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TRAVELING WAVE SOLUTIONS FOR A SHALLOW WATER MODEL

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Abstract. In this note, we seek traveling wave solutions of a shallow water model in a one dimensional space by a simple but rigorous calculation. From the profile equation of traveling wave solutions, we need to investigate the phase portrait of a one dimensional ordinary differential equation $\tilde{u}' = F(\tilde{u})$ connecting two end states of the traveling wave solution.

1. Introduction

We consider the following shallow water model in a one dimensional space:

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2)_x + \dfrac{g\cos\alpha}{2}(h^2)_x = -c_f u^2 + (g\sin\alpha)h, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$ is the direction of flow, t is time, h the height of the water layer and u its horizontal velocity. Moreover, $g > 0$ is the acceleration of gravity, $\alpha \in (0, \pi/2)$ is the constant slope angle, and $c_f > 0$ is the frictional coefficient. The first and second equations of (1.1) are the two conservation equations for h and u, in particular, they represent the conservation laws of mass and momentum, respectively. See [S, W] for the detailed derivation of a shallow water model.

The propagation of waves in shallow water has been studied in [PS, M] by using a linear perturbation method. Yong and Zumbrun investigated the existence of traveling wave solutions (relaxation shock profiles) for general strict hyperbolic conservation laws with relaxation under some

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structural assumption (see [YZ] and references cited therein). The existence and the time-asymptotic behavior of the outflow boundary layer solutions to (1.1) have been concerned in [KST].

Here, we investigate the existence of the traveling wave solution to (1.1). More precisely, we seek a traveling wave solution to (1.1) of the form

$$(h,u)(x,t) = (\tilde{h},\tilde{u})(x-st) \quad \text{with} \quad \lim_{\xi\to\pm\infty}(\tilde{h},\tilde{u})(\xi) = (h_\pm,u_\pm), \tag{1.2}$$

where $\xi = x - st$, $0 < h_+ < h_-$, and $0 < u_+ < u_-$. Here, $s \in \mathbb{R}$ is the wave speed of the profile $(\tilde{h},\tilde{u})$.

Applying the first equation (the conservation of mass) of (1.1) to the second equation (the conservation of momentum), (1.1) reduces to

$$\begin{cases} h_t + (hu)_x = 0, \\ u_t + (\frac{1}{2}u^2)_x + g\cos\alpha h_x = -c_f\dfrac{u^2}{h} + g\sin\alpha. \end{cases} \tag{1.3}$$

As a starting point, we first consider the profile equations of $(\tilde{h},\tilde{u})$ satisfying (1.3). By inserting $(h,u)(x,t) = (\tilde{h},\tilde{u})(x-st)$ into (1.3), we obtain the following first order ODEs (profile equations) :

$$-s\tilde{h}' + (\tilde{h}\tilde{u})' = 0, \tag{1.4}$$

$$-s\tilde{u}' + (\frac{1}{2}\tilde{u}^2)' + g\cos\alpha\tilde{h}' = -c_f\frac{\tilde{u}^2}{\tilde{h}} + g\sin\alpha, \tag{1.5}$$

where $'$ represents differentiation in $\xi \in (-\infty,\infty)$. The purpose of this paper is to show the existence of a trajectory $(\tilde{h},\tilde{u})$ connecting (h_-,u_-) on the left to (h_+,u_+) on the right.

1.1. Main result

We now state the main theorem in the present note.

Theorem 1.1. *For given constants* $c_f > 0$, $0 < \alpha < \dfrac{\pi}{2}$, *and* $g > 0$, *suppose that the end states* u_- *and* u_+ *satisfy* $0 < u_+ < u_-$ *and*

$$u_-^4 < c_f \cot\alpha (u_+ + u_-)^2 u_+^2. \tag{1.6}$$

Then there exists a traveling wave solution to (1.1) *of the form* (1.2) *with*

$$h_+ = \frac{c_f u_+^2}{g\sin\alpha}, \quad h_- = \frac{c_f u_-^2}{g\sin\alpha}, \quad \text{and} \quad s = u_- + \frac{u_+^2}{u_+ + u_-}. \tag{1.7}$$

Remark 1.2. *The strategy to prove Theorem 1.1 is as follows. We first solve the profile equation* (1.4) *for* $\tilde{h}$ *by integrating* (1.4) *from* $-\infty$ *to* x*; and then by plugging* $\tilde{h}$ *into* (1.5)*, the second profile equation* (1.5) *can be written as* $\tilde{u}' = F(\tilde{u})$*. Finally, we prove that there is a trajectory in the phase plane of* $\tilde{u}' = F(\tilde{u})$ *connecting the end states* u_+ *and* u_- *with* $0 < u_+ < u_-$*.*

Remark 1.3. *The relations between the end states* $h_\pm$ *and* $u_\pm$ *in* (1.7) *are trivially derived by solving*

$$-c_f \frac{u^2}{h} + g \sin\alpha = 0$$

in the second equation of (1.3) *because we consider* $(h_\pm, u_\pm)$ *as the constant solutions (equilibrium states) of* (1.3)*. Thus, by* (1.7)*, the condition* $0 < u_+ < u_-$ *implies that* $0 < h_+ < h_-$*.*

Remark 1.4. *Since the cotangent function is decreasing on* $(0, \pi/2)$*, Theorem 1.1 says that a larger slope angle* α *requires a smaller gap* $u_- - u_+$ *of two end states for the existence of traveling wave solutions of the form* (1.2)*. Moreover, the wave speed* s *must satisfy* $s = u_- + \frac{u_+^2}{u_+ + u_-} > u_- > u_+ > 0$.

2. Solving the profile equations

By integrating (1.4) from $-\infty$ to x, we obtain

$$\tilde{h} = \frac{u_- - s}{\tilde{u} - s} h_- \tag{2.1}$$

which leads to, by the first equation of (1.4),

$$\tilde{h}' = \frac{\tilde{u}'}{s - \tilde{u}} \tilde{h} = -\frac{h_-(u_- - s)\tilde{u}'}{(\tilde{u} - s)^2}. \tag{2.2}$$

By plugging (2.1) and (2.2) into (1.5) and by solving (1.5) for $\tilde{u}'$, we obtain a single ordinary differential equation of the form

$$\tilde{u}' = F(\tilde{u}) := \frac{-c_f \tilde{u}^2 (\tilde{u} - s)^3 + g \sin\alpha h_- (u_- - s)(\tilde{u} - s)^2}{h_-(u_- - s)(\tilde{u} - s)^3 - g \cos\alpha h_-^2 (u_- - s)^2}. \tag{2.3}$$

In order to solve the speed s, we again integrate (1.4) from $-\infty$ to ∞; so then

$$-s(h_+ - h_-) + (h_+ u_+ - h_- u_-) = 0. \tag{2.4}$$

We now prove that u_+ and u_- are fixed points of the ODE (2.3).

Proposition 2.1. *For given* $c_f > 0$, $0 < \alpha < \dfrac{\pi}{2}$, *and* $g > 0$, *assume* $u_\pm$ $(0 < u_+ < u_-)$ *and* $h_\pm$ *satisfy*

$$c_f u_+^2 - g\sin\alpha h_+ = 0 \quad \text{and} \quad c_f u_-^2 - g\sin\alpha h_- = 0, \tag{2.5}$$

$$(u_+ - s)^2 - g\cos\alpha h_+ \neq 0 \quad \text{and} \quad (u_- - s)^2 - g\cos\alpha h_- \neq 0. \tag{2.6}$$

Then u_+ *and* u_- *are fixed points of* (2.3), *that is,* $F(u_\pm) = 0$. *Moreover, there is no fixed point between* u_+ *and* u_-.

Proof. We first notice that, by (2.4) and (2.5),

$$s = u_- + h_+\frac{u_+ - u_-}{h_+ - h_-} = u_- + \frac{u_+^2}{u_+ + u_-} > u_- > u_+ > 0;$$

so then $u_\pm - s$ is nonzero. It is trivial that $F(u_-) = 0$ if $c_f u_-^2 - g\sin\alpha h_- = 0$ and $(u_- - s)^2 - g\cos\alpha h_- \neq 0$.

By (2.4), we have

$$h_-(u_- - s) = h_+(u_+ - s); \tag{2.7}$$

which leads to $F(\tilde{u}) = \dfrac{-c_f\tilde{u}^2(\tilde{u} - s)^3 + g\sin\alpha h_+(u_+ - s)(\tilde{u} - s)^2}{h_+(u_+ - s)(\tilde{u} - s)^3 - g\cos\alpha h_+^2(u_+ - s)^2}$. Thus, $F(u_+) = 0$ if $c_f u_+^2 - g\sin\alpha h_+ = 0$ and $(u_+ - s)^2 - g\cos\alpha h_+ \neq 0$.

Moreover, by directly solving $F(\tilde{u}) = 0$, there are three more fixed points of (2.3) :

$$\tilde{u} = s \quad \text{and} \quad \tilde{u} = \frac{1}{2}\left(s - u_- \pm \sqrt{(s + 3u_-)(s - u_-)}\,\right).$$

However, since $s = u_- + \dfrac{u_+^2}{u_+ + u_-}$, simple calculations give us

$$\begin{aligned}\frac{1}{2}\left(s - u_- - \sqrt{(s + 3u_-)(s - u_-)}\,\right) &< \frac{1}{2}\left(s - u_- + \sqrt{(s + 3u_-)(s - u_-)}\,\right)\\ &= u_+ < u_- < s\end{aligned}$$

which implies that there is no fixed point between u_+ and u_-. □

3. Proof of the main theorem

We now prove Theorem 1.1 by considering the phase plane of $\tilde{u}' = F(\tilde{u})$. Since we already proved in Proposition 2.1, u_+ and u_- are the only zeros of $F(\tilde{u})$ on a closed interval $[u_+, u_-]$, we need to prove that $F(\tilde{u}) < 0$ on (u_+, u_-) so that the flow is to the left for any initial data $u_0 \in (u_+, u_-)$.

Proof of Theorem 1.1 . In order to prove $F(\tilde{u}) < 0$ for all $\tilde{u} \in (u_+, u_-)$, it is enough to prove that $F'(u_+) < 0 < F'(u_-)$ in the phase space; so that u_+ is a stable fixed point and u_- is an unstable fixed point. By setting $F(\tilde{u}) = \dfrac{V(\tilde{u})}{W(\tilde{u})}$ and by the fact $V(u_\pm) = 0$,

$$\begin{aligned} F'(u_\pm) &= \frac{V'(u_\pm)W(u_\pm)}{W(u_\pm)^2} \\ &= \frac{(u_\pm - s)^4 c_f u_\pm h_\pm (2s - 3u_\pm)\{(u_\pm - s)^2 - g\cos\alpha h_\pm\}}{W(u_\pm)^2}. \end{aligned}$$

Since $0 < u_+ < u_-$ and $s = \dfrac{u_+^2 + u_+u_- + u_-^2}{u_+ + u_-}$,

$$2s - 3u_+ = \frac{(u_- - u_+)(2u_- + u_+)}{u_+ + u_-} > 0$$

and

$$2s - 3u_- = \frac{(u_+ - u_-)(2u_+ + u_-)}{u_+ + u_-} < 0;$$

so $F'(u_+) < 0 < F'(u_-)$ if and only if

$$(u_+ - s)^2 - g\cos\alpha h_+ < 0 \quad \text{and} \quad (u_- - s)^2 - g\cos\alpha h_- < 0. \tag{3.1}$$

However, since $(u_- - s)^2 < (u_+ - s)^2$ and $g\cos\alpha h_+ < g\cos\alpha h_-$, the condition

$$(u_+ - s)^2 < g\cos\alpha h_+ \tag{3.2}$$

is enough for $F'(u_+) < 0 < F'(u_-)$. By plugging $h_+ = \dfrac{c_f u_+^2}{g\sin\alpha}$ and $s = u_+ + \dfrac{u_-^2}{u_+ + u_-}$ into (3.2), we obtain the condition

$$u_-^4 < c_f \cot\alpha (u_+ + u_-)^2 u_+^2.$$

Moreover, this condition definitely replaces (2.6) in Proposition 2.1. We just proved that under the condition (1.6), there is a trajectory connecting u_+ and u_- such that the flow is to u_+ as $x \to \infty$ and to u_- as $x \to -\infty$ starting from an arbitrary initial data $u_0 \in (u_+, u_-)$. By inserting $\tilde{u}$ into (2.1), we also obtain $\tilde{h}$ connecting h_+ and h_- with $h_+ < \tilde{h} < h_-$. This completes the proof.

□

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