beta} Y_\alpha$ be a subbasic open set in the product space $\prod Y_\alpha$. Then $f^{-1}(V) = f^{-1}(V_\beta \times \prod_{\alpha\neq\beta} Y_\alpha) = f_\beta^{-1}(V_\beta) \times \prod_{\alpha\neq\beta} X_\alpha$. Since each f_β is R_θ-supercontinuous, $f_\beta^{-1}(V_\beta)$ is an r_θ-open subset of X_β and so $f^{-1}(V)$ is an r_θ-open subset of $\prod X_\alpha$ and hence f is R_θ-supercontinuous. □

Theorem 3.26. *Let $f, g : X \to Y$ be R_θ-supercontinuous functions from a topological space X into a Hausdorff space Y. Then the equalizer $E = \{x \in X : f(x) = g(x)\}$ of the functions f and g is an r_θ-closed subset of X.*

Proof. In order to prove that E is r_θ-closed, we shall show that its complement $X \setminus E$ is r_θ-open. To this end, let $x \in X \setminus E$. Then $f(x) \neq g(x)$. Since

Y is Hausdorff, there exist disjoint open sets G and H containing $f(x)$ and $g(x)$, respectively. Since f and g are R_θ-supercontinuous, $f^{-1}(G)$ and $g^{-1}(H)$ are r_θ-open sets containing x. Then $W = f^{-1}(G) \cap g^{-1}(H)$ is an r_θ-open set containing x and $W \cap E = \emptyset$. Thus E is r_θ-closed in X. □

4. Topological properties and R_θ-supercontinuity

Theorem 4.1. *If $f : X \to Y$ is an R_θ-supercontinuous open bijection, then X and Y are homeomorphic R_1-spaces.*

Proof. Let U be an open set in X and let $x \in U$. Then $f(U)$ is an open subset of Y containing $f(x)$. Now, since f is R_θ-supercontinuous, there exists an r_θ-open set G containing x such that $f(G) \subset f(U)$. Now, $x \in f^{-1}(f(G)) \subset f^{-1}(f(U))$. Again, since f is a bijection, $f^{-1}(f(G)) = G$ and $f^{-1}(f(U)) = U$. Thus U being expressible as a union of r_θ-open sets is r_θ-open. In view of Remark 3.1, X is an R_1-space. Since the property of being an R_1-space is a topological property and f is a homeomorphism, Y is also an R_1-space. □

Theorem 4.2. *Let $f : X \to Y$ be an R_θ-supercontinuous injection into a T_0-space Y. Then X is a Hausdorff space.*

Proof. Suppose $x, y \in X$, $x \neq y$. Then $f(x) \neq f(y)$. Since Y is a T_0-space, there exist an open set U containing one of the points $f(x)$ and $f(y)$ but not both. Without loss of generality, we assume that $f(x) \in U$, $f(y) \notin U$. Then $f^{-1}(U)$ is an r_θ-open set containing x such that $y \notin f^{-1}(U)$. So there exist a θ-closed set C_x in X such that $x \in C_x \subset f^{-1}(U)$. This implies that $y \in X \setminus C_x$. Since $X \setminus C_x$ is θ-open in X, in view of Lemma 2.1 there exist an open set V in X such that $y \in V \subseteq \overline{V} \subseteq X \setminus C_x$. Therefore, $X \setminus \overline{V}$ and V are two disjoint open sets in X containing x and y respectively. Hence X is a Hausdorff space. □

Definition 4.3. *A space X is said to be*

(a) *r_θ-**regular** if for every r_θ-closed set A and a point $x \notin A$, there exist disjoint open sets U and V in X containing x and A, respectively.*

(b) ***completely** r_θ-**regular** if for every r_θ-closed set A and a point $x \notin A$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$.*

Remark 4.4. *For the interested reader we point out that the properties of r_θ-regular spaces and completely r_θ-regular spaces can also be inferred directly by substituting for $P =$ the property of being an r_θ-closed set in the relevant results pertaining to P-regular spaces and completely P-regular spaces in* [10].

Theorem 4.5. *Let $f : X \to Y$ be an R_θ-supercontinuous open bijection from an r_θ-regular space X onto Y. Then X and Y are homeomorphic regular spaces.*

Proof. Let B be a closed subset of Y and let $y \notin B$. Then $f^{-1}(y)$ is a singleton and $f^{-1}(y) \notin f^{-1}(B)$. Since f is R_θ-supercontinuous, by Theorem 3.11 (iii) $f^{-1}(B)$ is an r_θ-closed subset of X. In view of r_θ-regularity of X, there exist disjoint open sets G and H containing $f^{-1}(y)$ and $f^{-1}(B)$ respectively. Since

f is an open bijection, $f(G)$ and $f(H)$ are disjoint open sets containing y and B, respectively and so Y is regular space. Again, since f is a homeomorphism and since regularity is a topological property, X is also a regular space. □

Definition 4.6. *A bijection $f : X \to Y$ is said to be an R_θ-**homeomorphism** if both f and f^{-1} are R_θ-supercontinuous.*

Theorem 4.7. *Let $f : X \to Y$ be an R_θ-homeomorphism from a completely r_θ-regular space X onto Y. Then X and Y are homeomorphic completely regular spaces.*

Proof. In order to prove that Y is a completely regular space, suppose F is a closed subset of Y and $y \notin F$. Then $x = f^{-1}(y)$ is a singleton and x does not belong to the r_θ-closed set $f^{-1}(F)$. Since X is completely r_θ-regular space, there exists a continuous function $h : X \to [0, 1]$ such that $h(x) = 0$ and $h(f^{-1}(F)) = 1$. Let $g = h \circ f^{-1}$. Since f is an R_θ-homeomorphism, g is a well defined continuous function from Y into $[0, 1]$. Moreover, $g(y) = 0$ and $g(F) = 1$. Thus Y is a completely regular space. Moreover, since f is a homeomorphism and complete regularity is a topological property, it follows that X is also a completely regular space. □

5. Properties of graph of an R_θ-supercontinuous function

Definition 5.1. *The graph $G(f)$ of a function $f : X \to Y$ is said to be r_θ-**closed with respect to** X if for each $(x, y) \notin G(f)$, there exist open sets U and V containing x and y, respectively, such that U is r_θ-open and $(U \times V) \cap G(f) = \emptyset$.*

Theorem 5.2. *If $f : X \to Y$ is an R_θ-supercontinuous function and Y is a Hausdorff space, then the graph of f is r_θ-closed with respect to X.*

Proof. Let $x \in X$ and let $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets V and W containing y and $f(x)$ respectively. Again, since f is R_θ-supercontinuous, there exists an r_θ-open set U containing x such that $f(U) \subset W \subset Y \setminus V$ and so $(U \times V) \cap G(f) = \emptyset$. Consequently, $G(f)$ is r_θ-closed with respect to X. □

Theorem 5.3. *Let $f : X \to Y$ be an injection such that its graph is r_θ-closed with respect to X. Then X is a Hausdorff-space.*

Proof. Let $x, y \in X$, $x \neq y$. Since f is an injection, $(x, f(y)) \notin G(f)$. Moreover, since the graph of f is r_θ-closed with respect to X, there exist open sets U and V containing x and $f(y)$ respectively, such that U is r_θ-open and $(U \times V) \cap G(f) = \emptyset$. This implies that $y \notin U$. For if $y \in U$, $(y, f(y)) \in (U \times V) \cap G(f)$ which implies that the graph of f is not r_θ-closed with respect to X. Since $x \in U$ and U is r_θ-open, there exists a θ-closed set C_x such that $x \in C_x \subset U$. Now $y \notin U$ implies $y \in X \setminus C_x$. Since $X \setminus C_x$ is θ-open, by Lemma 2.1, there exist an open set W in X such that $y \in W \subset \overline{W} \subset X \setminus C_x$. So $X \setminus \overline{W}$ and W are disjoint open sets in X containing x and y respectively. Hence X is a Hausdorff space. □

Theorem 5.4. *Let $f : X \to Y$ be a function such that its graph $G(f)$ is r_θ-closed with respect to X. Then $f^{-1}(K)$ is r_θ-closed in X for every compact subset K of Y.*

Proof. To prove that $f^{-1}(K)$ is r_θ-closed, we shall prove that its complement $X \setminus f^{-1}(K)$ is an r_θ-open subset of X. To this end, let $x \in X \setminus f^{-1}(K)$. Then $(x, y) \notin G(f)$ for each $y \in K$. Since the graph $G(f)$ is r_θ-closed with respect to X, there exist an r_θ-open set U_y containing x and an open set V_y containing y such that $(U_y \times V_y) \cap G(f) = \emptyset$. The collection $\{V_y : y \in K\}$ is an open cover of the compact set K. So there exist finitely many $y_1, ..., y_n \in K$ such that $K \subset \bigcup\{V_{y_i} : i = 1, ..., n\}$. Let $U = \bigcap_{i=1}^{n} U_{y_i}$, then U being a finite intersection of r_θ-open sets is r_θ-open. Moreover, $x \in U$ and $f(U) \cap K = \emptyset$. Thus $x \in U \subset X \setminus f^{-1}(K)$. So $X \setminus f^{-1}(K)$ being the union of r_θ-open sets is r_θ-open and so $f^{-1}(K)$ is r_θ-closed. □

6. r_θ-quotient topology and r_θ-quotient spaces

Let $f : X \to Y$ be a surjection from a topological space X onto a set Y. The quotient topology on Y is the finest topology on Y, which makes f continuous. Several variants of quotient topology have been defined and studied in mathematical literature (see [19, 20, 22, 23, 24, 26, 37, 38, 39, 43]) which in general are weaker than quotient topology and coincide with the quotient topology if the domain is suitably augmented. In this section, we introduce a new variant of quotient topology called r_θ-quotient topology which strictly lies between r_z-quotient topology [39] and r_δ-quotient topology [24].

Definitions 6.1. *Let $p : X \to Y$ be a function from a topological space X onto a set Y. The collection of all subsets $A \subset Y$ such that $p^{-1}(A)$ is an*

(a) r_θ***-open*** *in X is a topology on* Y *and is called the r_θ-quotient topology and the map p is called the r_θ-quotient map.*
(b) z***-open*** *in X is a topology on* Y *and is called the z-quotient topology* [19] *and the map p is called the z-quotient map.*
(c) r***-open*** *in X is a topology on* Y *and is called the r-quotient topology* [23] *and the map p is called the r-quotient map.*
(d) r_z***-open*** *in X is a topology on* Y *and is called the r_z-quotient topology* [39]. *The map p is called the r_z-quotient map and the set* Y *with r_z-quotient topology is called r_z-quotient space.*
(e) $r_{\mathbf{cl}}$***-open*** *in X is a topology on* Y *and is called the r_{cl}-quotient topology* [43] *and the map p is called the r_{cl}-quotient map.*
(f) r_δ***-open*** *in* X *is a topology on* Y *and is called the r_δ-quotient topology* [24] *and the map p is called the r_δ-quotient map.*

The following diagram illustrates the comparison among variants of quotient topologies defined in Definitions 6.1. For a detailed survey of variants of quotient topologies in the literature, we refer the interested reader to ([23, 26]).

However, none of the above inclusions is reversible in general as is shown by examples in ([23, 24, 26, 43]) and the following examples.

cl - quotient topology $\subset$ **r_{cl} - quotient topology**
$\cap$ $\cap$
z - quotient topology $\subset$ **r_z - quotient topology**
$\cap$
r_θ - quotient topology
$\cap$
r - quotient topology $\supset$ **r_δ - quotient topology**
$\cap$
quotient topology

Example 6.2. *Let $X = Y$ be the set of real numbers and* X *be endowed with the Smirnov topology* [42], τ_s. *Let f denote the identity function defined on X. Then r_θ-quotient topology on Y is τ_s, while* cl-*quotient topology* = r_{cl}-*quotient topology* = r_z-*quotient topology* = z-*quotient topology* = *indiscrete topology.*

Example 6.3. *Let $X = Y$ be the set of natural numbers and X be endowed with the $cofinite$ topology τ_c. Let f denote the identity function defined on X. Then r_δ-quotient topology on Y is identical with τ_c, while r_θ-quotient topology = indiscrete topology.*

Theorem 6.4. *Let $p : (X, \tau_1) \to (Y, \tau_2)$ be the surjection, where τ_2 is the r_θ-quotient topology on Y. Then p is R_θ-supercontinuous. Moreover, τ_2 is the largest topology on Y which makes $p : (X, \tau_1) \to Y$ R_θ-supercontinuous.*

The following result shows that a function out of r_θ-quotient space is continuous if and only if its composition with r_θ-quotient map is R_θ-supercontinuous

Theorem 6.5. *Let $p : X \to Y$ be an r_θ-quotient map. Then a function $g : Y \to Z$ is continuous if and only if $g \circ p$ is R_θ-supercontinuous.*

7. Change of topology and R_θ-supercontinuous functions

In this section, we study the behaviour of R_θ-supercontinuous function if its domain and/or range are retopologized in an appropriate way. This leads to an alternative proofs of certain results of preceding sections. Let (X, τ) be a topological space and let $\mathfrak{B}_{\mathfrak{r}_\theta}$ denote the collection of all r_θ-open subsets of (X, τ). Since arbitrary union and finite intersection of r_θ-open sets is r_θ-open, the collection $\mathfrak{B}_{\mathfrak{r}_\theta}$ is indeed a topology on X, which we denote by τ_{r_θ}. Then $\tau_{r_\theta} \subset \tau$ and the inclusion is proper if (X, τ) is not an R_1-space.

The technique of change of topology of a space is widely used in topology, functional analysis and several other branches of mathematics. For example, weak and weak* topologies of a Banach space, weak and strong operator topologies on $\mathfrak{B}(H)$ the space of operators on a Hilbert space, the hull kernel topology and the multitude of other topologies on $Id(A)$ the space of all closed two sided ideals of a Banach algebra A ([2], [3], [4], [40]). Furthermore, to taste the flavour of applications of the technique of change in topology see ([6], [7], [9], [14], [23], [38], [47]).

Theorem 7.1. *The space (X, τ) is an R_1-space if and only if $\tau = \tau_{r_\theta}$.*

Theorem 7.2. *A function $f : (X, \tau) \to (Y, \nu)$ is R_θ-supercontinuous if and only if $f : (X, \tau_{r_\theta}) \to (Y, \nu)$ is continuous.*

Proof. It can be proved very easily by Theorem 7.1. □

Many of the results of preceeding section follows from Theorem 7.2 and the corresponding properties of continuous functions.

Theorem 7.3. *For a topological space (X, τ) the following statements are equivalent:*

(i) (X, τ) is an R_1-space.
(ii) Every continuous function from (X, τ) into a space (Y, ν) is R_θ-supercontinuous.

Proof. $(i) \Longrightarrow (ii)$ is trivial.
$(ii) \Longrightarrow (i)$. Take $(Y, \nu) = (X, \tau)$. Then identity function 1_X on X is continuous and so R_θ-supercontinuous. By Theorem 7.2, the identity function $1_X : (X, \tau_{r_\theta}) \to (X, \tau)$ is continuous. Since $U \in \tau$ implies $1_X^{-1}(U) = U \in \tau_{r_\theta}$, therefore $\tau \subset \tau_{r_\theta}$. Hence, it follows that $\tau = \tau_{r_\theta}$ and thus (X, τ) is an R_1-space. □

As a byproduct of Theorem 7.3, we offer an alternative proof of the following well known result.

Theorem 7.4. *The product of any family $\{X_\alpha : \alpha \in \Lambda\}$ of R_1-spaces is an R_1-space.*

Proof. Let $X = \Pi X_\alpha$. In order to show that X is an R_1-space, in view of Theorem 7.3, it is sufficient to prove that every continuous function $f : X \to Y$ is R_θ-supercontinuous. Let $Y = \Pi Y_\alpha$ with product topology and define $f : X \to Y$ as $f((x_\alpha)) = (f_\alpha(x_\alpha))$ where $f_\alpha : X_\alpha \to Y_\alpha$ for each $\alpha \in \Lambda$. Since f is continuous, each f_α is also continuous. Moreover, since each X_α is an R_1-space, by Theorem 7.3, each f_α is R_θ-supercontinuous. Thus in view of Theorem 3.25, $f : X \to Y$ is R_θ-supercontinuous. Hence X is an R_1-space. □

Definition 7.5. *A function $f : X \to Y$ from a topological space X into a topological space Y is said to be R_θ-**continuous at** $x \in X$ if for each r_θ-open set V containing $f(x)$ there exists an open set U containing x such that $f(U) \subset V$. The function f is said to be R_θ-continuous if it is R_θ-continuous at each $x \in X$.*

Theorem 7.6. *For a function $f : (X, \tau) \to (Y, \nu)$ the following statements are true.*

(i) f is R_θ-continuous if and only if $f : (X, \tau) \to (Y, \nu_{r_\theta})$ is continuous.
(ii) f is R_θ-open if and only if $f : (X, \tau_{r_\theta}) \to (Y, \nu)$ is open.

In view of Theorems 7.2 and 7.6, Theorem 3.18 can be restated as follows:

If $f : (X, \tau_{r_\theta}) \to (Y, \nu)$ is a continuous open surjection and $g : (Y, \nu) \to (Z, \omega)$ is a function, then g is continuous if and only if $g \circ f$ is continuous. Further, if f maps open (closed) sets to r_θ-open (r_θ-closed) sets, then g is R_θ-supercontinuous.

Furthermore, r_θ-quotient topology on Y determined by the function $f : (X, \tau) \to Y$ in Section 6 is identical with the standard quotient topology on Y determined by $f : (X, \tau_{r_\theta}) \to Y$.

Remark 7.7. *For the interested reader we point out that the host of properties of R_θ-continuous functions can be inferred directly by simply substituting for $P \equiv$ the property of being an r_θ-closed set, in the relevant results pertaining to P-continuous functions and P^*-continuous functions in (*[11], [12], [13]*).*

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