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A NEW APPROACH ON THE CURVATURE DEPENDENT ENERGY FOR ELASTIC CURVES IN A LIE GROUP

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Abstract. Elastica is known as classical curve that is a solution of variational problem, which minimize a thin inextensible wire's bending energy. Studies on elastica has been conducted in Euclidean space firstly, then it has been extended to Riemannian manifold by giving different characterizations. In this paper, we focus on energy of the elastic curve in a Lie group. We attepmt to compute its energy by using geometric description of the curvature and the torsion of the trajectory of the elastic curve of the trajectory of the moving particle in the Lie group. Finally, we also investigate the relation between energy of the elastic curve and energy of the same curve in Frenet vector fields in the Lie group.

1. Introduction

Corresponding theory for the functional of curvature-based energy is considered to evolve in many research fields. Some prolific fields and pioneering studies for this theory can be found in mathematical physics, membrane chemistry, computer aided geometric design and geometric modeling, shell engineering, biology and thin plate [1, 5, 12, 13, 16, 17]. Wood [25] studied energy on the unit vector fields firstly. Gil-Medrano [11] worked on relation between energy and volume of vector fields. [6, 7] investigated on the energy of distributions and corrected energy of distributions on Riemannian manifolds. Altin [2] computed energy of Frenet vector fields for given nonlightlike curves. Körpınar [15] , discussed energy of the timelike biharmonic particle in Heisenberg spacetime.

Materials having the feature of deformable structure such as cloth, flexible metals, rubber, paper are the main subject and research field

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for the elasticity theory. However, elastica can be considered from a various of different perspective that enlighten broad range of physical and mathematical studies. Studies concerned about the elastica firstly focus on the research of mechanical equilibrium, the study of variational problems, and the solution of the elliptic integral.

One of the earliest approach on elastica yields prolific consequences on equilibrium of moments, which constitute elemantary principle of statics. Further, it is seen that elastica gives a natural solution for the variotional problem, which deal with the minimizing of bending energy of the elastic curve. Later, the equivalence between the motion of the simple pendulum and fundamental differential equation of elastica were investigated. Recently, numerical computation implemented on the elastica is used to develop mathematical spline theory [18].

Potential elastic energy takes place when materials are stretched, compressed or deformed in any way. That is, these deformed bodies store potential energy when there exists a force on them. This potential energy is exerted to bring the deformed body back to its neutral position prior to deformation [24]. In this study, we attempt to carry this concept into the elastic curves lying on Lie group $\mathbf{R}$.

The innovation that Lie group brings to mathematics is that it has three different structures of mathematical form. It enables setting a connection between these different forms. Primarily, it has structure of group. Further, the elements belonging to this group form a topological space. Lastly, the elements also form an analytic manifold.

Lie groups play a key role not only in physical systems but also in mathematical studies. It is highly significant in loop groups, gauge groups, and Fourier integral's groups operators that occurs as symmetry groups and phase spaces. Lie groups are also useful in mechanics. Since incomprehensible inviscid fluid motion and rigid body motion correspond to geodesic flow of left (or right) invariant metric defined on the Lie group [3, 10, 14, 19].

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In this study, we approach the concept on the energy of the elastic materials from a different point of view. We firstly define kinematics of the particle lying on a Lie group. Then we characterize the energy on each Frenet vector fields in Lie gorup. Moreover, we also determine formula of the elastic curve in Lie gorup in terms of Frenet vector elements using variational method. Finally, we compute energy of elastic curves in this space. The method we use for computing energy in this study is that considering a vector field as a map from manifold M to the Riemannian manifold $(TM,\ \rho_s)$, where TM is tangent bundle of a Riemannian manifold and ρ_s is a Sasaki metric induced from TM naturally.

2. Kinematics of the Particle in a Lie Gorup R

Let Γ be a particle moving in a space $\mathbf{R}$ such that the precise location of the particle is specified by $\Gamma = \Gamma(t)$, where t is a time parameter. Changing time parameter describes the motion and trajectory of the particle, ultimately. In most cases, this trajectory corresponds to a particular curve in the space. It is convenient to remind that arc-length parameter s is used to compute the distance traveled by a particle along its trajectory. It is defined by

$$\frac{ds}{dt} = \|\mathbf{v}\|,$$

where $\mathbf{v} = \mathbf{v}(t) = \frac{d\zeta}{dt}$ is the velocity vector and $\frac{d\zeta}{dt} \neq 0$. In particle dynamics, the arc-length parameter s is considered as a function of t. Thanks to the arc-length, it is also determined Serret-Frenet frame, which allows us determining characterization of the intrinsic geometrical features of the regular curve. This coordinate system is constructed by three orthonormal vectors $\mathbf{e}_{(\alpha)}$, assuming the curve is sufficiently smooth at each point. The index within the parenthesis is the tetrad index that describes particular member of the tetrad. In particular, $\mathbf{e}_{(0)}$ is the unit tangent vector, $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$ is the unit normal and binormal vector of the curve ζ, respectively. Orthonormality conditions are summarized by $\mathbf{e}_{(\alpha)}\mathbf{e}_{(\beta)} = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is Euclidean metric such that: $\mathrm{diag}(1,1,1)$. For nonnegative coefficients κ, τ, and vectors $\mathbf{e}_{(i)}$ $(i = 0,1,2)$ following equations and properties are satisfied [8, 20].

$$\begin{aligned} \frac{D\mathbf{e}_{(0)}}{ds} &= \kappa\mathbf{e}_{(1)}, \\ \frac{D\mathbf{e}_{(1)}}{ds} &= -\kappa\mathbf{e}_{(0)} + (\tau - \tau_{\mathbf{R}})\,\mathbf{e}_{(2)}, \\ \frac{D\mathbf{e}_{(2)}}{ds} &= (\tau_{\mathbf{R}} - \tau)\,\mathbf{e}_{(1)}, \end{aligned} \tag{1}$$

where Lie group $\mathbf{R}$ has the Levi-Civita connection D and

$$\tau_{\mathbf{R}} = \frac{1}{2}([\mathbf{e}_{(0)},\mathbf{e}_{(1)}],\mathbf{e}_{(2)}).$$

Proposition 2.1 *For a 3-dimensional Lie group* $\mathbf{R}$ *induced with a bi-invariant metric we have following statements.*
(i) $\tau_{\mathbf{R}} = 0$ *if* $\mathbf{R}$ *is Abelian group.*
(ii) $\tau_{\mathbf{R}} = 1$ *if* $\mathbf{R}$ *is* SU^2.
(iii) $\tau_{\mathbf{R}} = \frac{1}{2}$ if $\mathbf{R}$ *is* SO^3 $[8, 20]$.

3. Energy on the Classical Bernoulli-Euler Elastica

3.1. Energy on the Unit Frenet Vector Fields

We first give the fundamental definitions and propositions, which are used to compute the energy of the vector field.

Definition 3.1 *Let* (M, ρ) *and* (N, h) *be two Riemannian manifolds, then the energy of a differentiable map* $f : (M, \rho) \to (N, h)$ *can be defined as*

$$\varepsilon nergy\,(f) = \frac{1}{2}\int_M \sum_{a=1}^{n} h\left(df\left(e_a\right), df\left(e_a\right)\right) v, \tag{2}$$

where $\{e_a\}$ *is a local basis of the tangent space and* v *is the canonical volume form in* M $[2, 6]$.

Proposition 3.2 *Let* $Q : T\left(T^1M\right) \to T^1M$ *be the connection map. Then following two conditions hold:*
i) $\omega \circ Q = \omega \circ d\omega$ *and* $\omega \circ Q = \omega \circ \tilde{\omega}$, *where* $\tilde{\omega} : T\left(T^1M\right) \to T^1M$ *is the tangent bundle projection;*
ii) *for* $\varrho \in T_xM$ *and a section* $\xi : M \to T^1M$; *we have*

$$Q\left(d\xi\left(\varrho\right)\right) = D_{\varrho}\xi, \tag{3}$$

where ∇ *is the Levi-Civita covariant derivative* $[6]$.

Definition 3.3 *Let* $\varsigma_1, \varsigma_2 \in T_{\xi}\left(T^1M\right)$, *then we define*

$$\rho_S\left(\varsigma_1, \varsigma_2\right) = \rho\left(d\omega\left(\varsigma_1\right), d\omega\left(\varsigma_2\right)\right) + \rho\left(Q\left(\varsigma_1\right), Q\left(\varsigma_2\right)\right). \tag{4}$$

This yields a Riemannian metric on TM. *As known* ρ_S *is called the Sasaki metric that also makes the projection* $\omega : T^1M \to M$ *a Riemannian submersion.*

Theorem 3.4 *Let* Γ *be a moving particle in a Lie group* $\mathbf{R}$ *such that it corresponds to a unit speed space curve* ζ. *Energy on the particle in* *tangent, normal, and binormal vector field is stated by*

$$\begin{aligned}\varepsilon nergy \mathbf{e}_{(0)} &= \frac{1}{2}(s+\int_0^s \kappa^2 ds),\\ \varepsilon nergy \mathbf{e}_{(1)} &= \frac{1}{2}(s+\int_0^s \left(\kappa^2+(\tau-\tau_{\mathbf{R}})^2\right) ds),\\ \varepsilon nergy \mathbf{e}_{(2)} &= \frac{1}{2}(s+\int_0^s (\tau-\tau_{\mathbf{R}})^2 ds).\end{aligned}$$

Proof. Here, we only prove energy of the particle in the tangent vector field, since the rest of the proof can be completed similarly. From the Eq. (2) and Eq. (3) we know that

$$\varepsilon nergy \mathbf{e}_{(0)} = \frac{1}{2}\int_0^s \rho_S \left(d\mathbf{e}_{(0)}(\mathbf{e}_{(0)}), d\mathbf{e}_{(0)}(\mathbf{e}_{(0)})\right) ds.$$

Using Eq. (4) we have

$$\begin{aligned}\rho_S \left(d\mathbf{e}_{(0)}(\mathbf{e}_{(0)}), d\mathbf{e}_{(0)}(\mathbf{e}_{(0)})\right) = \rho(d\omega(\mathbf{e}_{(0)}(\mathbf{e}_{(0)})), d\omega(\mathbf{e}_{(0)}(\mathbf{e}_{(0)})))\\ +\rho(Q(\mathbf{e}_{(0)}(\mathbf{e}_{(0)})), Q(\mathbf{e}_{(0)}(\mathbf{e}_{(0)}))).\end{aligned}$$

Since $\mathbf{e}_{(0)}$ is a section, we get

$$d(\omega)\circ d(\mathbf{e}_{(0)}) = d(\omega\circ \mathbf{e}_{(0)}) = d(id_C) = id_{TC}.$$

We also know

$$Q(\mathbf{e}_{(0)}(\mathbf{e}_{(0)})) = D_{\mathbf{e}_{(0)}}\mathbf{e}_{(0)} = \kappa \mathbf{e}_{(1)}.$$

Thus, we find from the Eq. (1)

$$\begin{aligned}\rho_S \left(d\mathbf{e}_{(0)}(\mathbf{e}_{(0)}), d\mathbf{e}_{(0)}(\mathbf{e}_{(0)})\right) &= \rho\left(\mathbf{e}_{(0)}, \mathbf{e}_{(0)}\right) + \rho(D_{\mathbf{e}_{(0)}}\mathbf{e}_{(0)}, D_{\mathbf{e}_{(0)}}\mathbf{e}_{(0)})\\ &= 1+\kappa^2.\end{aligned}$$

So we can easily obtain

$$\varepsilon nergy \mathbf{e}_{(0)} = \frac{1}{2}(s+\int_0^s \kappa^2 ds).$$

This completes the proof.

3.2. Energy on the Elastic Curves in a Lie Group

The research on the curvature-based energies for space curves began with Bernoulli and Euler's studies on elastic thin beams and rods. This type of energy is both essential in the mechanical context and also significant in computer vision, image processing and computer vision besides mathematical and physical importance [4, 9, 21, 22].

The elastica in Riemannian manifold has been formulated with the help of variational problem. It is used an integral of the squared curvature of the curve with a specific boundary conditions [23].

Let $\eta : I \subset \mathbb{R} \to \mathbf{R}$ be an immersed unit speed curve in a Lie group $\mathbf{R}$, then it has vector of a velocity $\mathcal{V} = v\mathbf{e}_{(0)}$ and squared geodesic curvature

$$\left\| D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)} \right\| = \kappa^2.$$

For the family of curves with Frenet characterization in the Eq. (1), we have $\eta_w(t) = g(w,t)$. Thus, we are allowed to write

$$\begin{aligned} \mathcal{W}(w,t) &= \frac{\partial \eta}{\partial w}, \\ \mathcal{V}(w,t) &= \frac{\partial \eta}{\partial t} = v(w,t)\,\mathbf{e}_{(0)}(w,t), \end{aligned}$$

where $v = \frac{ds}{dt}$ is speed, $\mathcal{V}$ is velocity, $\mathcal{W}$ is an infinitesimal variation of the curve. Also we need following formulas to calculate Euler equations

$$\begin{aligned} [\mathcal{W},\mathcal{V}] &= \frac{\mathcal{W}(v)}{v}\mathbf{e}_{(0)} = g\mathbf{e}_{(0)},\ g = -\left\langle D_{\mathbf{e}_{(0)}}\mathcal{W}, \mathbf{e}_{(0)} \right\rangle, \\ \mathcal{W}\left(\kappa^2\right) &= 2\left\langle D_{\mathbf{e}_{(0)}} D_{\mathbf{e}_{(0)}} \mathcal{W} \right\rangle + 4g\kappa^2 + 2\left\langle \mathcal{R}\left(\mathcal{W}, \mathbf{e}_{(0)}\right)\mathbf{e}_{(0)}, D_{\mathbf{e}_{(0)}}\mathbf{e}_{(0)} \right\rangle. \end{aligned}$$

Another significant formula that we should take into account is the Riemannian curvature $\mathcal{R}$, which is given as the following.

$$\mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{Z} = D_{\mathcal{X}} D_{\mathcal{Y}} \mathcal{Z} - D_{\mathcal{Y}} D_{\mathcal{X}} \mathcal{Z} - D_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z}.$$

where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are vector fields on any Riemannian manifold M.

Let $\eta : [0,1] \to \mathbf{R}$ be a curve of unit length. For any fixed constant Ω, let

$$\frac{d}{dw}\mathcal{F}^{\Omega}(w) = \frac{1}{2}\int_0^L k^2 + \Omega ds = \frac{1}{2}\int_0^1 (\left\| D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)} \right\|^2 + \Omega) v(t)\, dt.$$

For a variation η_w and variational field $\mathcal{W}$, we have

$$\begin{aligned} \frac{d}{dw}\mathcal{F}^{\Omega}(\eta_w) &= \frac{1}{2}\int_0^1 W\left(\kappa^2\right) v + \left(\kappa^2 + \Omega\right) W(v)\, dt \\ &= \frac{1}{2}\int_0^1 W\left(\kappa^2\right) - \left(\kappa^2 + \Omega\right) g ds. \end{aligned}$$

Based on this method it is obtained that elastic curves satisfy following statement.

$$\mathbf{E} = D^3_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)} - D_{\mathbf{e}_{(0)}}\left(\Upsilon \mathbf{e}_{(0)}\right) + \mathcal{R}(D_{\mathbf{e}_{(0)}}\mathbf{e}_{(0)}, \mathbf{e}_{(0)})\mathbf{e}_{(0)},$$

where $g = -\left\langle D_{\mathbf{e}_{(0)}}\mathcal{W}, \mathbf{e}_{(0)}\right\rangle$ and $\Upsilon = \frac{\Omega - 3\kappa^2}{2}$.

Let η be a unit speed curve lying on $\mathbf{R}$ that has the Frenet characterization in (1),then we obtain for the elastic curve in a Lie group $\mathbf{R}$ following statetment.

$$\begin{aligned}\mathbf{E} =&(\kappa'' - \kappa\left(\tau_R - \tau\right)^2 - \kappa^3 - \Upsilon\kappa - \kappa\tau_R^2)\mathbf{e}_{(1)} \\ &+(2\kappa'\left(\tau - \tau_R\right) + \kappa\left(\tau - \tau_R\right)' + \frac{\kappa}{2}\tau_R')\mathbf{e}_{(2)}.\end{aligned} \tag{5}$$

Theorem 3.5 *Energy of the unit speed elastic curve in Lie group* $\mathbf{R}$ *is stated by using Sasaki metric as follows.*

$$\begin{aligned}\varepsilon nergy\left(\mathbf{E}\right) = \frac{1}{2}s + \frac{1}{2}\int_0^s &((-\kappa\kappa'' + \kappa^2\left(\tau_R - \tau\right)^2 + \kappa^4 + \Upsilon\kappa^2 + \kappa^2\tau_R^2)^2 \\ &+(\kappa''' - 3\kappa'\left(\tau_R - \tau\right)^2 - 3\kappa\left(\tau_R - \tau\right)\left(\tau_R - \tau\right)' \\ &-\Upsilon\kappa' - \kappa'\tau_R^2 - 2\kappa\tau_R\tau_R' + \frac{\kappa}{2}\tau_R'\left(\tau_R - \tau\right))^2 \\ &+((\tau - \tau_R)\left(3\kappa'' - \kappa\tau_R^2 - \Upsilon\kappa - \kappa^3 - \kappa\left(\tau_R - \tau\right)^2\right) \\ &+3\kappa'\left(\tau - \tau_R\right)' + \kappa\left(\tau - \tau_R\right)'' + \frac{\kappa'}{2}\tau_R' + \frac{\kappa}{2}\tau_R'')^2)ds.\end{aligned}$$

Proof. From the Eq. (2) and Eq. (3) we know

$$\varepsilon nergy\left(\mathbf{E}\right) = \frac{1}{2}\int_0^s \rho_S\left(d\mathbf{E}(\mathbf{e}_{(0)}), d\mathbf{E}(\mathbf{e}_{(0)})\right)ds.$$

Using Eq. (4) we have

$$\begin{aligned}\rho_S\left(d\mathbf{E}(\mathbf{e}_{(0)}), d\mathbf{E}(\mathbf{e}_{(0)})\right) &= \rho(d\omega(E(\mathbf{e}_{(0)})), d\omega(\mathbf{E}(\mathbf{e}_{(0)}))) \\ &+\rho(Q(\mathbf{E}(\mathbf{e}_{(0)})), Q(\mathbf{E}(\mathbf{e}_{(0)}))).\end{aligned}$$

Since $\mathbf{E}$ is a section, we get

$$d(\omega) \circ d(\mathbf{E}) = d(\omega \circ \mathbf{E}) = d(id_C) = id_{TC}.$$

Then we have by using Eq. (5)

$$\begin{aligned}Q(\mathbf{E}(\mathbf{e}_{(0)})) = D_{\mathbf{e}_{(0)}}\mathbf{E} =&(-\kappa\kappa'' + \kappa^2\left(\tau_R - \tau\right)^2 + \kappa^4 + \Upsilon\kappa^2 + \kappa^2\tau_R^2)\mathbf{e}_{(0)} \\ &+ (\kappa''' - 3\kappa'\left(\tau_R - \tau\right)^2 - 3\kappa\left(\tau_R - \tau\right)\left(\tau_R - \tau\right)' - \Upsilon\kappa' - \kappa'\tau_R^2 \\ &- 2\kappa\tau_R\tau_R' + \frac{\kappa}{2}\tau_R'\left(\tau_R - \tau\right))\mathbf{e}_{(1)} \\ &+ ((\tau - \tau_R)\left(3\kappa'' - \kappa\tau_R^2 - \Upsilon\kappa - \kappa^3 - \kappa\left(\tau_R - \tau\right)^2\right) \\ &+ 3\kappa'\left(\tau - \tau_R\right)' + \kappa\left(\tau - \tau_R\right)'' + \frac{\kappa'}{2}\tau_R' + \frac{\kappa}{2}\tau_R'')\mathbf{e}_{(2)}.\end{aligned}$$

Thus, we find that

$$\begin{aligned}
\rho_S\left(d\mathbf{E}(\mathbf{e}_{(0)}), d\mathbf{E}(\mathbf{e}_{(0)})\right) &= \rho\left(\mathbf{e}_{(0)}, \mathbf{e}_{(0)}\right) + \rho(D_{\mathbf{e}_{(0)}}\mathbf{E}, D_{\mathbf{e}_{(0)}}\mathbf{E}) \\
&= 1 + (-\kappa\kappa'' + \kappa^2\left(\tau_R - \tau\right)^2 + \kappa^4 + \Upsilon\kappa^2 + \kappa^2\tau_R^2)^2 \\
&+(\kappa''' - 3\kappa'\left(\tau_R - \tau\right)^2 - 3\kappa\left(\tau_R - \tau\right)\left(\tau_R - \tau\right)' \\
&-\Upsilon\kappa' - \kappa'\tau_R^2 - 2\kappa\tau_R\tau_R' + \frac{\kappa}{2}\tau_R'\left(\tau_R - \tau\right))^2 \\
&+((\tau - \tau_R)\left(3\kappa'' - \kappa\tau_R^2 - \Upsilon\kappa - \kappa^3 - \kappa\left(\tau_R - \tau\right)^2\right) \\
&+3\kappa'\left(\tau - \tau_R\right)' + \kappa\left(\tau - \tau_R\right)'' + \frac{\kappa'}{2}\tau_R' + \frac{\kappa}{2}\tau_R'')^2.
\end{aligned}$$

So we can easily compute the energy of the elastic curve, if we plug obtained data.

Remark 3.6 *Let η be the unit speed elastic curve in a Lie group* $\mathbf{R}$. *Then we have following relation between energy of the elastic curve in Frenet vector fields and energy on the same curve in* $\mathbf{E}$ *is given as follows, respectively.*

$$\begin{aligned}
\varepsilon nergy\mathbf{e}_{(0)} - \varepsilon nergy\left(\mathbf{E}\right) &= \frac{1}{2}\int_0^s (\kappa^2 - (-\kappa\kappa'' + \kappa^2\left(\tau_R - \tau\right)^2 + \kappa^4 \\
&+\Upsilon\kappa^2 + \kappa^2\tau_R^2)^2 - (\kappa''' - 3\kappa'\left(\tau_R - \tau\right)^2 - 3\kappa\left(\tau_R - \tau\right)\left(\tau_R - \tau\right)' \\
&-\Upsilon\kappa' - \kappa'\tau_R^2 - 2\kappa\tau_R\tau_R' + \frac{\kappa}{2}\tau_R'\left(\tau_R - \tau\right))^2 \\
&-((\tau - \tau_R)\left(3\kappa'' - \kappa\tau_R^2 - \Upsilon\kappa - \kappa^3 - \kappa\left(\tau_R - \tau\right)^2\right) \\
&+3\kappa'\left(\tau - \tau_R\right)' + \kappa\left(\tau - \tau_R\right)'' + \frac{\kappa'}{2}\tau_R' + \frac{\kappa}{2}\tau_R'')^2)ds,
\end{aligned}$$

$$\begin{aligned}
\varepsilon nergy\mathbf{e}_{(1)} - \varepsilon nergy\left(\mathbf{E}\right) &= \frac{1}{2}\int_0^s (\kappa^2 + (\tau - \tau_R)^2 - (-\kappa\kappa'' + \kappa^2\left(\tau_R - \tau\right)^2 + \kappa^4 \\
&+\Upsilon\kappa^2 + \kappa^2\tau_R^2)^2 - (\kappa''' - 3\kappa'\left(\tau_R - \tau\right)^2 - 3\kappa\left(\tau_R - \tau\right)\left(\tau_R - \tau\right)' \\
&-\Upsilon\kappa' - \kappa'\tau_R^2 - 2\kappa\tau_R\tau_R' + \frac{\kappa}{2}\tau_R'\left(\tau_R - \tau\right))^2 \\
&-((\tau - \tau_R)\left(3\kappa'' - \kappa\tau_R^2 - \Upsilon\kappa - \kappa^3 - \kappa\left(\tau_R - \tau\right)^2\right) \\
&+3\kappa'\left(\tau - \tau_R\right)' + \kappa\left(\tau - \tau_R\right)'' + \frac{\kappa'}{2}\tau_R' + \frac{\kappa}{2}\tau_R'')^2)ds,
\end{aligned}$$

$$\varepsilon nergy\mathbf{e}_{(2)} - \varepsilon nergy\,(\mathbf{E}) = \frac{1}{2}\int_0^s ((\tau - \tau_R)^2 - (-\kappa\kappa'' + \kappa^2\,(\tau_R - \tau)^2 + \kappa^4$$
$$+\Upsilon\kappa^2 + \kappa^2\tau_R^2)^2 - (\kappa''' - 3\kappa'\,(\tau_R - \tau)^2 - 3\kappa\,(\tau_R - \tau)\,(\tau_R - \tau)'$$
$$-\Upsilon\kappa' - \kappa'\tau_R^2 - 2\kappa\tau_R\tau_R' + \frac{\kappa}{2}\tau_R'\,(\tau_R - \tau))^2$$
$$-((\tau - \tau_R)\,(3\kappa'' - \kappa\tau_R^2 - \Upsilon\kappa - \kappa^3 - \kappa\,(\tau_R - \tau)^2)$$
$$+3\kappa'\,(\tau - \tau_R)' + \kappa\,(\tau - \tau_R)'' + \frac{\kappa'}{2}\tau_R' + \frac{\kappa}{2}\tau_R'')^2)ds.$$

Proof. It is obvious from Theorem 3.4 and Theorem 3.5.

Corollary 3.7 *Let assume that η lies on the 3-dimensional Lie group $\mathbf{R}$ such that $\mathbf{R}$ is Abelian group. Then for the energy of the elastic curve, we have*

$$\varepsilon nergy\,(\mathbf{E}) = \frac{1}{2}s + \frac{1}{2}\int_0^s ((-\kappa\kappa'' + \kappa^2\tau^2 + \kappa^4 + \Upsilon\kappa^2 + \kappa^2)^2$$
$$+(\kappa''' - 3\kappa'\tau^2 - 3\kappa\tau\tau' - \Upsilon\kappa' - \kappa' - 2\kappa + \frac{\kappa}{2})^2$$
$$+(\tau(3\kappa'' - \kappa - \Upsilon\kappa - \kappa^3 - \kappa\tau^2) + 3\kappa'\tau' + \kappa\tau'' + \frac{\kappa'}{2} + \frac{\kappa}{2})^2)ds.$$

Corollary 3.8 *Let assume that η lies on the 3-dimensional Lie group $\mathbf{R}$ such that $\mathbf{R}$ is SU^2. Then for the energy of the elastic curve, we have*

$$\varepsilon nergy\,(\mathbf{E}) = \frac{1}{2}s + \frac{1}{2}\int_0^s ((-\kappa\kappa'' + \kappa^2\,(1-\tau)^2 + \kappa^4 + \Upsilon\kappa^2 + \kappa^2)^2$$
$$+(\kappa''' - 3\kappa'\,(1-\tau)^2 - 3\kappa\,(1-\tau)\,(1-\tau)' - \Upsilon\kappa' - \kappa')^2$$
$$+((\tau - 1)\,(3\kappa'' - \kappa - \Upsilon\kappa - \kappa^3 - \kappa\,(1-\tau)^2) + 3\kappa'\,(\tau-1)' + \kappa\,(\tau-1)'')^2)ds.$$

Corollary 3.9 *Let assume that η lies on the 3-dimensional Lie group $\mathbf{R}$ such that $\mathbf{R}$ is SO^3. Then for the energy of the elastic curve, we have*

$$\varepsilon nergy\,(\mathbf{E}) = \frac{1}{2}s + \frac{1}{2}\int_0^s ((-\kappa\kappa'' + \kappa^2(\frac{1}{2} - \tau)^2 + \kappa^4 + \Upsilon\kappa^2 + \frac{\kappa^2}{4})^2$$
$$+(\kappa''' - 3\kappa'(\frac{1}{2} - \tau)^2 - 3\kappa(\frac{1}{2} - \tau)(\frac{1}{2} - \tau)' - \Upsilon\kappa' - \frac{\kappa'}{4})^2$$
$$+((\tau - \frac{1}{2})(3\kappa'' - \frac{\kappa}{4} - \Upsilon\kappa - \kappa^3 - \kappa(\frac{1}{2} - \tau)^2) + 3\kappa'(\tau - \frac{1}{2})' + \kappa(\tau - \frac{1}{2})'')^2)ds.$$

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