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A TANGENTIAL INTERPOLATION PROBLEM FOR RATIONAL MATRIX FUNCTIONS FROM A CLASSICAL POINT OF VIEW

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Abstract. By applying a classical approach for scalar rational interpolation problem, certain type of minimal interpolation problem for rational matrix functions is solved. It is shown that an appropriately defined matrix plays a key role in solving the minimal problem.

1. Introduction

Interpolation problems for scalar rational functions have been studied by using Pade approximation, Euclidean algorithm, Lowner matrix as main tools. For a review of many approaches to this problem, see [9]. Also, interpolation problems for rational matrix functions have been studied through classical matrix theories, module theories, or through the null-pole structure of rational matrix functions. This paper hires the last approach which is well presented in [6]. Interpolation problems for rational matrix functions play an important role in systems theory, network theory and control theory. A classical paper on the occurrence of interpolation problems in the network theory and systems theory is [10].

For a scalar rational function, the spectral data consisting of zeros and poles with their respective multiplicities uniquely determines the function up to a nonzero multiplicative factor. But due to the richness of the spectral structure of a rational matrix function, reconstruction of a rational matrix function from a given spectral data is not that simple.

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In this paper, a classical approach to solve scalar rational interpolation problem is applied to a tangential interpolation problem for rational matrix functions. By a classical approach, we mean the approach which considers the solvability of the system of linear equations derived from the interpolation data to solve the scalar interpolation problem. In particular, we are interested in a solution of minimal complexity, measured by *McMillan degree*. Here by an $M \times N$ rational matrix fuction, we mean an $M \times N$ matrix with a scalar raional function as its each entry. If we define the poles of a rational matrix function as the poles of each entry, a rational matrix function has a finite number of poles. The McMillan degree of a raional matrix function $W(z)$ is defined to be the number of the poles of $W(z)$ and denoted by $\delta(W)$.

Here, an important basic idea to deal with rational matrix functions is to represent a proper (i.e., analytic at infinity) rational matrix function $W(z)$ by

$$W(z) = D + C(zI - A)^{-1}B.$$

Then, the zero and pole data for $W(z)$ is encoded in constant matrices A, B, C, D. This approach was spurred by diverse applications in many engineering context. For the details of this approach including how to count the number of the poles of a rational matrix function , readers are referred to [6].

The interpolation problem for the scalar case is the following. Given distinct points $z_1, \cdots, z_n$ in the complex plain $\mathbb{C}$ and given complex numbers $\{w_{ij}\}_{i=1}^{\mu_j}{}_{j=1}^{n}$, find a rational function $y(z)$ in the form of

$$y(z) := \frac{n(z)}{d(z)}, \quad gcd(n, d) = 1 \tag{1.1}$$

such that

$$y^{(i-1)}(z_j) = w_{ij}, \quad i = 1, \cdots, \mu_j, \quad j = 1, \cdots, n \tag{1.2}$$

where *gcd* means the greatest common divisor of polynomials. In the scalar case, the McMillan degree of $y(z)$ in (1.1) is defined by

$$\delta(y) = max\{deg\, n(z),\ deg\, d(z)\}.$$

Some special cases of this problem were studied by Belevich(see [4]) and Donoghue(see [7]) and the general case was understood by Antoulas and Anderson(see [1]). In the latter approach, the *Lowner matrix* is a key notion. For simplicity, if we assume, in (1.2), $\mu_i = 1,\ i = 1, \cdots, n$ and $n = 2m + 1$ for some nonnegative integer m, then the associated *Lowner* *matrix* is given by

$$L := \left[\frac{w_{m+1+i}-w_j}{z_{m+1+i}-z_j}\right]_{1\le i\le m,\, 1\le j\le m+1}. \tag{1.3}$$

This paper consists of two sections except the first introductory section. The second section introduces the already existing results for the scalar case: If there exists a solution of McMillan degree no more than $\frac{n}{2}$, then it is the unique solution of McMillan degree q, where q is the rank of the matrix L given by(1.3). Otherwise, the minimal possible McMillan degree for the solutions of (1.2) is $n-q$. The proof of the main results for the scalar case, Theorem 2.2, is redone by the author. In the third section, this author's efforts are made to extend the approach of the second section to the matrix case. In the matrix case, the *null-pole coupling matrix* Γ which is introduced in (3.3) plays the role of the *Lowner matrix* of the scalar case. For given (tangential) interpolation conditions for rational matrix functions, matrix anlogue of the scalar generic case is solved and a special solution with low McMillan is found in realization form. If the involved *null -pole coupling matrix* has full column rank, this special solution has the minimal possible McMillan degree among all the solutions. Even if the problem considered in the third section is a nice special case, it turns out the classical approach depending on the Lowner matrix and the solvability of linear equations derived from the interpolation data is not very enlighening for the matrix case. The major difficulty is the collapse of Theorem 2.1 (compare Theorem 2.1 and Theorem 3.1).

2. Scalar rational interplation problems

In this section, summary of some results for the scalar rational interpolation problem is given. For the details of the contents in this section, readers are referred to [1]. The problem considered here is the following. Given distinct points $z_1, \cdots, z_n$ in the complex plane $\mathbb{C}$ and given complex numbers $\{w_j\}_{j=1}^n$, find a rational function $f(z)$ in the form of (1.1) which has the minimal possible McMillan degree among the rational functions satisfying the interpolating conditions

$$f(z_j) = w_j,\ j = 1, \cdots, n. \tag{2.1}$$

To fix the notation, we assume

$$n = 2m + 1$$

(For the results of the general case, readers are referred to [1]).

The classical approach to this problem is based on the naive (but in general false) hope that the interpolation problem (2.1) can be solved by a raional function of McMillan degree at most m. Indeed, if we put

$$f(z) := \frac{n(z)}{d(z)}$$

with

$$n(z) := a_0 + a_1 z + \cdots + a_m z^m$$
$$d(z) := b_0 + b_1 z + \cdots + b_m z^m$$

then from (2.1) the following system of linear equations are derived;

$$n(z_j) - w_j d(z_j) = 0, \quad j = 1, \cdots, n \tag{2.2}$$

or equivalently,

$$\begin{bmatrix} 1 & z_1 & \cdots & z_1^m & w_1 & w_1 z_1 & \cdots & w_1 z_1^m \\ 1 & z_2 & \cdots & z_2^m & w_2 & w_2 z_2 & \cdots & w_2 z_2^m \\ \vdots & & & \vdots & & & & \vdots \\ 1 & z_n & \cdots & z_n^m & w_n & w_n z_n & \cdots & w_n z_n^m \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdots \\ a_m \\ -b_0 \\ -b_1 \\ \cdots \\ -b_m \end{bmatrix} = 0. \tag{2.3}$$

The system of linear equations (2.2) or (2.3) is called the modified interpolation problem in [9] and has often been studied as an intermediate tool in solving the interpolation problem (2.1). The system of linear equations (2.3) always has a nontrivial solution since the system of linear equations to be solved has $(2m+1)$ equations and $(2m+2)$ variables. But, in this approach, the difficulty arises if in the solution of the modified in terpolation problem (2.2) $d(z)$ has a zero at one of the prescribed points $z_1, \cdots, z_n$. Then the polynomials $n(z)$ and $d(z)$ have common zero at the point and the function $f(z) = \frac{n(z)}{d(z)}$ may not interpolate at the point. Those points are called *unattainable* or *inaccessible* points. By the *generic case*, we refer to the case where there is no inaccessible point. Hence, in the generic case, a solution of McMillan degree at most m exists. In [4], the problem of finding a solution of McMillan degree at most m is analyzed more systematically by using the Lowner matrix L defined by (1.3). To introduce his approach, we start with the following theorem which is crucial in his analysis.

Theorem 2.1. *Let*

$$f(z) = \frac{n(z)}{d(z)}, \qquad gcd(n, d) = 1$$

be a rational function of McMillan degree q and L be any $p \times l$ Lowner matrix built on $(p + l)$ distinct points in $\mathbb{C}$ with $p \geq q,\ l \geq q$. Then

$$rank\, L = q.$$

For the proof, see [1] or [4].

Let L be the $m \times (m + 1)$ Lowner matrix given by (1.3) and

$$rank\, L = q.$$

Then a solution of the modified interpolation problem (2.2) or (2.3) with $deg\, n(z) \leq m,\ \ deg\, d(z) \leq m$ can be constructed in the following way, if it exists. Let $\bar{L}$ be an $(n - q - 1) \times (q + 1)$ Lowner matrix defined by

$$\bar{L} = [\frac{w_{q+1+i} - w_j}{z_{q+1+i} - z_j}]_{1 \leq i \leq n-q-1,\ 1 \leq j \leq q+1}. \tag{2.4}$$

By Corollart 2.24 of [1] which states that any $r \times l$ Lowner matrix constructed from the same data for L with $r \geq q$, $l \geq q$ has *rank* q,

$$rank \bar{L} = q. \tag{2.5}$$

Since $(n - q - 1) \times (q + 1)$ matrix $\bar{L}$ has *rank* q, there exists a $(q + 1)$ -dimensional nonzero vector

$$c := [c_1, \cdots, c_{q+1}]^T$$

satisfying

$$\bar{L}c = 0. \tag{2.6}$$

Then, (2.6) is expressed as

$$\sum_{j=1}^{q+1} c_j \frac{w_k - w_j}{z_k - z_j} = 0, \quad k = q + 2,\ q + 3, \cdots, n \tag{2.7}$$

Let

$$\bar{d}(z) := \sum_{j=1}^{q+1} c_j \prod_{i=1,\, i \neq j}^{q+1} (z - z_i) \tag{2.8}$$

$$\bar{n}(z) := \sum_{j=1}^{q+1} c_j w_j \prod_{i=1\,,i\neq j}^{q+1} (z - z_i) \tag{2.9}$$

and $\bar{f}(z)$ be a rational function satisfying

$$\bar{f}(z)\bar{d}(z) = \bar{n}(z). \tag{2.10}$$

Since at least one c_i is not zero, by our choice of vector c, $\bar{d}(z)$ is not identically zero. Thus, $\bar{f}(z)$ in (2.10) is well-defined, and

$$w_k\bar{d}(z_k) - \bar{n}(z_k) = 0 \;\; for \;\; k = 1, \cdots, n. \tag{2.11}$$

Indeed, for $k = 1, \cdots, q+1$, the equalty (2.11) is obtained as soon as we plug $z = z_k$ in (2.8) and (2.9). For $k \geq q+2$, (2.11) is derived from (2.7) by multiplying both sides by $\prod_{j=1}^{q+1}(z_k - z_i)$. Hence, if $\bar{d}(z)$ in (2.8) has no zeroes at $\{z_1, z_2, \cdots, z_n\}$, then $\bar{f}(z)$ given by (2.10) is a solution for (2.1) with $\delta(\bar{f}) = rankL$. As we will see in the next theorem, if there exists a solution of McMillan degree less than $\frac{n}{2}$, $\bar{f}(z)$ is the unique minimal solution.

The following theorem, due to [1], gives a solution of minimal possible McMillan degree when $\bar{f}(z)$ in (2.10) is not a solution of (2.1). This presented proof is newly done by the author. The main difference of the author's proof arises by applying the approach by [6].

Theorem 2.2. *Let L be an $m \times (m+1)$ Lowner matrix given by (1.3) and $\bar{d}(z)$ and $\bar{f}(z)$ be given as in (2.8)- (2.10).*
(a) If $\bar{d}(z)$ has no zeros at z_j, $j = 1, \cdots, n$, then the minimal possible McMillan degree for the solutions to (2.1) is $rank\, L$ and $\bar{f}$ is the unique such solution.
(b) Otherwise, $n - rank\, L$ is the minimal possible McMillan degree and there is more than one solution of McMillan degree $n - rank\, L$.

Proof. (a) By Theorem 2.1, there is no solution of McMillan degree less than $q := rank\, L$. Now, we show that if $f(z)$ is a solution of McMillan degree at mot m, then $f(z) = \bar{f}(z)$. Suppose $\bar{f}(z)$ interpolates $n - \alpha (= 2m+1-\alpha)$ points ($\alpha = 0$ if and only if $\bar{f}(z)$ is a solution). Then, by the previous arguments, there exist points $z_{i_1}, \cdots, z_{i_\alpha}$ for which

$$\bar{d}(z_{i_j}) = 0, \quad j = 1, \cdots, \alpha. \tag{2.12}$$

Because of (2.10)(2.12),

$$\bar{n}(z_{i_j}) = 0, \quad j = 1, \cdots, \alpha \tag{2.13}$$

Upon combining (2.12)(2.13), we can see that $\delta(\bar{f}) \leq q - \alpha$. Hence a rational function $f(z) - \bar{f}(z)$ has degree no more than $m + q - \alpha$ and has at least $n - \alpha$ zeros. But, the fact that $m + q - \alpha \leq 2m - \alpha < n - \alpha$ forces that $f(z) = \bar{f}(z)$. Hence $\bar{f}(z)$ is the unique solution of McMillan degree at most m.

(b) Suppose $\bar{f}(z)$ is not a solution. By the arguments in the proof of (a) saying that if there is a solution of McMillan degree at most m, then it should be $\bar{f}$, there is no solution of McMillan degree at most m. Suppose there exists a solution $\tilde{f}(z)$ of McMillan degree $\tilde{q} > m$. Let $\tilde{L}$ be a $l \times n$ Lowner matix constructed from $\tilde{f}(z)$ so that $l \geq \tilde{q}$ and $n \geq \tilde{q}$. Then by Theorem 2.1, any $\tilde{q} \times \tilde{q}$ submatrix of $\tilde{L}$ is invertible. Observing that the Lowner matrix here is the null-pole coupling matrix in [6], we apply Theorem 4.5.1 of [6] to conclude the minimal possible such $\tilde{q}$ is $n - rank\, L = n - q$. For the proof of nonuniqueness of the solutions of McMillan degree $n - q$, readers are referred to [1].

□

Remarks

(a) Theorem 2.1 for the multiple point interpolation problems is proved in [1].

(b) If n is even and unless the rank of the $\frac{n}{2} \times \frac{n}{2}$ Lowner matrix L is $\frac{n}{2}$, all the arguments are the same. But if $rank\, L = \frac{n}{2}$, then $\frac{n}{2}$ is the minimal possible McMillan degree for the solutions and a solution of McMillan degree $\frac{n}{2}$ is obtained in [1] by exactly the same way to construct a minimal solution of McMillan degree $n - q$.

(c) Both the multiple point case and even n case, Theorem 2.2 can be proved by the same arguments since the results in [6] covers the general cases including the both.

3. A tangential matrix interpolation problem from classical point of view

In this section, we consider a generalization of the interpolation conditions (1.2) with $\mu_i = 1$ to the matrix case and try to find a solution of minimal possible McMillan degree.

The problem we consider in this section is the following. For given distinct points $\{z_1, z_2, \cdots, z_m, w_1, w_2, \cdots, w_{m+N}\}$ in the complex plain $\mathbb{C}$, and nonzero vectors $x_1, \cdots, x_m$ in $\mathbb{C}^{1\times M}$ and vectors $y_i \in C^{1\times N,}$ (for $i = 1, \ldots, m$), nonzero vectors $u_1, \cdots, u_{m+N}$ in $\mathbb{C}^{N\times 1}$ and vectors

$v_j \in C^{M\times 1,}$ (for $j = 1, \ldots, m+N$), find an $M \times N$ rational matrix function $W(z)$ for which

$$x_i W(z_i) = y_i, \quad i = 1, \cdots, m \tag{3.1}$$

$$W(w_j)u_j = v_j, \quad j = 1, \cdots, m+N, \tag{3.2}$$

and analytic in σ.

Then we can derive $MN + (M+N)m$ equations from the conditions (3.1), (3.2). On the other hand, it is known that an $M \times N$ rational matrix function of McMillan degree d is determined by $MN + (M+N)d$ parameters (see [5]). Hence, by comparing the number of constraints and that of parameters, we can expect to find a solution of (3.1) and (3.2) which has McMillan degree at most m. But, as in the scalar case, it is not always the case. In this paper, we find a sufficient condition for a solution of McMillan degree at most m to exist and, if it exists, a solution of McMillan degree at most m is described in terms of given interpolation data (see Theorm 3.3 and Theorem 3.4.).

Let Γ be an $m \times (m+N)$ matrix whose (i,j) entry, denoted by Γ_{ij}, is given by

$$\Gamma_{ij} = \frac{x_i v_j - y_i u_j}{w_j - z_i} \tag{3.3}$$

where vectors x_i, y_i, u_j, v_j are as in (3.1)(3.2). Then Γ is an analogue of the Lowner matrix L in(1.3). In [6], Γ in (3.3) is called a *null-pole coupling matrix*.

Now we introduce some basic notions for rational matrix functioins. We follow the notions of [6]. By an $M \times N$ rational matrix function, we understand an $M \times N$ matrix with rational functions as its entries and shall regard it as a meromorphic matrix function over the extended complex plane $\mathbb{C}^\infty$. For an $M \times N$ proper (i.e., analytic at infinity) rational matrix function $W(z)$, we define a *realization* of $W(z)$ to be a representation of the form

$$W(z) = D + C(zI - A)^{-1}B, \quad z \notin \sigma(A) \tag{3.4}$$

where A, B, C, D are matrices of sizes $n\times n, n\times N, M\times n, M\times N$ respectively, and $\sigma(A)$ refers to the spectrum of the matrix A. A realization (3.4) is said to be *minimal* if (C, A) is a *null-kernel pair* and (A, B) is a *full-range pair*, that is

$$\bigcap_{j=0}^{n-1} Ker\ CA^j = \{0\}$$

$$\sum_{j=0}^{n-1} Im\ A^j B = \mathbb{C}^n.$$

It is known that if (3.4) is a minimal realization for W, then $\delta(W) = n$. If D is invertible in (3.4), then

$$W^{-1}(z) = D^{-1} - D^{-1}C(zI - A^{\times})^{-1}BD^{-1} \tag{3.5}$$

with $A^{\times} = A - BD^{-1}C$ is a realization of $W^{-1}(z)$ and (3.4) is minimal if and only if (3.5) is. When W has singularities at infinity, the realization (3.4) cannot be used and is replaced by

$$W(z) = D + C_F(zI - A_F)^{-1}B_F + D + zC_\infty(I - zA_\infty)^{-1}B_\infty \tag{3.6}$$

with $\sigma(A_\infty) = \{0\}$. Here $(A_F, B_F, C_F, D, A_\infty, B_\infty, C_\infty)$ of sizes $n_F \times n_F$, $n_F \times N$, $M \times n_F$, $M \times N$, $n_\infty \times n_\infty$, $n_\infty \times N$, $M \times n_\infty$ are said to be a realization of W. Realizations for a rational matrix function always exist. For more details, readers are referred to [6].

A realization of the form (3.6) has the disadvantage that the value of W at any point in $\mathbb{C}$ is not displayed in an obvious way. This situatoin can be mended by representing (3.6) as follows. For $\alpha \notin \sigma(A_F)$,

$$\begin{aligned} W(z) = D_\alpha - (z-\alpha)C_F(zI - A_F)^{-1}(\alpha I - A_F)^{-1}B_F + \\ +(z-\alpha)C_\infty(I - zA_\infty)^{-1}(I - \alpha A_\infty)^{-1}B_\infty \end{aligned} \tag{3.7}$$

with $D_\alpha = W(\alpha)$. The realization in the form of (3.7) is called *a realization for W centered at α* .

If we put

$$G := \begin{bmatrix} I & 0 \\ 0 & A_\infty \end{bmatrix}, \qquad A := \begin{bmatrix} A_F & 0 \\ 0 & I \end{bmatrix},$$

$$B := \begin{bmatrix} -(\alpha I - A_F)^{-1}B_F \\ (I - \alpha A_\infty)^{-1}B_\infty \end{bmatrix}, \quad C := \begin{bmatrix} C_F & C_\infty \end{bmatrix},$$

then (3.7) can be represented as

$$W(z) = D_\alpha + (z-\alpha)C(zG - A)^{-1}B. \tag{3.8}$$

Now, we are ready to state the following theorem which is a counterpart of Theorem 2.1. Note that the equality in Theorem 2.1 falls apart in the next theorem.

Theorem 3.1. *Let $W(z)$ be a given $M \times N$ rational matrix function with $\delta(W) = q$ and let*

$$\Gamma = \left[x_i \frac{W(w_j)-W(z_i)}{w_j - z_i} u_j\right]_{1\le i\le n,\ 1\le j\le l} \tag{3.9}$$

where $n \ge q$, $l \ge q$ and $z_1, \cdots, z_n, w_1, \cdots, w_l$ are distinct points in the complex plane at which $W(z)$ is analytic and $x_1, \cdots, x_n$ are nonzero $1 \times M$ vectors, $u_1, \cdots, u_l$ are $N \times 1$ nonzero vectors. Then,

$$rank\, \Gamma \le \delta(W).$$

Proof. Let

$$W(z) = D + (z-\alpha)C(zG - A)^{-1}B \tag{3.10}$$

be a minimal realization for $W(z)$. Then, it is easily seen

(3.11)

$$W(w_j) - W(z_i) = (w_j - z_i)C(z_iG - A)^{-1}(\alpha G - A)(w_jG - A)^{-1}B.$$

By substituting (3.11) in (3.9), we have

$$\begin{aligned}\Gamma &= \left[x_i C(z_iG - A)^{-1}(\alpha G - A)(w_jG - A)^{-1}Bu_j\right]_{1\le i\le n,\ 1\le j\le l}\\ &= \begin{bmatrix} x_1 & & 0\\ & \ddots & \\ 0 & & x_n\end{bmatrix} \begin{bmatrix} C(z_1G - A)^{-1}\\ \vdots \\ C(z_nG - A)^{-1}\end{bmatrix}\\ &\quad \times (\alpha G - A)\left[(w_1G - A)^{-1}B \quad \cdots \quad (w_lG - A)^{-1}B\right] \begin{bmatrix} u_1 & & 0\\ & \ddots & \\ 0 & & u_l\end{bmatrix}.\end{aligned}$$

From the above equation, we see

$$rank\, \Gamma \le rank(\alpha G - A) \le \text{ the size of } A.$$

By the minimality of the realization of $W(z)$ given in (3.10), the size of the square matrix A is $\delta(W)$. This completes the proof. □

The following example shows that the inequality

$$rank\Gamma \le \delta(W)$$

in Theorem 3.1 is, unlike the scalar case, as sharp as we can get .

Exmaple 3.1. Let

$$W = \begin{bmatrix} 1 & \frac{1}{z^2} \\ 0 & 1 \end{bmatrix}.$$

Then, $W(z)$ can be written as

$$W(z) = I + C(zI - A)^{-1}B$$

with

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since (C, A) is a *null-kernel pair* and (A, B) is a *full range pair*, the above equation for $W(z)$ is a minimal realization, and hence $\delta(W) = 2$.

On the other hand, let $z_1 = 1,\ z_2 = 2,\ x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ x_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $w_1 = -1,\ w_2 = -2,\ u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then, by(3.1)(3.2), we get $y_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $y_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$. By (3.3), we get

$$\Gamma = \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & 0 \end{bmatrix}$$

whose rank is 1. Thus, in this case, $rank\,\Gamma < \delta(W)$.

Now, we want to find a condition for a solution of McMillan degree at most m to exist. And, if it exists, we describe the solution in terms of the given interpolation data. To do this, we need to introduce a parametrization of all solutions. Let

$$(3.12) \qquad A_\zeta := \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_m \end{bmatrix}, \quad B_+ := \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \quad B_- := -\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix},$$

$$(3.13) \qquad \begin{bmatrix} C_+ \\ C_- \end{bmatrix} := \begin{bmatrix} v_1, \cdots\cdots, v_{m+N} \\ u_1, \cdots\cdots, u_{m+N} \end{bmatrix},$$

$$(3.14) \qquad A_\pi := \begin{bmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_{m+N} \end{bmatrix},$$

where z_i, x_i, y_i, w_j, u_j, v_j are the same as in (3.1),(3.2). Then, the data set $\tilde{\gamma} := (C_+, C_-, A_\pi; A_\zeta, B_+, B_-; \Gamma)$ satisfies the conditions

$$(A_\zeta, B_+) \text{ is a full range pair },$$

$$(C_-, A_\pi) \text{ is a null-kernel pair},$$

$$\Gamma \text{ satisfies the following Sylvester equation}$$
$$\Gamma A_\pi - A_\zeta \Gamma = B_+ C_+ + B_- C_-,$$

where Γ is given by (3.3).

Let σ be a subset of $\mathbb{C}$ satisfying $\sigma(A_\pi) \cup \sigma(A_\zeta) \subset \sigma$. Let

$$\Theta(z) = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \tag{3.15}$$

be any $(M+N)+(M+N)$ rational matrix function having the set

$$\tilde{\tau} := (\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; A_\zeta, \begin{bmatrix} B_+ & B_- \end{bmatrix}; \Gamma)$$

as a σ-null-pole triple and φ^{-1} be a regular $N \times N$ matrix function having the set

$$\tilde{\tau}_- := (C_-, A_\pi; 0,\ 0\, ;\ 0)$$

as a σ-null-pole triple. Then, due to [6], a solution for (3.1)(3.2) exists and all the solutions are parametrized as follows:

$$W = (\Theta_{11} P + \Theta_{12} Q)(\Theta_{21} P + \Theta_{22} Q)^{-1}, \tag{3.16}$$

where P, Q are rational matrix functions of sizes $M \times N$, $N \times N$ respectively and analytic in σ and

$$\varphi(\Theta_{21} P + \Theta_{22} Q) = I. \tag{3.17}$$

Upon combining (3.16)(3.17), all solutions can be parametrized as follows: a rational matrix function $W(z)$ analytic in σ is a solution if and only if

$$\begin{bmatrix} W \\ I \end{bmatrix} \varphi^{-1} = \Theta \begin{bmatrix} P \\ Q \end{bmatrix}, \tag{3.18}$$

where P, Q are as in (3.16)(3.17).

On the other hand, since Θ has $\tilde{\tau}$ as its σ-null-pole triple, by [6], there exist an $(m+N)\times N$ matrix $\tilde{B}$ and rational functions H, K analytic in σ for which

$$\Theta \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (zI - A_\pi)^{-1}\tilde{B} + \begin{bmatrix} H(z) \\ K(z) \end{bmatrix}, \tag{3.19}$$

where $H(z)$, $K(z)$ satisfy the residue equation

$$\sum_{z_0 \in C} Res_{z=z_0}(zI - A_\zeta)^{-1} \begin{bmatrix} B_+ & B_- \end{bmatrix} \begin{bmatrix} H(z) \\ K(z) \end{bmatrix} = \Gamma\tilde{B} \tag{3.20}$$

and $(A_\pi, \tilde{B})$ is a full-range pair.

To use this parametrization of all the solutions, we construct a candidate of φ^{-1} having $\tilde{\tau}_-$ as its σ-null-pole triple. Since $m\times(m+N)$ matrix Γ has *rank* at most m, we can find $(m+N)\times N$ matrix B satisfying

$$\Gamma B = 0 \text{ with } rank\, B = N. \tag{3.21}$$

Let

$$\begin{aligned} \varphi^{-1}(z) &= C_-(zI - A_\pi)^{-1}B \\ &= \sum_{j=1}^{m+N} \frac{1}{z - w_j} u_j \tilde{b}_j, \end{aligned} \tag{3.22}$$

where $\tilde{b}_j$ represents the j^{th} row of B. Now we have

$$\varphi^{-1}(z)p(z) = \sum_{j=1}^{m+N} p_j(z)u_j\tilde{b}_j := D(z), \tag{3.23}$$

where

$$p(z) := \prod_{k=1}^{m+N}(z - w_k), \qquad p_j(z) := \prod_{k=1,\, k\neq j}^{m+N}(z - w_k).$$

The next Lemma gives a refinement of (3.19)(3.20).

Lemma 3.2. *If $\varphi^{-1}(z)$ in (3.22) has $\tilde{\tau}_-$ as its σ-null-pole triple, then all solutions $W(z)$ can be parametrized as (3.16), where P, Q are rational matrix functions of sizes $M\times N$, $N\times N$ and analytic in σ satisfying*

$$\Theta \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} C_+ \\ C_- \end{bmatrix} (zI - A_\pi)^{-1}\, B + \begin{bmatrix} H(z) \\ 0 \end{bmatrix}, \tag{3.24}$$

where $H(z)$, analytic in σ, satisfies the residue equation

$$\sum_{z_0 \in C} Res_{z=z_0}(zI - A_\zeta)^{-1} B_+ H(z) = 0. \tag{3.25}$$

Proof. Since the parametrization of all solutions are given by (3.16)-(3.20), it is enough to show that $\tilde{B} = B$, $K(z) = 0$ in (3.19). By (3.18)(3.19),

$$\varphi^{-1}(z) = C_-(zI - A_\pi)^{-1}\tilde{B} + K(z),$$

where $K(z)$, analytic in σ, is subject to the condition (3.20). Since $\varphi^{-1}(z)$ has poles only inside of σ and $K(z)$ has poles only outside of σ, the above equation forces $K(z) = 0$. Combining (3.22) and the above equation with $K(z) = 0$, we have

$$C_-(zI - A_\pi)^{-1}(\tilde{B} - B) = 0$$

Since (C_-, A_π) is a null-kernel pair, $\tilde{B} = B$.

□

The following is our main theorem which gives a sufficient condition for a solution of McMillan degree at most m to exist. We see that the genericity condition $\bar{d}(z)$ in (2.8) having no zeros at $\{z_1, \cdots, z_n\}$ in the scalar case corresponds to the condition φ^{-1} in (3.22) has $\tilde{\tau}_- := (C_-, A_\pi; 0,\, 0\,; 0)$ as its σ-null-pole triple. Equivalently, $N \times N$ matrix polynomial $D(z)$ in (3.23) has full rank (that is, having no zero) at $\{z_1, \cdots, z_m\}$ and $\tilde{b}_j \neq 0$ for $j = i, \cdots, m+N$.

Theorem 3.3. *If $D(z)$ in (3.23) has full rank at $\{z_1, \cdots, z_m\}$ and $\tilde{b}_j \neq 0$ for $j = i, \cdots, m+N$, then*

$$W_0(z) := N(z)D^{-1}(z)$$

is a solution of (3.1)(3.2) with McMillan degree at most m, where

$$N(z) = \sum_{j=1}^{m+N} p_j(z) v_j \tilde{b}_j. \tag{3.26}$$

Proof. Let $W_0(z)$ be a rational matrix function satisfying

$$W_0(z)D(z) = N(z), \tag{3.27}$$

and analytic at σ. Then, W_0 is well-defined, since $D(z)$ is regular (that is, its determinant is not identically zero). Upon using the formula (3.3) for Γ, for $1 \le i \le m$,

$$\text{the } i^{th} \text{ row of } \Gamma B = \sum_{j=1}^{m+N} \frac{y_i u_j - x_i v_j}{z_i - w_j} \tilde{b}_j.$$

Multiplying the both sides of $\Gamma B = 0$ by $p(z)$, we have

$$y_i D(z_i) = x_i N(z_i), \ i = 1, \cdots, m.$$

By hyphothesis that $D(z_i)$ has full rank for each i, we see (3.1) holds with W_0 in place of W.

On the other hand, upon plugging $z = w_k$ in (3.23)(3.26), we have

$$\begin{aligned} &W_0(w_k)D(w_k) - N(w_k) \\ &= p_k(w_k)[W_0(w_k)u_k - v_k]\tilde{b}_k, \end{aligned}$$

for $k = 1, \cdots, m+N$. Since $p_k(w_k)$ and the row vector $\tilde{b}_k$ is not zero for each k, by (3.27), the column vector $W_0(w_k)u_k - v_k = 0$ for each k. Thus, the condition (3.2) holds with W_0 in place of W.

Now we compute the McMillan degree of $W_0(z)$. By dividing $D(z)$ and $N(z)$ by $p(z)$, we have

$$W_0(z) = N(z)D^{-1}(z) = C_+(zI - A_\pi)^{-1}B\varphi(z) \tag{3.28}$$

with $\varphi^{-1}(z) = C_-(zI - A_\pi)^{-1}B$. Note that all poles of $W_0(z)$ come from those of $\varphi(z)$ since $W_0(z)$ and $\varphi(z)$ are analytic at σ and $\sigma(A_\pi) \subset \sigma$. But there may occur some cancellation of poles of $\varphi(z)$ by premultiplying $R(z) := C_+(zI - A_\pi)^{-1}B$. Indeed, at ∞, at least N poles of $\varphi(z)$ are cancelled by the zeros of $R(z)$. To see this, we apply Mobius Transformation to $R(z)$ and $\varphi^{-1}(z)$ to get

$$R(\frac{1}{z}) = zC_+(I - zA_\pi)^{-1}B$$

and

$$\varphi^{-1}(\frac{1}{z}) = zC_-(I - zA_\pi)^{-1}B.$$

The above two formula show that $\varphi(\frac{1}{z})$ has the pole factor $\frac{1}{z}I_N$, which is cancelled out by the zeros of $R(\frac{1}{z})$ at $z = 0$. This means at least N poles of $\varphi(z)$ at infinity are cancelled out by $R(z)$. Thus,

$$\delta(W_0) \le \delta(\varphi) - N \le m.$$

□

Now, we express $W_0(z)$, constructed in Theorem 3.3., in a realization form.

Theorem 3.4. *If we choose $\alpha \in \mathbb{C}$ so that $\varphi^{-1}(\alpha)$ is invertible, then*

$$W_0(z) := C_+(z\tilde{G} - \tilde{A})^{-1}BD^{-1}$$

where B satisfies (3.21) and

$$\begin{aligned} D =& \varphi^{-1}(\alpha) = C_-(\alpha I - A_\pi)^{-1}B \\ \tilde{G} =& I - BD^{-1}C_-(\alpha I - A_\pi)^{-1} \\ \tilde{A} =& A_\pi - \alpha BD^{-1}C_-(\alpha I - A_\pi)^{-1}, \end{aligned}$$

and where C_+, C_-, A_π, Γ are given by (3.13),(3.14) (3.3).

Proof. Since $\varphi(z)$ is regular, there exists $\alpha \in C$ for which $D := \varphi^{-1}(\alpha)$ is invertible. Then, $\varphi^{-1}(z)$ can be expressed in a realization form centered at α as follows:

$$\varphi^{-1}(z) = D - (z - \alpha)C_-(zI - A_\pi)^{-1}(\alpha I - A_\pi)^{-1}B,$$

and $\varphi(z)$ is given by

$$\varphi(z) = D^{-1} + (z - \alpha)D^{-1}C_-(zG^\times - A^\times)^{-1}(\alpha I - A_\pi)^{-1}BD^{-1}, \tag{3.29}$$

where

$$G^\times := I - (\alpha I - A_\pi)^{-1}BD^{-1}C_- \tag{3.30}$$

$$A^\times := A_\pi - \alpha(\alpha I - A_\pi)^{-1}BD^{-1}C_-. \tag{3.31}$$

Substituting (3.29) in (3.28), we have

$$\begin{aligned} W_0(z) =& C_+(zI - A_\pi)^{-1}BD^{-1} + (z - \alpha)C_+(zI - A_\pi)^{-1}B \\ & \times D^{-1}C_-(zG^\times - A^\times)^{-1}(\alpha I - A_\pi)^{-1}BD^{-1} \\ =& C_+(zI - A_\pi)^{-1}\{(\alpha I - A_\pi)(zG^\times - A^\times) + (z - \alpha)BD^{-1}C_-\} \\ & \times (zG^\times - A^\times)^{-1}(\alpha I - A_\pi)^{-1}BD^{-1}. \end{aligned} \tag{3.32}$$

Substituting (3.30)(3.31) to have

$$(\alpha I - A_\pi)(zG^\times - A^\times) + (z - \alpha)BD^{-1}C_- = (\alpha I - A_\pi)(zI - A_\pi).$$

So, (3.32) becomes

$$\begin{aligned} W_0(z) =& C_+(\alpha I - A_\pi)(zG^\times - A^\times)^{-1}(\alpha I - A_\pi)^{-1}BD^{-1} \\ =& C_+\{(\alpha I - A_\pi)(zG^\times - A^\times)(\alpha I - A_\pi)^{-1}\}^{-1}BD^{-1}. \end{aligned}$$

Again, by substituting (3.30)(3.31) in places of $G^{\times}, A^{\times}$, the above formula is reduced to

$$W_0(z) = C_+(z\tilde{G} - \tilde{A})^{-1}BD^{-1}$$

with

$$\begin{aligned}\tilde{G} =& I - BD^{-1}C_-(\alpha I - A_\pi)^{-1}\\ \tilde{A} =& A_\pi - \alpha BD^{-1}C_-(\alpha I - A_\pi)^{-1}.\end{aligned}$$

This completes the proof. □

Remember that, in the scalar case, if there exists a solution $y(z)$ of McMillan degree no more than $\frac{n}{2}$, then $\delta(y) = rank\Gamma$, where n is the number of interpolating conditions. But, The following example shows this is not true for rational matrix functions.

Exmaple 3.2. Let z_i, x_i, y_i, w_j, u_j, v_j are given as in Example 3.1 for $i = 1, 2$, $j = 1, 2$. Let $w_3 = 3$, $w_4 = -3$, $u_3 = u_4 = v_3 = v_4 = \begin{bmatrix}1\\0\end{bmatrix}$. Then, $W(z)$ in Example 3.1 is an interpolant satisfying the interpolation condition (3.1)(3.2) with $m = M = N = 2$. By simple computation, we get

$$\Gamma = \begin{bmatrix}0 & \frac{1}{4} & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Thus, $rank\Gamma < \delta(W) = 2 = m < (2m + N)/2$. This example shows W_0 obtained in Theorem3.2 with $\delta(W_0) \leq m$ may have $\delta(W_0) < m$.

According to the next collary, the equality holds if Γ has full rank.

Corollary 3.5. *If $rank\,\Gamma = m$, then $W_0(z)$ in Theorem3.3 satisfies $\delta(W_0) = m$. That is, $W_0(z)$ has the minimal possible McMillan degree among all the solutions.*

Proof. Applying Theorem 3.1 to $W_0(z)$ constructed in Theorem 3.3, we see $\delta(W_0) = m = rank\,\Gamma$ and, in this case, m is the minimal possible McMillan degree among all the solutions. □

Remarks Even for the very special case of matrix interpolation problem which we consider in (3.1)(3.2), it seems difficult to go beyond Theorem 3.3 by this classical approach. In [2], the minimal possible McMillan degree for the different type of matrix interpolation problem is found by this approach under some additional highly restrictive hypothesis on the structure of Lowner Matrix. It seems this approach is not very fruitful for the matrix case. We need new perspective for the scalar interpolation problem which can be extended to the matrix case more effectively. This new perspective in understanding the scalar interpolation problem will be addressed elsewhere.

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