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ON THE SEQUENCE GENERATED BY A CERTAIN TYPE OF MATRICES

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Abstract. Several properties of the sequence generated from a kind of Danzer matrices were examined and proved using already known facts about the Chebyshev polynomials. Asymptotic behavior of our interest sequence also discussed.

1. Introduction

In this section we introduce a particular type of Danzer matrices and a sequence generated from one of them.

Suppose M be a square matrix with nonnegative integer entries. We also let $s(M)$ be the sum of all the entries of the matrix M. Let matrix $B(n)$ be a matrix of size $n \times n$ with a value 1 over the anti-diagonal entries or above, and with a value 0 elsewhere. This matrix is a kind of *Danzer matrices*.(See [3] and [6] for details.) For example,

$$B(5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $b(m,n) := s((B(n))^m)$. This sequence with double indices is given in A050446 of OEIS([6]). This matrix arises and appears in several different areas of mathematics and has some interesting properties.(See [1], [2] and [5].) Below we list the first few columns and rows of this

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sequence.

$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 1 & 3 & 6 & 10 & 15 & 21 & 28 & \cdots \\ 1 & 5 & 14 & 30 & 55 & 91 & 140 & \cdots \\ 1 & 8 & 31 & 85 & 190 & 371 & 658 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

There are four conjectures on this sequence. We can see A205497 of OEIS([6]) for those conjectures.

2. Main Results

This section introduces and shows the main results and related facts of this article.

Fact 2.1. [2] *The nth column has the rational function of the form*

$$F_n(x) = \frac{P_n(x)}{Q_n(x)},$$

where $P_n(x)$ and $Q_n(x)$ are polynomials without any nontrivial common factors.

Example 2.2. [2]

$$F_1(x) = \frac{1}{-x+1} = \frac{1}{1-x} = \frac{P_1(x)}{Q_1(x)},$$

$$F_2(x) = \frac{1}{-x + \cfrac{1}{x+1}} = \frac{1+x}{1-x-x^2} = \frac{P_2(x)}{Q_2(x)},$$

$$F_3(x) = \frac{1}{-x + \cfrac{1}{x + \cfrac{1}{-x+1}}} = \frac{1+x-x^2}{1-2x-x^2+x^3} = \frac{P_3(x)}{Q_3(x)},$$

$$F_4(x) = \frac{1}{-x + \cfrac{1}{x + \cfrac{1}{-x + \cfrac{1}{x+1}}}} = \frac{1+2x-x^2-x^3}{1-2x-3x^2+x^3+x^4} = \frac{P_4(x)}{Q_4(x)}$$

$$\vdots$$

The x and $-x$ at the bottom left of the continued fraction appear alternatively.

Fact 2.3. [7]

$$Q_n(x) = \det(I - xB(n))$$

Fact 2.3 comes from the following. Let M be a square matrix of size $m \times m$. We also denote $M_{ij}(n) := (M^n)_{ij}$, (i,j)-entry of matrix M^n. We consider a generating function $F_{ij}(M,t)$ given by the sequence $(M_{ij}(n))_{n\geq 0}$ as follows:

$$F_{ij}(M,t) := \sum_{n\geq 0} M_{ij}(n)t^n.$$

Then we have the following result ([7], Ch.4), so-called Transfer-Matrix Method:

Theorem 2.4.

$$F_{ij}(M,t) = \frac{(-1)^{i+j}\det(I - tM : j,i)}{\det(I - tM)},$$

where $(B : j,i)$ denotes the matrix obtained by removing the j-th row and the i-th column of the matrix B.

Theorem 2.5. *$Q_n(x)$ satisfies the following recurrence relation*

$$\begin{aligned} &Q_0(x) = 1, \quad Q_1(x) = 1 - x, \\ &Q_n(x) = -xQ_{n-1}(-x) + Q_{n-2}(x) \quad (n \geq 2). \end{aligned}$$

Proof. By convention, $Q_0(x) = 1$. Example 2.2 shows that $Q_1(x) = 1 - x$.

$$\begin{aligned} \frac{P_n(x)}{Q_n(x)} = F_n(x) &= \frac{1}{-x + F_{n-1}(x)} = \frac{1}{-x + \dfrac{P_{n-1}(-x)}{Q_{n-1}(-x)}} \\ &= \frac{Q_{n-1}(-x)}{P_{n-1}(-x) - xQ_{n-1}(-x)} \end{aligned}$$

$$\begin{cases} P_n(x) &= Q_{n-1}(-x), \\ Q_n(x) &= -xQ_{n-1}(-x) + P_{n-1}(-x) \end{cases}$$

From the last recursive system, we get the desired recurrence relation

$$Q_n(x) = -xQ_{n-1}(-x) + Q_{n-2}(x).$$

□

It is immediate to obtain the next result from the proof of Theorem 2.5.

Corollary 2.6. *For $n \geq 1$,*

$$P_n(x) = Q_{n-1}(-x).$$

Conjecture 2.7. *The mth row has the rational function of the form*

$$G_m(t) = \frac{H_m(t)}{(1-t)^m} \quad (m = 3, 4, 5, \dots),$$

where $H_m(t)$ are polynomials without any non-trivial common factors.

Example 2.8.

$$G_3(t) = \frac{H_3(t)}{(1-t)^3} = \frac{1}{(1-t)^3},$$
$$G_4(t) = \frac{H_4(t)}{(1-t)^4} = \frac{1+t}{(1-t)^4},$$
$$G_5(t) = \frac{H_5(t)}{(1-t)^5} = \frac{1+3t+t^2}{(1-t)^5},$$
$$G_6(t) = \frac{H_6(t)}{(1-t)^6} = \frac{1+7t+7t^2+t^3}{(1-t)^6}$$
$$\vdots$$

In particular, what we should note here is the function $H_m(t)$ for $n = 3, 4, 5, \dots$. Coefficients of $H_m(t)$ for a few m are listed on the A205497 in OEIS([6]). Below is a list of the first few of them.

1					
1	1				
1	3	1			
1	7	7	1		
1	14	31	14	1	
1	26	109	109	26	1

$$\vdots$$

Conjecture 2.9. *$H_m(t)$ is symmetric for all m. That is,*

$$H_m(t) = t^{m-3} H_m(1/t)$$

for $m = 3, 4, 5, \dots$.

Notice that row sums $H_m(t)$ of the previous table seem to be Euler's updown number. (See A000111 in [OEIS].) One could try to give combinatorial interpretation of $H_m(t)$. Conjecture 2.7 and Conjecture 2.9 were discussed in the article [4]. We mainly focus on the polynomial $Q_n(x)$. From now on we will clarify and reveal the several properties concerning $Q_n(x)$.

Let $U_n(x)$ be the Chebyshev polynomial of the second kind. They are generated by the recurrence relation with initial conditions as shown below ([8]):

$$\begin{aligned} &U_0(x) = 1, U_1(x) = 2x, \\ &U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad (n \geq 2). \end{aligned}$$

Fact 2.10. *The generating function of the polynomial sequence $U_n(x)$ is as follows:*

$$\sum_{n\geq 0} U_n(x)t^n = \frac{1}{1+t^2-2xt}.$$

Fact 2.11. $U_n(x) = 0$ *if and only if* $x = x_k = 2\cos\left(\frac{k\pi}{n+1}\right)$ *for* $k = 1, 2, 3, \ldots, n$. *In addition,*

$$U_n(x) = 2^n \prod_{k=1}^{n} (x - x_k)$$

also holds.

Let $W_n(x) := U_{2n}\left(\dfrac{x}{2}\right)$. Then, next results hold.

Lemma 2.12. $W_0(x) = 1, W_1(x) = x^2 - 1$, *and*

$$W_n(x) = (x^2 - 2)W_{n-1}(x) - W_{n-2}(x).$$

Its generating function is

$$\sum_{n=0}^{\infty} W_n(x)t^n = \frac{1+t^2}{(1+t^2)^2 - x^2t^2}.$$

Proof. $W_0(x) = U_0\left(\frac{x}{2}\right) = 1$. $W_1(x) = U_2\left(\frac{x}{2}\right) = 4\left(\frac{x}{2}\right)^2 - 1 = x^2 - 1$.

$$\begin{aligned} W_n(x) = U_{2n}\left(\frac{x}{2}\right) &= 2\left(\frac{x}{2}\right)U_{2n-1}\left(\frac{x}{2}\right) - U_{2n-2}\left(\frac{x}{2}\right) \\ &= x\left(xU_{2n-2}\left(\frac{x}{2}\right) - U_{2n-3}\left(\frac{x}{2}\right)\right) - U_{2n-2}\left(\frac{x}{2}\right) \\ &= (x^2-1)W_{n-1}(x) - xU_{2n-3}\left(\frac{x}{2}\right) \\ &= (x^2-1)W_{n-1}(x) - U_{2n-2}\left(\frac{x}{2}\right) - U_{2n-4}\left(\frac{x}{2}\right) \\ &= (x^2-2)W_{n-1}(x) - W_{n-2}(x). \end{aligned}$$

Fact 2.11 states that

$$\sum_{n\geq 0} U_n(x)t^n = \frac{1}{1+t^2-2xt}.$$

$$\sum_{n\geq 0} U_{2n}(x)t^{2n} = \frac{1}{2}\left(\frac{1}{1+t^2-2xt} + \frac{1}{1+(-t)^2+2xt}\right) = \frac{1+t^2}{(1+t^2)^2-4x^2t^2}.$$

Hence,

$$\sum_{n=0}^{\infty} W_n(x)t^n = \frac{1+t^2}{(1+t^2)^2-x^2t^2}$$

follows. □

Note that $W_n(x) = 0 \leftrightarrow x = x_k = 2\cos\left(\frac{k\pi}{2n+1}\right)$ $(1 \leq k \leq 2n)$. Let

$$\begin{aligned} L_n(x) &:= \prod_{k=1}^{n}(x - x_{2k-1}) = \prod_{k=1}^{n}\left(x - 2\cos\left(\frac{(2k-1)\pi}{2n+1}\right)\right), \\ L_n^*(x) &:= \frac{W_n(x)}{L_n(x)} = \prod_{k=1}^{n}\left(x - 2\cos\left(\frac{2k\pi}{2n+1}\right)\right). \end{aligned}$$

Fact 2.13. *(A108299 of [OEIS])*

$$\sum_{n\geq 0} L_n(x)t^n = \frac{1-t}{1+t^2-xt}.$$

In particular, $L_n(x)$ is the characteristic polynomial of the form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

(See A108299 of OEIS for the rest of details on $L_n(x)$.)

Lemma 2.14. *$L_n^*(x) = L_n(-x)(-1)^n$ for all n. Hence, its generating function is*

$$\sum_{n\geq 0} L_n^*(x)t^n = \frac{1+t}{1+t^2-xt}.$$

Proof.

$$\begin{aligned} \sum_{n\geq 0} L_n^*(x)t^n &= \sum_{n\geq 0} (-1)^n L_n(-x)t^n \\ &= \sum_{n\geq 0} L_n(-x)(-t)^n \\ &= \frac{1-(-t)}{1+(-t)^2-(-x)(-t)} \\ &= \frac{1+t}{1+t^2-xt}. \end{aligned}$$

□

Example 2.15.

$$\begin{aligned} W_4(x) &= 1 - 10x^2 + 15x^4 - 7x^6 + x^8. \\ L_4(x) &= 1 + 2x - 3x^2 - x^3 + x^{4‘}, \\ L_4^*(x) &= L_n(-x)(-1)^4 = 1 - 2x - 3x^2 + x^3 + x^4, \end{aligned}$$

Hence, $W_4(x) = L_4(x)L_4^*(x) = (1-3x^2+x^4)^2 - (2x-x^3)^2$.

$$\begin{aligned} W_5(x) &= -1 + 15x^2 - 35x^4 + 28x^6 - 9x^8 + x^{10}. \\ L_5(x) &= -1 + 3x + 3x^2 - 4x^3 - x^4 + x^5, \\ L_5^*(x) &= L_5(-x)(-1)^5 = 1 + 3x - 3x^2 - 4x^3 + x^4 + x^5. \end{aligned}$$

Hence, $W_5(x) = L_5(x)L_5^*(x) = (3x-4x^3+x^5)^2 - (-1+3x^2-x^4)^2$.

There might be certain relationship between those three generating functions $\frac{1+t^2}{(1+t^2)^2-x^2t^2}$, $\frac{1-t}{1+t^2-xt}$, and $\frac{1+t}{1+t^2-xt}$.

Theorem 2.16.

$$\sum_{n=0}^{\infty} Q_n(x)t^n = \frac{(1+t)(1-t^2-xt)}{(1-t^2)^2+x^2t^2}.$$

Proof. Let $H(x,t) = \sum_{n\geq 0} Q_n(x)t^n$. From the recurrence relation in Theorem 2.5,

$$H(x,t) - Q_0(x) - Q_1(x)t = -xt(H(-x,t) - Q_0(-x)) + t^2 H(x,t)$$

$$(1-t^2)H(x,t) + xtH(-x,t) = 1+t \tag{1}$$

$$(1-t^2)H(x,-t) - xtH(-x,-t) = 1-t \tag{2}$$

We can rewrite the formula (2) as follows:

$$(1-t^2)H(-x,t) - xtH(x,t) = 1+t \tag{3}$$

From formulae (1) and (3), we obtain the desired generating function on $Q_n(x)$. □

Theorem 2.17. *$Q_n(x)$ and $L_n(x)$ have the following relationship:*

$$Q_n(x) = (-1)^{\lfloor \frac{n+1}{2} \rfloor} L_n((-1)^{n+1}x)$$

for $n = 0, 1, 2, \ldots$. In other words,

$$Q_n(x) = \begin{cases} L_n(-x), & n \equiv 0 \pmod 4 \\ -L_n(x), & n \equiv 1 \pmod 4 \\ -L_n(-x), & n \equiv 2 \pmod 4 \\ L_n(x), & n \equiv 3 \pmod 4). \end{cases}$$

Proof. Let $U(x,t) := \dfrac{1-t}{1+t^2-xt}$ and $V(x,t) := \dfrac{(1+t)(1-t^2-xt)}{(1-t^2)^2+x^2t^2}$.

$$\begin{aligned}
&\sum_{n\geq 0} L_{4n}(-x)t^{4n} - \sum_{n\geq 0} L_{4n+1}(x)t^{4n+1} - \sum_{n\geq 0} L_{4n+2}(-x)t^{4n+2} + \sum_{n\geq 0} L_{4n+3}(x)t^{4n+3} \\
&= \sum_{n\geq 0} L_{2n}(-x)(it)^{2n} + i\sum_{n\geq 0} L_{2n+1}(x)(it)^{2n+1} \\
&= \frac{1}{2}\left(U(-x,it)+U(-x,-it)\right) + \frac{i}{2}(U(x,it)-U(x,-it)) \\
&= \frac{1-t^2-xt^2}{(1-t^2)^2+x^2t^2} + \frac{t-xt-t^3}{(1-t^2)^2+x^2t^2} \\
&= V(x,t)
\end{aligned}$$

□

The roots of $Q_n(x)$ are obtained immediately by Theorem 2.17.

Corollary 2.18. *The roots of $Q_n(x)$ are as follows:*

$$x_k = (-1)^{n+1} 2\cos\left(\frac{2k-1}{2n+1}\pi\right), \qquad k = 1, 2, \ldots, n.$$

Theorem 2.19.

$$\min\left\{|x_k| : x_k = 2\cos\left(\frac{2k-1}{2n+1}\pi\right), k = 1,2,\ldots,n\right\} = x_{\lfloor\frac{n+2}{2}\rfloor} = 2\sin\left(\frac{\pi}{2(2n+1)}\right).$$

Proof. The root with the smallest absolute value is

$$\begin{aligned}
x &= (-1)^{n+1}2\cos\left(\frac{n+\frac{1+(-1)^n}{2}}{2n+1}\pi\right) = (-1)^{n+1}2\cos\left(\frac{2n+1+(-1)^n}{2(2n+1)}\pi\right) \\
&= (-1)^{n+1}2\cos\left(\frac{\pi}{2}+\frac{(-1)^n\pi}{2(2n+1)}\right) = (-1)^n 2\sin\left(\frac{(-1)^n\pi}{2(2n+1)}\right) = 2\sin\left(\frac{\pi}{2(2n+1)}\right).
\end{aligned}$$

□

Roots of the polynomial $Q_n(x)$ are the reciprocals of the eigenvalues of the matrix $B(n)$. Asymptotic ratio of the sequence $\{b(m,n)\}_{n=0}^{\infty}$ is dominated by the largest eigenvalue of the matrix $B(n)$. Let

$$\alpha(n) := \frac{1}{2\sin\left(\dfrac{\pi}{2(2n+1)}\right)}.$$

Then, this fact gives us the following conclusion.

Theorem 2.20. $b(m,n) = O((\alpha(n))^m)$ *as* $m \to \infty$. *Equivalently,*

$$\lim_{m\to\infty} \frac{\ln(b(m,n))}{m} = \ln(\alpha(n)).$$

Proof. Since $b(m,n) = \sum_{k=1}^{n} c_k \left(\frac{1}{2}\csc\left(\frac{(2k-1)\pi}{2n+1}\right)\right)^m$ for some constants $c_1, \ldots, c_n$,

$$\lim_{m\to\infty} \frac{b(m+1,n)}{b(m,n)} = \frac{1}{2\sin\left(\frac{\pi}{2(2n+1)}\right)} = \alpha(n),$$

the conclusion follows. □

Example 2.21.

$$F_5(x) = \sum_{m=0}^{\infty} b(m,5)x^m = \frac{P_5(x)}{Q_5(x)} = \cfrac{1}{-x + \cfrac{1}{x + \cfrac{1}{-x + \cfrac{1}{x + \cfrac{1}{-x+1}}}}}.$$

$$\alpha(5) = \frac{1}{2}\csc\left(\frac{\pi}{2(2\cdot 5+1)}\right) = 3.513347091\cdots$$

$$b(m,5) : 1, 5, 15, 55, 190, 671, 2353, 8272, 29056, 102091, 358671, \cdots$$

$$\frac{b(m+1,5)}{b(m,5)} : \frac{5}{1} = 5, \frac{15}{5} = 3, \frac{55}{15} \cong 3.666667, \frac{190}{55} \cong 3.454545, \frac{671}{190} \cong 3.531579,$$

$$\frac{2353}{671} \cong 3.506706, \frac{8272}{2353} \cong 3.515512, \cdots, \frac{358671}{102091} \cong 3.513248, \ldots.$$

3. Concluding Remarks and Further Questions

First, as already mentioned earlier, it might be necessary to clarify the relationship between the three functions $W_n(x), L_n(x)$ and $L_n^*(x)$. Next, if we change second-order linear recurrence relation on $Q_n(x)$ to the linear system, then, through matrix analysis, maybe we can get more information and understand better about $Q_n(x)$ and $b(m,n)$. Finally, it would be very valuable attempt to provide a combinatorial interpretation of $H_n(x)$.

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