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SYMMETRY ABOUT CIRCLES AND CONSTANT MEAN CURVATURE SURFACE

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Abstract. We show that a closed curve invariant under inversions with respect to two intersecting circles intersecting at angle of an irrational multiple of 2π is a circle. This generalizes the well known fact that a closed curve symmetric about two lines intersecting at angle of an irrational multiple of 2π is a circle. We use the result to give a different proof of that a compact embedded cmc surface in $\mathbb{R}^3$ is a sphere. Finally we show that a closed embedded cmc surface which is invariant under the spherical reflections about two spheres, which intersect at an angle that is an irrational multiple of 2π, is a sphere.

1. Introduction

Let C be a closed curve in $\mathbb{R}^2$. If, for each vector $v \in \mathbb{S}^1$, there is a line l_v with direction vector v about which C is symmetric, then C is a circle. More precisely, a closed curve symmetric about two lines, which intersect at an angle of irrational multiple of 2π, is a circle. In [5], McCuan generalized this result. McCuan defined a new notion of *symmetry* for a compact set in the upper half plane. Let $S_\rho(x)$ be a circle of radius ρ with center x.

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Definition 1. A compact set K in the upper half plane is symmetric if for each x on the x-axis there is some $\rho = \rho(x) > 0$ such that K is invariant under inversion about $S_\rho(x)$.

McCuan showed that a *symmetric* set K in the upper half plane is a circle. We weaken McCuan's condition and show that a closed curve $C \subset \mathbb{R}^2$ which is invariant under inversions about two circles that intersects at an angle of an irrational multiple of 2π is a circle.

For surfaces in $\mathbb{R}^3$, Alexandrov developed the moving plane argument to show that a compact embedded cmc surface S in $\mathbb{R}^3$ is a round sphere [1], [3]. Alexandrov first showed that, for each $n \in \mathbb{S}^2$, there is a symmetry plane Π_n of S with normal vector n. Then for two intersecting symmetry planes Π_{n_1} and Π_{n_2} of S which intersects at an angle of irrational multiple of 2π, S is invariant under rotation about the line $\Pi_{n_1} \cap \Pi_{n_2}$. Since $n_1, n_2 \in \mathbb{S}^2$ can be chosen arbitrarily, S is invariant under rotation about a line ℓ_v for each direction vector $v \in \mathbb{S}^2$. It follows that S is a round sphere.

McCuan used spheres and spherical reflections instead of the planes and reflections about planes to prove Alexandrov's result [4], [5]. We show that a compact embedded cmc surface in $\mathbb{R}^3$ which is invariant under the spherical reflections about two spheres which intersect at an angle of an irrational multiple of 2π is a round sphere.

2. Inversion and stereographic projection

Let C be a circle in $\mathbb{R}^2$ centered at the origin with radius r. The inversion $I_C : \mathbb{R}^2 \setminus \{O\} \to \mathbb{R}^2 \setminus \{O\}$ about C is defined by

$$I_C(p) = \frac{r^2}{|p|^2} p. \tag{1}$$

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin and $N = (0, 0, 1)$ be the north pole. Let $\pi : \mathbb{S}^2 \to \mathbb{R}^2$ be the stereographic projection from N onto the xy-plane Π. Then, for $(X, Y) = \pi(x, y, z)$,

$$\begin{aligned} (X, Y) &= \left(\frac{x}{1-z}, \frac{y}{1-z} \right), \\ (x, y, z) &= \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2} \right). \end{aligned} \tag{2}$$

Let V be a set invariant under I_C in Π. Then the scaled set $\lambda V = \{\lambda x : x \in V\}$, $\lambda > 0$, is invariant under the inversion $I_{\lambda C}$. Suppose that $r = 1$. Then

$$\pi^{-1}\left(I_C(p)\right) = \pi^{-1}\left(\frac{p}{|p|^2}\right) = R_z\left(\pi^{-1}(p)\right),$$

where R_z is the reflection about Π. Hence $\pi^{-1}(V)$ is invariant under the reflection about Π in $\mathbb{R}^3$.

Lemma 1. *Let Γ be a closed curve in Π invariant under two inversions I_{Γ_1} and I_{Γ_2} about two circles Γ_1 and Γ_2, where the angle between Γ_1 and Γ_2 is an irrational multiple of 2π. Then Γ is a circle.*

Proof. Since $\lambda\Gamma$ is invariant under $I_{\lambda\Gamma_i}$, $i = 1, 2$, we may assume that the radius of Γ_1 is 1. Hence $\pi^{-1}(\Gamma)$ is symmetric about Π in $\mathbb{R}^3$ as above. Let

$$Rot_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $(Rot_1 \circ \pi^{-1})(\Gamma_1)$ is the great circle of $\mathbb{S}^2$ on the xz-plane and $(Rot_1 \circ \pi^{-1})(\Gamma)$ is invariant under the reflection about the xz-plane in $\mathbb{R}^3$. We note that $(\pi \circ Rot_1 \circ \pi^{-1})(\Gamma_1)$ is the x-axis, and the inversion about Γ_1 corresponds to the reflection about the x-axis in Π after $\pi \circ Rot_1 \circ \pi^{-1}$. It is clear that $(\pi \circ Rot_1 \circ \pi^{-1})(\Gamma)$ is invariant under the reflection about the x-axis in Π.

Now we use a translation T and a dilation D of Π so that the center of $(T \circ \pi \circ Rot_1 \circ \pi^{-1})(\Gamma_2)$ is symmetric about the y-axis, and $(D \circ T \circ \pi \circ Rot_1 \circ \pi^{-1})(\Gamma_2)$ intersects the x-axis at $(1, 0)$ and $(-1, 0)$. For simplicity, we call the x-axis as $\tilde{\Gamma}_1$ and $(D \circ T \circ \pi \circ Rot_1 \circ \pi^{-1})(\Gamma_2)$ as $\tilde{\Gamma}_2$ and $(D \circ T \circ \pi \circ Rot_1 \circ \pi^{-1})(\Gamma)$ as $\tilde{\Gamma}$. We note that the inversion about Γ_2 corresponds to the inversion about $\tilde{\Gamma}_2$ after $D \circ T \circ \pi \circ Rot_1 \circ \pi^{-1}$.

We see that $\pi^{-1}(\tilde{\Gamma}_1)$ and $\pi^{-1}(\tilde{\Gamma}_2)$ are great circles in $\mathbb{S}^2$ with $\pi^{-1}(\tilde{\Gamma}_1) \cap \pi^{-1}(\tilde{\Gamma}_2)$={ (0,1,0),(0,-1,0)}. Let

$$Rot_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $(\pi \circ Rot_2 \circ \pi^{-1})(\tilde{\Gamma}_1)$ and $(\pi \circ Rot_2 \circ \pi^{-1})(\tilde{\Gamma}_2)$ are straight lines through the origin in Π, and the inversions about Γ_1 and Γ_2 corresponds to the reflections R_1 and R_2 about $(\pi \circ Rot_2 \circ \pi^{-1})(\tilde{\Gamma}_1)$ and $(\pi \circ Rot_2 \circ \pi^{-1})(\tilde{\Gamma}_2)$

respectively. Since all the mappings used above are conformal, the angle θ between $(\pi \circ Rot_2 \circ \pi^{-1})(\tilde{\Gamma}_1)$ and $(\pi \circ Rot_2 \circ \pi^{-1})(\tilde{\Gamma}_2)$ is an irrational multiple of 2π. Then $R_2 \circ R_1$ is the rotation of angle 2θ. For $p \in \Gamma$, the point $(\pi \circ Rot_2 \circ \pi^{-1})(\tilde{p})$ is mapped to a dense subset of a circle by $R_2 \circ R_1$. Since $(\pi \circ Rot_2 \circ \pi^{-1})(\tilde{\Gamma})$ is also a closed curve, it follows that $(\pi \circ Rot_2 \circ \pi^{-1})(\tilde{\Gamma})$ is a circle. Since the stereographic projection, inverse stereographic projection, rotations of the plane, translation, dilation sends a circle or a line to a circle or a line, Γ is a circle. $\square$

3. Spherical reflection and embedded closed cmc surface in $\mathbb{R}^3$

Let Σ be a closed embedded surface in $\mathbb{R}^3$ and let W be the (bounded) region bounded by Σ. We suppose that $W \subset \mathbb{R}^3 \setminus \{O\}$. Let S_ρ be the sphere centered at the origin with radius ρ in $\mathbb{R}^3$. The spherical reflection SR_ρ of $\mathbb{R}^3 \setminus \{O\}$ about S_ρ is given by

$$X \mapsto \frac{\rho^2}{|X|^2} X. \tag{3}$$

Let $\Sigma_\rho^- = \{X \in \Sigma : |X| \geq \rho\}$ and $\Sigma_\rho^+ = \{X \in \Sigma : |X| \leq \rho\}$. Since Σ is closed, $\Sigma_\rho^- = \emptyset$ for large ρ. As ρ decreases, there is ρ_0 for which Σ_ρ^- is nonempty for the first time. We denote by $\hat{\Sigma}_\rho^-$ the reflection of Σ_ρ^- for $\rho \leq \rho_0$. Decreasing ρ, we find $\rho_1 > 0$ for which $\hat{\Sigma}_\rho^-$ and Σ_ρ^+ are tangent at the image of some $X \in \Sigma_\rho^-$ for the first time, that is $T_{\hat{X}}\hat{\Sigma}_\rho^- = T_{X'}\Sigma_\rho^+$ with $X' \in \Sigma_\rho^+$ corresponding to $\hat{X}$. We call X the *first touch point*.

Let N be the unit normal vector field on Σ pointing into W. The mean curvature H of Σ is computed with respect to N. From now on, we suppose that Σ is a closed embedded cmc surface in $\mathbb{R}^3$. We recall the following results from [5].

LEMMA 2. *Let X be a closed embedded cmc surface in $\mathbb{R}^3$.*

(I) The mean curvature $\hat{H}(X,\rho)$ of $\hat{\Sigma}$ at the image of X under the map (3) *is given by*

$$\hat{H}(X,\rho) = \frac{1}{\rho^2}\left(|X|^2 H + 2X \cdot N\right).$$

(II) For $\rho \geq \rho_1$, $\hat{H}(X,\rho)$ is subharmonic. Therefore $\hat{H}(X,\rho)$ attains maximum at $\partial\hat{\Sigma}_\rho^-$. Moreover, for $\rho \geq \rho_1$, we have $\hat{H}(X,\rho) \leq H$.

Let X be the first touch point of Σ. Then $\hat{\Sigma}^-_\rho$ lies in the region bounded by Σ^+_ρ and S_{ρ_1}. Since $\hat{H}(X,\rho) \le H$ by Lemma 2 and $T_{\hat{X}}\hat{\Sigma}^-_\rho = T_{X'}\Sigma^+_\rho$, one can use the comparison principles for quasilinear elliptic partial differential equations of second order [2] to see that $\hat{\Sigma}^-_{\rho_1}$ and $\Sigma^+_{\rho_1}$ are congruent.

We can repeat the above argument for spheres centered at an arbitrary point of $\mathbb{R}^3 \setminus \bar{W}$. In fact, for each fixed point $P \in \mathbb{R}^3 \setminus \bar{W}$, we can find a radius $\rho_1(P)$ such that Σ is invariant under the spherical reflection $SR_{\rho_1}(P)$ about $S_{\rho_1(P)}(P)$:

$$X \mapsto \frac{\rho_1^2(P)}{|X-P|^2}(X-P).$$

Note that $\rho_1(P)$ is a continuous function of P.

Let ℓ be the line through the origin and P. We suppose that ℓ does not intersect Σ. Let Π_P be a plane that contains ℓ.

Lemma 3. *For $P \in \mathbb{R}^3$, $\Pi_P \cap \Sigma$ is either empty, or a single point or a circle.*

Proof. Suppose that $\Pi_P \cap \Sigma$ contains a point Q different from $\Pi_P \cap (S_{\rho_1} \cap S_{\rho_1(P)}(P))$. If the angle between S_{ρ_1} and $S_{\rho_1(P)}(P)$ is a rational multiple of 2π, then we use a point P' on ℓ close to P for which the angle between S_{ρ_1} and $S_{\rho_1(P')}(P')$ is an irrational multiple of 2π. Hence we suppose that the angle between S_{ρ_1} and $S_{\rho_1(P)}(P)$ is an irrational multiple of 2π. Then the angle between $\Pi_P \cap S_{\rho_1}$ and $\Pi_P \cap S_{\rho_1(P)}(P)$ is an irrational multiple of 2π. Arguing as in the proof of Lemma 1, the inversions $SR_{\rho_1}|_{\Pi_P}$ and $SR_{\rho_1}(P)|_{\Pi_P}$ sends Q into a dense subset of a circle. Hence $\Pi_P \cap \Sigma$ is a circle.

If $\Pi_P \cap \Sigma = \Pi_P \cap (S_{\rho_1} \cap S_{\rho_1(P)}(P))$, then $\Pi_P \cap \Sigma$ is fixed by $SR_{\rho_1}|_{\Pi_P}$ and $SR_{\rho_1}(P)|_{\Pi_P}$. If $\Pi_P \cap \Sigma$ contains more than one point, then one point is different from $(\Pi_P \cap S_{\rho_1}) \cap (\Pi_P \cap S_{\rho_1(P)}(P))$. Therefore $\Pi_P \cap \Sigma$ is a circle. □

It follows that Σ is foliated by circles. In [6], the author showed that a cmc surface in $\mathbb{R}^3$, which is foliated by circles, is either a sphere, or a surface of rotation with constant mean curvature, that is, the Delaunay surface. We give a different proof of the following theorem using the foliations by circles.

Theorem 1. *A closed embedded cmc surface* Σ *in* $\mathbb{R}^3$ *is a round sphere.*

Proof. We may assume that Σ is in the upper half space $\mathbb{R}^3_+$. As observed above, each line ℓ in $\mathbb{R}^3 \setminus \Sigma$ gives a foliation $\mathcal{F}_\ell$ of Σ by circles. Let ℓ be the x-axis and ℓ' be the line through $(0,1,0)$ and parallel to ℓ. Let C_ℓ and $C_{\ell'}$ be the circles of biggest radius in $\mathcal{F}_\ell$ anf $\mathcal{F}_{\ell'}$ respectively. Since $\Sigma \subset \mathbb{R}^3_+$, C_ℓ and $C_{\ell'}$ are different. Moreover $C_\ell \cap C_{\ell'}$ is the end point of the diameter of C_ℓ and $C_{\ell'}$.

Let $\Pi^\perp$ be the plane through the center of C_ℓ and perpendicular to ℓ. For a circle C' of $\mathcal{F}_{\ell'}$ intersecting C_ℓ, $C' \cap C_\ell$ is symmetric about $\Pi^\perp$. Hence C' is also symmetric about $\Pi^\perp$. It is easy to see that the distance between the center of C_ℓ and points on C' is the radius of C_ℓ. Hence part of Σ is spherical. Since Σ has constant mean curvautre, Σ is a sphere by the comparison principle of the quasilinear elliptic partial differential equation of second order. □

Theorem 2. *Let* Σ *be a closed embedded cmc surface* $\mathbb{R}^3$. *If* Σ *is invariant under two spherical reflections* $SR(P_1)$ *and* $SR(P_2)$ *about spheres* $S_{\rho_1}(P_1)$ *and* $S_{\rho_2}(P_2)$. *If the angle between* $S_{\rho_1}(P_1)$ *and* $S_{\rho_2}(P_2)$ *is an irrational multiple of* 2π, *then* Σ *is a sphere.*

Proof. We may suppose that P_1 and P_2 is on the x-axis. We denote by Π_ϕ the plane containing x-axis with angle to the xy-plane ϕ. Since the angle between $S_{\rho_1}(P_1)$ and $S_{\rho_2}(P_2)$ is an irrational multiple of 2π, $\Pi_\phi \cap \Sigma$ is either empty, or a single point, or a circle. Hence Σ is foliated by circles. The conclusion follows from Theorem 1. in [6] □

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