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A STUDY ON κ-AP, κ-WAP SPACES AND THEIR RELATED SPACES

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Abstract. In this paper we define AP_c and AP_{cc} spaces which are stronger than the property of approximation by points(AP). We investigate operations on their subspaces and study function theorems on AP_c and AP_{cc} spaces. Using those results, we prove that every continuous image of a countably compact Hausdorff space with AP is AP. Finally, we prove a theorem that every compact κ-WAP space is κ-pseudoradial, and prove a theorem that the product of a compact κ-radial space and a compact κ-WAP space is a κ-WAP space.

1. Introduction

Throughout this paper, all spaces are assumed as Hausdorff, the cardinality of any set X is denoted by $|X|$. $\aleph_0$ is the first countably infinite cardinal and $\aleph_1$ is the first uncountable cardinal. Undefined notions and terminologies can be found in [4].

Approximation by points (AP) and weak approximation by points (WAP) were introduced by A. Pultr and A. Tozzi in 1993 ([7]) and studied by A. Bella ([1]) and P. Simon ([8]) etc.

A space X is said to have the property of *Approximation by Points* ([7]), for short, *AP*, if for every non-closed subset A of X and for every point $x \in \overline{A}\backslash A$, there exists a subset F of A such that $\overline{F} = F \cup \{x\}$. A space X is said to have the property of *Weak Approximation by Points*, for short, *WAP*, if for every non-closed subset A of X, there exist a point $x \in \overline{A}\backslash A$ and a subset F of A such that $\overline{F} = F \cup \{x\}$. Such a

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set F is said to be *almost closed*. In a space with a unique non-isolated point, any non-closed set is almost closed. So any space with a unique non-isolated point is AP.

W. Hong ([5]) introduced many properties by generalizing AP and WAP. Specially, he defined approximation by countable points (ACP) and countably approximation by points, for short, countably AP.

Here, we state the well-known classical definitions concerned with AP and WAP. A space X is called *compact* (resp. *countably compact*) if every (resp. countable) open cover X has a finite subcover. A space X is called *Fréchet-Urysohn* if for any subset A of X and any $x \in \overline{A}$ there is a sequence S in A which converges to x. A space X is called *sequential* if for any non-closed subset A of X there is a sequence S in A which converges to some $x \in \overline{A} \backslash A$. We say that a space X has *countable tightness*, i.e., $t(X) \leq \aleph_0$, if whenever $A \subseteq X$ and $x \in \overline{A}$, there exists a countable subset B of A such that $x \in \overline{B}$.

A. Bella and J. Gerlits ([2]) studied radiality, pseudoradiality, and semiradiality, which are some kind of approximations by points defined in view of convergence of transfinite sequences. The definitions are as follows. A space X is said to be *radial* if for every non-closed subset A of X and for every point $x \in \overline{A} \setminus A$, there exists a transfinite sequence which converges to x. A space X is said to be *pseudoradial* if for every non-closed subset A of X, there exist a point $x \in \overline{A} \setminus A$ and a transfinite sequence which converges to x. We say that a subset A of a space X is *κ-closed* whenever $B \subseteq A$ and $|B| \leq \kappa$ imply $\overline{B} \subseteq A$. A space X is said to be *semiradial* if for every non-κ-closed subset A of X, there exists a transfinite sequence S in A which converges to a point outside A and satisfies $|S| \leq \kappa$.

The relations Fréchet-Urysohn $\to$ radial $\to$ semiradial $\to$ pseudoradial always hold. But in general the arrows cannot be reversed even for compact spaces. For more details on pseudoradial and related spaces, see [6]. AP and WAP spaces are closely related to radial and pseudoradial spaces. Likewise, it will be turned out that κ-AP and κ-WAP spaces are closely related to κ-radial and κ-pseudoradial spaces.

This paper is organized as follows: This section is to provide some notions and definitions which we will need afterwards. In section 2, we define AP_c and AP_{cc} spaces which are stronger than the property of approximation by points (AP). We investigate operations on their subspaces and we also study function theorems on AP_c and AP_{cc} spaces. Using those results, we prove that every continuous image of a countably compact Hausdorff space with AP is AP.

Section 3 is devoted to studying κ-radiality and κ-pseudoradiality which streng-then radiality and pseudoradiality. We modify a theorem by A. Bella ([1]) that every compact WAP space is pseudoradial to get a theorem [Theorem 3.4] that every compact κ-WAP space is κ-pseudoradial. We also modify a theorem by A. Bella ([1]) that the product of a compact semiradial space and a compact WAP space is a WAP space to get a theorem [Theorem 3.9] that the product of a compact κ-radial space and a compact κ-WAP space is a κ-WAP space.

2. AP$_c$ and AP$_{cc}$ Spaces

A space X is said to be κ-*AP* ([3]) if for every non-closed subset A of X and for every point $x \in \overline{A}\backslash A$, there exists a subset F of A such that $\overline{F} = F \cup \{x\}$ and $|F| \leq \kappa$. Such a set F is called κ-*almost closed*.

A space X is said to be κ-*WAP* ([3]) if for every non-closed subset A of X, there exist a point $x \in \overline{A}\backslash A$ and a subset F of A such that $\overline{F} = F \cup \{x\}$ and $|F| \leq \kappa$.

We say that a subset A of a space X is κ-*AP-closed* if for every $F \subseteq A$ with $|F| \leq \kappa$ the relation $|\overline{F}\backslash A| \neq 1$ holds. It is clear that X is a κ-WAP space if and only if every κ-AP-closed subset of X is closed. Recall that a subset A of a space X is κ-*closed* if whenever $B \subseteq A$ and $|B| \leq \kappa$ imply $\overline{B} \subseteq A$.

Definition 2.1. A space X is said to be AP_c provided that for any non-closed subset A of X and $x \in \overline{A} \setminus A$, there exists a subset F of A such that $\overline{F} = F \cup \{x\}$ and $\overline{F}$ is compact.

Definition 2.2. A space X is said to be AP_{cc} provided that for any non-closed subset A of X and $x \in \overline{A} \setminus A$, there exists a subset F of A such that $\overline{F} = F \cup \{x\}$ and $\overline{F}$ is countably compact.

Theorem 2.3. *The following statements hold:*

(1) *every subspace of an AP_c space is an AP_c space;*
(2) *every subspace of an AP_{cc} space is an AP_{cc} space;*
(3) *every Fréchet-Urysohn space is an AP_c space;*
(4) *every AP_c space is an AP_{cc} space; and*
(5) *every AP_{cc} space is an AP space.*

Proof. (1) Let Y be a subspace of an AP$_c$ space X and let A be a non-closed subset of Y. If $x \in \overline{A}^Y \setminus A$, then there exists a subset F of

A such that $\overline{F} = F \cup \{x\}$ and $\overline{F}$ is compact since X is AP$_c$. Then it follows from $x \in Y$ that $\overline{F}^Y = \overline{F}$. Therefore $\overline{F}^Y$ is compact.

(2) It can be proved similarly.

(3) Let A be a non-closed subset of a Fréchet-Urysohn space X and let $x \in \overline{A} \setminus A$. Since X is Fréchet-Urysohn, there exists a sequence S in A which converges to the point x. Then the subspace $S \cup \{x\}$ is compact. Therefore the space X is AP$_c$.

(4) and (5) They are immediate from the definitions. □

Theorem 2.4. *The following are equivalent for a Hausdorff space X:*

(1) *X is Fréchet-Urysohn;*
(2) *X is AP_c; and*
(3) *X is AP_{cc}.*

Proof. It is enough to show the implication (3) ⇒ (1) to complete the proof.

Without loss of generality, we may assume that $x \in \overline{A} \setminus A$. Since X is AP$_{cc}$, there exists a subset F of A such that $\overline{F} = F \cup \{x\}$ and $\overline{F}$ is countably compact. Since every subspace of an AP space is AP [10, Proposition 2.1 (1)], the subspace $\overline{F}$ is AP. Since every countably compact AP space is Fréchet-Urysohn [10, Theorem 2.2], $\overline{F}$ is Fréchet-Urysohn. So there exists a sequence S in F (hence in A) which converges to x. Therefore X is Fréchet-Urysohn. □

When X is a sequential Hausdorff space, the following statements are equivalent ([5], Corollary 2.4):

X is Fréchet-Urysohn ⇔ X is ACP ⇔ X is countably AP.

Therefore we have the following corollary.

Corollary 2.5. *Let X be a sequential Hausdorff space. Then the following statements are equivalent:*

(1) *X is Fréchet-Urysohn;*
(2) *X is AP_c;*
(3) *X is AP_{cc};*
(4) *X is ACP; and*
(5) *X is countably AP.*

V. V. Tkachuk and I. V. Yaschenko ([10]) have proved that every countably compact Hausdorff space with AP is Fréchet-Urysohn. So we have the following corollary.

Corollary 2.6. *Every countably compact Hausdorff space with AP is AP$_c$ and AP$_{cc}$.*

The following is an improvement of the result in [10] that every closed continuous image of an AP space is AP.

Theorem 2.7. *The following statements hold:*

(1) *every continuous image of an AP$_c$ space is AP$_c$; and*
(2) *every continuous image of an AP$_{cc}$ space is AP$_{cc}$.*

Proof. (1) Suppose that X is an AP$_c$ space and $f : X \to Y$ is a continuous onto map. Let $A \subseteq Y$ and $y \in \overline{A} \setminus A$. Take $B = f^{-1}(A)$. Since $f^{-1}(y) \cap \overline{B} \neq \emptyset$, B is a non-closed subset of X. Hence for some $x \in f^{-1}(y)$, there exists a subset C of B such that $\overline{C} = C \cup \{x\}$ and $\overline{C}$ is compact. Let $F = f(C)$. Then $\overline{F} = F \cup \{y\}$. Furthermore, $\overline{F}$ is compact because $\overline{F} = f(\overline{C})$ and f is continuous.

(2) can be proved similarly. □

Remark 2.8. Notice that there exists an AP space whose continuous image is not AP. The space $X = \omega_1 + 1$ with interval topology is not AP, and it is a continuous image of a discrete space D of cardinality ω_1 which is AP ([9]).

However we have the following.

Theorem 2.9. *Every continuous image of a countably compact Hausdorff space with AP is AP.*

Proof. Let X be a countably compact Hausdorff space with AP, and let $f : X \to Y$ be a continuous map from the space X onto a space Y. Then, by Corollary 2.6, X is AP$_c$. Theorem 2.7 implies that Y is AP$_c$, and hence it is AP. □

3. κ-Radial and κ-Pseudoradial spaces

A space X is called *κ-radial* ([3]) if for every non-closed subset A of X and for every point $p \in \overline{A} \setminus A$, there exists a transfinite sequence $\{x_\alpha \in A : \alpha < \kappa\}$ which converges to p.

A space X is called *κ-pseudoradial* ([3]) if for every non-closed subset A of X, there exist a point $p \in \overline{A} \setminus A$ and a transfinite sequence $\{x_\alpha \in A : \alpha < \kappa\}$ which converges to p.

By definitions, we have the following diagram.

$$
\begin{tikzcd}
\text{Fréchet-Urysohn} \arrow[r, leftrightarrow] \arrow[d] & \aleph_0\text{-radial} \arrow[r] \arrow[d] & \kappa\text{-radial} \arrow[r] \arrow[d] & \kappa\text{-AP} \arrow[d] \\
\text{sequential} \arrow[r, leftrightarrow] & \aleph_0\text{-pseudoradial} \arrow[r] & \kappa\text{-pseudoradial} \arrow[r] & \kappa\text{-WAP}
\end{tikzcd}
$$

Diagram 1. κ is an infinite cardinal

We introduce a definition which is an extension of sequential compactness: a space X is called *κ-sequentially compact* if every transfinite sequence has a convergent transfinite subsequence with length $\leq \kappa$.

Theorem 3.1 ([1]). *Every compact WAP space is pseudoradial.*

Theorem 3.2 ([1]). *Every compact AP space is radial.*

Theorem 3.3 ([3]). *Every countably compact κ-WAP space is κ-sequentially compact.*

We have two improvements of the above theorems as follows:

Theorem 3.4. *Every compact κ-WAP space is κ-pseudoradial.*

Proof. Let X be a compact κ-WAP space and let A be a non-closed subset of X. Since X is κ-WAP, there exist $B \subseteq A$ and $x \in X \backslash A$ such that $\overline{B} \backslash A = \{x\}$ and $|B| \leq \kappa$. Select a minimal family $\{U_\xi : \xi \in \kappa_0\}(\kappa_0 \leq \kappa)$ of open subsets in the subspace $\overline{B}$ satisfying $\bigcap_{\xi \in \kappa_0} \overline{U}_\xi = \{x\}$. By the minimality, $\bigcap_{\nu \in \xi} \overline{U}_\nu \backslash \{x\} \neq \emptyset$ for every $\xi \in \kappa_0$. Choose a point $x_\xi \in \bigcap_{\nu \in \xi} \overline{U}_\nu \backslash \{x\}$. Since $\overline{B}$ is compact, it is easy to check that the well-ordered net $\langle x_\xi : \xi \in \kappa_0 \rangle$ so obtained converges to x. By construction, the set $F = \{x_\xi : \xi \in \kappa_0\}$ is a subset of A. So the proof is complete. $\square$

Theorem 3.5. *Every compact κ-AP space is κ-radial.*

Proof. Let X be a compact κ-AP space. Let A be a non-closed subset of X and $x \in \overline{A} \backslash A$. Then there exists $B \subseteq A$ such that $\overline{B} \backslash A = \{x\}$ and $|B| \leq \kappa$. With a similar argument in Theorem 3.4, there exists a well-ordered net $\langle x_\xi : \xi \in \kappa_0 \rangle$, $\kappa_0 \leq \kappa$, of A which converges to x. Hence X is κ-radial. $\square$

Theorem 3.6 ([1]). *Every compact scattered AP space is Fréchet-Urysohn.*

Corollary 3.7. *For every infinite cardinal κ, every compact scattered κ-AP space is $\aleph_0$-radial.*

Proof. It follows from Theorem 3.6. $\square$

Theorem 3.8 ([1]). *The product of compact semiradial space and compact WAP space is a WAP space.*

Finally, we prove the following theorem.

Theorem 3.9. *The product of compact κ-radial space and compact κ-WAP space is a κ-WAP space.*

Proof. Let X be a compact κ-radial space and let Y be a compact κ-WAP space. By way of contradiction, assume that $X \times Y$ is not κ-WAP. Then there exists a non-closed subset A of $X \times Y$ such that for every $(x, y) \in \overline{A} \backslash A$ and for every subset F of A with $|F| \leq \kappa, \overline{F} \neq F \cup \{(x, y)\}$, i.e., there is a κ-AP-closed set $A \subseteq X \times Y$ which is not closed. Let κ be the minimal cardinal such that the set A is not κ-closed and choose a set $B \subseteq A$ satisfying $|B| = \kappa$ and $\overline{B} \backslash A \neq \emptyset$. Select a point $(x, y) \in \overline{B} \backslash A$. Since $\{x\} \times Y$ is κ-WAP and $A \cap (\{x\} \times Y)$ is κ-AP-closed (and hence it is closed in $\{x\} \times Y$), there exists a closed neighbourhood V of (x, y) in $X \times Y$ such that $V \cap [A \cap (\{x\} \times Y)] = \emptyset$(because of $(x, y) \notin A \cap (\{x\} \times Y)$). Let $\pi_X : X \times Y \to X$ be a projection map on X. Since $\pi_X^{-1}(x) = \{x\} \times Y$ and $(x, y) \in V$, $V \cap \pi_X^{-1}(x) \neq \emptyset$. Hence $A \cap (\{x\} \times Y) = \emptyset$, i.e., $A \cap \pi_X^{-1}(x) = \emptyset$. Thus $\pi_X^{-1}(x) \nsubseteq A$ and so $x \notin \pi_X(A)$. Since $(x, y) \in \overline{B} \backslash A$, it follows from the continuity of π_X that $x \in \overline{\pi_X(B)}$. Hence $x \in \overline{\pi_X(B)} \backslash \pi_X(A)$. Also since $|B| = \kappa, |\pi_X(B)| \leq |B| = \kappa$. Hence $\pi_X(A)$ is not κ-closed.

Now since X is κ-radial, there exists a transfinite sequence $\{x_\alpha : \alpha \in \kappa\}$ in $\pi_X(A)$ which converges to a point $\hat{x} \in X \backslash \pi_X(A)$. For each $\alpha \in \kappa$, choose $y_\alpha \in Y$ such that $(x_\alpha, y_\alpha) \in A$. Since Y is compact and the set $\{y_\alpha : \alpha \in \kappa\}$ is infinite, there exists a complete accumulation point p of $\{y_\alpha : \alpha \in \kappa\}$ in Y. Since the point $(\hat{x}, p) \notin A$, we get $p \notin \pi_Y(A)$ as before. For any $\alpha \in \kappa$, we denote $C_\alpha = \overline{\{y_\beta : \beta \in \alpha\}}$ and put $C = \bigcup_{\alpha \in \kappa} C_\alpha$. Since $\pi_Y(A)$ is κ-closed, $\{y_\beta : \beta \in \alpha\} \subseteq \pi_Y(A)$, and $|\{y_\beta : \beta \in \alpha\}| < \kappa$, we have $\overline{\{y_\beta : \beta \in \alpha\}} \subseteq \pi_Y(A)$, i.e., $C_\alpha \subseteq \pi_Y(A)$ for every $\alpha \in \kappa$. Hence $C = \bigcup_{\alpha \in \kappa} C_\alpha \subseteq \pi_Y(A)$. Moreover, since $p \in \overline{C} \backslash \pi_Y(A)$, $p \in \overline{C} \backslash C$ and thus C is not closed in Y. Hence by κ-WAP of Y, there exist a point $\hat{y} \in \overline{C} \backslash C$ and a subset D of C such that $\overline{D} = D \cup \{\hat{y}\}$ and $|D| \leq \kappa$. Clearly, we can write $\overline{D} = \{\hat{y}\} \cup (\bigcup_{\alpha \in \kappa} \overline{D} \cap C_\alpha)$. Since $\hat{y} \notin C$, $\hat{y} \notin C_\alpha$ for every $\alpha \in \kappa$. So for each $\alpha \in \kappa$, $\hat{y} \notin \overline{D} \cap C_\alpha$ and $\overline{D} \cap C_\alpha$ is closed in Y. Hence for each $\alpha \in \kappa$, choose a closed neighborhood U_α of $\hat{y}$ in the subspace $\overline{D}$ satisfying $U_\alpha \cap \overline{D} \cap C_\alpha = \emptyset$ and let $V_\alpha = \bigcap_{\beta \in \alpha} U_\beta$. Since the subspace $\overline{D}$ is compact and $\overline{D} \backslash \{y\}$ is κ-closed, it follows that every

$V_\alpha \backslash \{\hat{y}\}$ is not empty. (If $V_\alpha \backslash \{\hat{y}\} = \emptyset$ for some $\alpha \in \kappa$, then $V_\alpha = \{\hat{y}\}$ and so $\bigcap_{\beta \in \alpha} U_\beta = \{\hat{y}\}$. By κ-closedness of $\overline{D} \backslash \{y\}$, $\bigcap_{\beta \in \alpha} U_\beta \backslash \{\hat{y}\} \neq \emptyset$ for every $\beta \in \alpha$. Picking a point $y_\beta \in \bigcap_{\beta \in \alpha} U_\beta \backslash \{\hat{y}\}$ and taking into account the compactness of $\overline{D}$, it is easy to check that the transfinite sequence so obtained converges to $\hat{y}$. Hence $\hat{y} \in \overline{D} \subseteq C$. This is a contradiction.) Now picking a point $y'_\alpha \in V_\alpha \backslash \{\hat{y}\}$ for every $\alpha \in \kappa$, we obtain a transfinite sequence converging to $\hat{y}$. Next, fix a function $f : \kappa \to \kappa$ such that $y'_\alpha \in \overline{\{y_\beta : \alpha \in \beta \in f(\alpha)\}}$ for any $\alpha \in \kappa$. Using the fact that A is $< \kappa$-closed, we can select a point $x'_\alpha \in \overline{\{x_\beta : \alpha \in \beta \in f(\alpha)\}}$ in such way that $(x', y') \in A$ for any $\alpha \in \kappa$. To finish, observe that the transfinite sequence $F = \{(x'_\alpha, y'_\alpha) : \alpha \in \kappa\}$ must converges to the point $(\hat{x}, \hat{y}) \notin A$. Every point in the closure of F and distinct from $(\hat{x}, \hat{y})$ is actually in the closure of an initial segment of F and hence in A. Thus we have $\overline{F} \backslash A = (\hat{x}, \hat{y})$ and $|F| \leq \kappa$, in contradiction with the fact that A is κ-AP-closed. □

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