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(CO)RETRACTABILITY AND (CO)SEMI-POTENCY

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Abstract. This paper is a continuation of study semi-potentness endomorphism rings of module. We give some other characterizations of endomorphism ring to be semi-potent. New results are obtained including necessary and sufficient conditions for the endomorphism ring of semi(injective) projective module to be semi-potent. Finally, we characterize a module M whose endomorphism ring it is semi-potent via direct(injective) projective modules. Several properties of the endomorphism ring of a semi(injective) projective module are obtained. Besides to that, many necessary and sufficient conditions are obtained for semi-projective, semi-injective modules to be semi-potent and co-semi-potent modules.

1. Introduction.

Throughout in this paper R will be an associative ring with identity and all modules are unitary right $R-$modules. For a ring R, we write $J(R)$ for the *Jacobson radical* of R, and for a module M we denote $J(M)$ for the Jacobson radical of M. By notations, $N \leq_e M$, $N \ll M$ we mean that N is a *large* (*essential*) submodule and a *small* submodule of M, respectively. We denote $S = End_R(M)$ the endomorphism ring for an $R-$module M.

The concept I_0-*rings* or *semi-potent rings*, was first introduced by Nicholson [6] in 1975, and has been extensively studied by Tuganbaev,

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Kasch, Hamza, and others (see for example [5] and [8]). For example, Hamza in [4] shows that every projective module P over a semi-potent ring is semi-potent, i.e. any submodule of P not contained in $J(P)$ contains a nonzero direct summand of P. In the study of the concept semi-potency, one of the interesting questions is when the endomorphism ring of some module is semi-potent. Toward this question, many results have been obtained. In section 2, we study the semi-potentness of the endomorphism ring of a module, several necessary and sufficient conditions for the endomorphism ring of a module to be semi-potent are given. In section 3, we studied semi-potentness endomorphism ring of semi-(injective) projective modules. It is proved that endomorphism ring of semi-projective module M is semi-potent if and only if $Im(\alpha)$ contains a nonzero direct summand of M for every $\alpha \in S \setminus J(S)$. Also, it is proved that endomorphism ring of semi-injective module M is semi-potent if and only if $Ker(\alpha)$ is contained in a direct summand $N \neq M$ of M for every $\alpha \in S \setminus J(S)$. In section 4, we characterize the module M for which endomorphism ring of M is semi-potent in cases $J(S) = 0$, $J(S) = \nabla S$ and $J(S) = \Delta S$. It is proved that the endomorphism ring of a module M is semi-potent and $J(S) = 0$ if and only if M is direct-projective and for every $0 \neq \alpha \in S$, $Im(\alpha)$ contains a nonzero direct summand of M if and only if M is direct-injective and for every $0 \neq \alpha \in S$, $Ker(\alpha)$ is contained in a direct summand $N \neq M$ of M. Also, it is proved that the endomorphism ring of a module M is semi-potent and $J(S) = \nabla S$ if and only if M is direct-projective and for every $\alpha \in S$ which $Im(\alpha)$ is not small in M, contains a nonzero direct summand of M. Finally, it is proved that the endomorphism ring of a module M is semi-potent and $J(S) = \Delta S$ if and only if M is direct-injective and for every $\alpha \in S$ which $Ker(\alpha)$ is not large in M, is contained in a direct summand $N \neq M$ of M. In section 5, we study the semi-projective retractable and the semi-injective co-retractable modules. We find that the concept of retractability preserve semi-potency and co-semi-potency between the semi-projective modules and the endomorphism ring of this modules. While the concept of co-retractability dissent between semi-potency and co-semi-potency for semi-injective modules and the endomorphism ring of this modules.

2. Semi-potent rings.

Recall that a ring R is a *semi-potent* ring, also called I_0-ring by Nicholson [6] and Hamza [4], if every principal left (resp. right) ideal not contained in $J(R)$ contain a nonzero idempotent. For any non-empty subset X of a ring R, we denote the left annihilator of X in R by $\ell(X)$. Similarly the right annihilator of X in R is denoted by $r(X)$. Next we present a characterization of semi-potent rings:

PROPOSITION 2.1. *For any ring R the following statements are equivalent:*
(1) *R is semi-potent.*
(2) *For every $a \in R \setminus J(R)$, $b = bab$ for some $0 \neq b \in R$.*
(3) *For every $a \in R \setminus J(R)$, $\ell(1 - ab) = Re$ for some $0 \neq b \in R$ and idempotent $0 \neq e \in R$.*
(4) *For every $a \in R \setminus J(R)$, $\ell(1 - ba) = Rg$ for some $0 \neq b \in R$ and idempotent $0 \neq g \in R$.*
(5) *For every $a \in R \setminus J(R)$ there exists a nonzero idempotent $e \in R$ such that $e \in \ell(1 - ab)$ for some $0 \neq b \in R$.*
(6) *For every $a \in R \setminus J(R)$ there exists a nonzero idempotent $e \in R$ such that $e \in \ell(1 - ba)$ for some $0 \neq b \in R$.*
$(6 + i)$ *The left-right symmetry of $(2 + i)$, $i = 1, 2, 3, 4$.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R \setminus J(R)$, then there exists $0 \neq e^2 = e \in R$ such that $e \in aR$. So $e = az$ for some $z \in R$. For $b = zaz$, $b = bab$ and $0 \neq b \in R$.
(2) $\Rightarrow$ (3). Let $a \in R \setminus J(R)$, then $b = bab$ for some $0 \neq b \in R$. For $e = ab$, $\ell(1 - ab) = \ell(1 - e) = Re$ and so $0 \neq e \in R$ is an idempotent.
(3) $\Rightarrow$ (5). It is clear.
(5) $\Rightarrow$ (1). Let $a \in R \setminus J(R)$, then there exists $0 \neq b \in R$ and idempotent $0 \neq e \in R$ such that $e \in \ell(1 - ab)$, so $e = eab$ and $be = (be)a(be)$. For $g = abe$, $g \in aR$ is an idempotent. Similarly, we can prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (6) $\Rightarrow$ (1). □

THEOREM 2.2. *Let M_R be a module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *S is a semi-potent ring.*
(2) *For every $\alpha \in S \setminus J(S)$ there exists $\beta \in S$ such that $Im(\alpha\beta) \neq 0$ and $Ker(\alpha\beta) \neq M$ are direct summands of M.*
(2′) *For every $\alpha \in S \setminus J(S)$ there exists $\gamma \in S$ such that $Im(\gamma\alpha) \neq 0$*

and $Ker(\gamma\alpha) \neq M$ are direct summands of M.
(3) For every $\alpha \in S \setminus J(S)$ there exists $\beta \in S$ such that $Im(1-\alpha\beta) \neq M$ is a direct summand of M.
(3′) For every $\alpha \in S \setminus J(S)$ there exists $\gamma \in S$ such that $Im(1-\gamma\alpha) \neq M$ is a direct summand of M.
(4) For every $\alpha \in S \setminus J(S)$ there exists $\beta \in S$ such that $Ker(1-\alpha\beta)$ is a nonzero direct summand of M.
(4′) For every $\alpha \in S \setminus J(S)$ there exists $\gamma \in S$ such that $Ker(1-\gamma\alpha)$ is a nonzero direct summand of M.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (2′). By [4, Theorem 2.2].
(1) $\Rightarrow$ (3). Let $\alpha \in S \setminus J(S)$. Then by proposition 2.1 there exists $0 \neq \beta \in S$ such that $\beta = \beta\alpha\beta$. For $e = \alpha\beta$, $0 \neq e \in S$ is an idempotent and so $Im(1-\alpha\beta) = Im(1-e) \neq M$ is a direct summand of M.
(3) $\Rightarrow$ (1). Let $\alpha \in S \setminus J(S)$, then by assumption there exists $\beta \in S$ such that $Im(1-\alpha\beta) \neq M$ is a direct summand of M. Let $e : M \to Im(1-\alpha\beta)$ be the projection, then $1 \neq e \in S$ is an idempotent. Since for every $x \in M$, $x = \alpha\beta(x) + (1-\alpha\beta)(x)$ implies $e(x) = (1-\alpha\beta)(x)$ and so $e = 1-\alpha\beta$. Therefore $1-e = \alpha\beta$ and so $1-e \in S$ is a nonzero idempotent.
(1) $\Rightarrow$ (4). Let $\alpha \in S \setminus J(S)$. Then by proposition 2.1 there exists $0 \neq \beta \in S$ such that $\beta = \beta\alpha\beta$. For $e = \alpha\beta$, $0 \neq e \in S$ is an idempotent and so $Ker(1-\alpha\beta) = Ker(1-e) \neq 0$ is a direct summand of M.
(4) $\Rightarrow$ (1). Let $\alpha \in S \setminus J(S)$, then by assumption there exists $\beta \in S$ such that $Ker(1-\alpha\beta) \neq 0$ is a direct summand of M. Let $e : M \to Ker(1-\alpha\beta)$ be the projection. Then $e \in$ S is a nonzero idempotent and $Im(e) = Ker(1-\alpha\beta)$. So $(1-\alpha\beta)e = 0$ which implies $e = \alpha\beta \in \alpha S$, thus S is semi-potent. □

THEOREM 2.3. *Let M_R be a module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) S is a semi-potent ring.
(2) For every $\alpha \in S \setminus J(S)$ there exists $\beta \in S$ such that $Im(1-\alpha\beta)$ contained in a direct summand $N \neq M$ of M.
(2′) For every $\alpha \in S \setminus J(S)$ there exists $\gamma \in S$ such that $Im(1-\gamma\alpha)$ contained in a direct summand $N \neq M$ of M.
(3) For every $\alpha \in S \setminus J(S)$ there exists $\beta \in S$ such that $Ker(1-\alpha\beta)$ contains a nonzero direct summand of M.
(3′) For every $\alpha \in S \setminus J(S)$ there exists $\gamma \in S$ such that $Ker(1-\gamma\alpha)$ contains a nonzero direct summand of M.

Proof. (1) $\Rightarrow$ (2). Is similar to the prove of (1) $\Rightarrow$ (3) of the Theorem 2.2. (2) $\Rightarrow$ (1). Let $\alpha \in S \setminus J(S)$. By assumption there exists a direct summand $N \neq M$ of M such that $Im(1-\alpha\beta) \subseteq N$. Let $\pi : M \to N$ the projection, then for every $m \in M$, $\pi(1-\alpha\beta)(m) = (1-\alpha\beta)(m)$, therefore $\pi(1-\alpha\beta) = 1-\alpha\beta$ and so $(1-\pi)\alpha\beta = 1-\pi$, $1-\pi \neq 0$ which implies that $(1-\pi)\alpha\beta(1-\pi) = 1-\pi$ and so $\beta(1-\pi)\alpha\beta(1-\pi) = \beta(1-\pi)$. Let $\mu = \beta(1-\pi)$, then $\mu \in S$, $\mu\alpha\mu = \mu$, moreover $\mu \neq 0$, if $\mu = 0$, $1-\pi = (1-\pi)\alpha\beta(1-\pi) = (1-\pi)\alpha\mu = 0$ a contradiction. Thus S is semi-potent. Similarly we can prove the equivalent (1) $\Leftrightarrow$ (2$'$).
(1) $\Rightarrow$ (3). Let $\alpha \in S \setminus J(S)$. By proposition 2.1 $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in S$. For $e = \alpha\beta$, $e \in S$ is a nonzero idempotent and so $Ker(1-\alpha\beta) = Ker(1-e) \neq 0$ is a direct summand of M.
(3) $\Rightarrow$ (1). Let $\alpha \in S \setminus J(S)$. By assumption there exists a direct summand $K \neq 0$ of M such that $K \subseteq Ker(1-\alpha\beta)$ for some $\beta \in S$. Let $\pi : M \to K$ be the projection, then $\pi \neq 0$ and $Im(\pi) = K \subseteq Ker(1-\alpha\beta)$, therefore $(1-\alpha\beta)\pi = 0$ and so $\pi = \alpha\beta\pi$, $\beta\pi = (\beta\pi)\alpha(\beta\pi)$. Let $\mu = \beta\pi$, then $\mu \in S$ and that $\mu = \mu\alpha\mu$, $\mu \neq 0$ hence if $\mu = 0$, $\pi = \alpha\beta\pi = \alpha\mu = 0$ a contradiction. Thus S is semi-potent. Similarly we can prove the equivalent (1) $\Leftrightarrow$ (3$'$). □

Let M_R be a module and $S = End_R(M)$. The *co-singular ideal* of S is $\nabla S = \{\alpha : \alpha \in S; Im(\alpha) \ll M\}$ and the *singular ideal* of S is $\triangle S = \{\alpha : \alpha \in S; Ker(\alpha) \leq_e M\}$. Toward this ideals we define:

$$\widehat{\nabla} S = \{\alpha : \alpha \in S; Im(1-\alpha\beta) = M \text{ for all } \beta \in M\}$$

$$\widehat{\Delta} S = \{\alpha : \alpha \in S; Ker(1-\alpha\beta) = 0 \text{ for all } \beta \in M\}$$

Since for each $\alpha, \beta \in S$, $Im(1-\alpha\beta) = M$ if and only if $Im(1-\beta\alpha) = M$ and also, $Ker(1-\alpha\beta) = 0$ if and only if $Ker(1-\beta\alpha) = 0$,

$$\widehat{\nabla} S = \{\alpha : \alpha \in S; Im(1-\beta\alpha) = M \text{ for all } \beta \in M\}$$

$$\widehat{\Delta} S = \{\alpha : \alpha \in S; Ker(1-\beta\alpha) = 0 \text{ for all } \beta \in M\}$$

there is relation ship between the substructures ∇S, $\widehat{\nabla} S$, ΔS, $\widehat{\Delta} S$, $J(S)$ of S we derive in the following:

LEMMA 2.4. *Let M_R be a module and $S = End_R(M)$. Then:*
(1) $\nabla S \subseteq \widehat{\nabla} S$ *and* $\Delta S \subseteq \widehat{\Delta} S$.
(2) $J(S) \subseteq \widehat{\nabla} S$ *and* $J(S) \subseteq \widehat{\Delta} S$.

Proof. (1). Let $\alpha \in \nabla S$. Since for each $\beta \in S$, $M = Im(\alpha) + Im(1 - \alpha\beta) = Im(1 - \alpha\beta)$, so $\alpha \in \widehat{\nabla} S$. Let $\alpha \in \Delta S$. Since for each $\beta \in S$, $Ker(\alpha) \cap Ker(1 - \beta\alpha) = 0$, $Ker(1 - \beta\alpha) = 0$, so $\alpha \in \widehat{\Delta} S$. (2) it is clear. □

LEMMA 2.5. [9, Lemma 3.1] *Let M_R be a module and $\alpha \in S = End_R(M)$. Then the following are equivalent:*
(1) *There exists $\beta \in S$ such that $\alpha = \alpha\beta\alpha$*
(2) *$Im(\alpha)$ and $Ker(\alpha)$ are direct summand of M.*

3. Semi-projective (injective) modules.

Recall that a module M_R is *semi-projective* [10], if for every submodule N of M and every epimorphism $\alpha : M \to N$, homomorphism $\lambda : M \to N$ there exists $\beta \in End_R(M)$ such that $\alpha\beta = \lambda$.

LEMMA 3.1. [7, Theorem 2.7]. *Let M_R be a module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The module M is semi-projective.*
(2) *For every $\alpha \in S$, $\alpha S = Hom_R(M, Im(\alpha))$.*
(3) *If for $\alpha, \beta \in S$, $Im(\alpha) \subseteq Im(\beta)$, then $\alpha S \subseteq \beta S$.*

LEMMA 3.2. *Let M_R be a semi-projective module and $S = End_R(M)$. Then $\nabla S \subseteq J(S) = \widehat{\nabla} S$.*

Proof. By Lemma 2.4 we have $J(S) \subseteq \widehat{\nabla} S$. Let $\alpha \in \widehat{\nabla} S$, then for every $\beta \in S$ $Im(1-\alpha\beta) = M$. Since M is semi-projective $(1-\alpha\beta)\lambda = 1_M$ for some $\lambda \in S$, so $\alpha \in J(S)$. □

PROPOSITION 3.3. *Let M_R be a semi-projective module and $S = End_R(M)$. Then the following are equivalent:*
(1) *The ring S is semi-potent.*
(2) *For every $\alpha \in S \setminus J(S)$, $Im(\gamma\alpha)$ is a nonzero direct summand of M for some $\gamma \in S$.*
(3) *For every $\alpha \in S \setminus J(S)$, $Im(\alpha\beta)$ is a nonzero direct summand of M for some $\beta \in S$.*
(4) *For every $\alpha \in S \setminus J(S)$, $Im(\alpha)$ contains a nonzero direct summand of M.*

Proof. (1) $\Rightarrow$ (2). By Theorem 2.2. (2) $\Rightarrow$ (3). Let $\alpha \in S \setminus J(S)$, then by assumption $Im(\gamma\alpha)$ is a nonzero direct summand of M, so $Im(\gamma\alpha) =$

$Im(e)$ for some nonzero idempotent $e \in S$. Then by Lemma 3.1, $\gamma\alpha S = eS$, hence is semi-projective. So $\gamma\alpha\lambda = e$ for some $\lambda \in S$ and so $e = e\gamma\alpha\lambda e$ therefor $\lambda e\gamma = (\lambda e\gamma)\alpha(\lambda e\gamma)$. For $\beta = \lambda e\gamma$ we found that $\beta = \beta\alpha\beta$. Thus $\alpha\beta \in S$ is a nonzero idempotent and so $Im(\alpha\beta)$ is a nonzero direct summand of M. (3) $\Rightarrow$ (4). It is obvious. (4) $\Rightarrow$ (1). Let $\alpha \in S \setminus J(S)$ and N be a nonzero direct summand of M, $N \subseteq Im(\alpha)$. Suppose that $e : M \to N$ the projection, then $e \in S$ is a nonzero idempotent and $Im(e) = N \subseteq Im(\alpha)$ by Lemma 3.1 $e \in eS \subseteq \alpha S$, so S is semi-potent. □

THEOREM 3.4. *Let M_R be a semi-projective module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The ring S is semi-potent and $J(S) = \nabla S$.*
(2) *For every $\alpha \in S$ which $Im(\alpha)$ is not small in M, $Im(\alpha)$ contains a nonzero direct summand of M.*

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in S$ with $Im(\alpha)$ is not small in M. Then $\alpha \notin \nabla S = J(S)$, by assumption $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in S$. Let $e = \alpha\beta$, then $e \in S$ is a nonzero idempotent and $Im(e) = Im(\alpha\beta) \subseteq Im(\alpha)$, where $Im(e) \neq 0$ is a direct summand of M.
(2) $\Rightarrow$ (1). First we will prove that $J(S) = \nabla S$. By Lemma 3.2 we have $\nabla S \subseteq J(S)$. Let $\alpha \in J(S)$. If $\alpha \notin \nabla S$, $Im(\alpha)$ not small in M, by assumption there exists a nonzero direct summand N of M such that $N \subseteq Im(\alpha)$. Let $e : M \to N$ be the projection. Then $e \in S$ is a nonzero idempotent and $Im(e) \subseteq Im(\alpha)$, by Lemma 3.1 $e \in eS \subseteq \alpha S \subseteq J(S)$, so $e = 0$ a contradiction, thus $\alpha \in \nabla S$ and so $J(S) = \nabla S$. Let $\alpha \in S \setminus J(S)$. Then there exists a nonzero direct summand N of M, $N \subseteq Im(\alpha)$. Since M is semi-projective $e \in \alpha S$ where $e : M \to N$ the projection and so $0 \neq e \in S$ is an idempotent, so S is semi-potent. □

From Theorem 3.4 we conclude the following:

COROLLARY 3.5. *Let M_R be a semi-projective module and $S = End_R(M)$. Then the following are equivalent:*
(1) *The ring S is semi-potent and $J(S) = 0$.*
(2) *For every nonzero $\alpha \in S$, $Im(\alpha)$ contains a nonzero direct summand of M.*

Recall that a module M_R is *semi-injective* [7] if for every factor module N of M and every monomorphism $\alpha : N \to M$, homomorphism $\lambda : N \to M$ there exists $\beta \in End_R(M)$ such that $\beta\alpha = \lambda$.

Lemma 3.6. [10, p.260]. *Let M_R be a module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The module M is semi-injective.*
(2) *For every $\alpha \in S$, $S\alpha = Hom_R(\frac{M}{Ker(\alpha)}, M)$.*
(3) *If for $\alpha, \beta \in S$, $Ker(\alpha) \subseteq Ker(\beta)$, then $S\beta \subseteq S\alpha$.*

Lemma 3.7. *Let M_R be a semi-injective module and $S = End_R(M)$. Then $\Delta S \subseteq J(S) = \widehat{\Delta}S$.*

Proof. By Lemma 2.4 we have $J(S) \subseteq \widehat{\Delta}S$. Let $\alpha \in \widehat{\Delta}S$, then for every $\beta \in S$ $Ker(1-\beta\alpha) = 0$ that is $1_M - \beta\alpha$ is a monomorphism. Since M is semi-injective $\lambda(1-\beta\alpha) = 1_M$ for some $\lambda \in S$, so $\alpha \in J(S)$. □

Proposition 3.8. *Let M_R be a semi-injective module and $S = End_R(M)$. Then the following are equivalent:*
(1) *The ring S is semi-potent.*
(2) *For every $\alpha \in S \setminus J(S)$, $Ker(\alpha\beta) \neq M$ is a direct summand of M for some $\beta \in S$.*
(3) *For every $\alpha \in S \setminus J(S)$, $Ker(\gamma\alpha) \neq M$ is a direct summand of M for some $\gamma \in S$.*
(4) *For every $\alpha \in S \setminus J(S)$, $Ker(\alpha)$ is contained in a direct summand of $N \neq M$ of M.*

Proof. (1) ⇒ (2). By Theorem 2.2. (2) ⇒ (3). Let $\alpha \in S \setminus J(S)$. Then by assumption $Ker(\alpha\beta) \neq M$ is a direct summand of M for some $\beta \in S$. So $Ker(\alpha\beta) = Im(e)$ for some idempotent $1 \neq e \in S$. By Lemma 3.6, $S\alpha\beta = Se$, hence M is semi-injective, so $e = \lambda\alpha\beta$ for some $\lambda \in S$ and so $e = e\lambda\alpha\beta e$, therefore $\beta e\lambda = (\beta e\lambda)\alpha(\beta e\lambda)$. For $\gamma = \beta e\lambda \in S$ we found that $\gamma = \gamma\alpha\gamma$ and $1 \neq \gamma\alpha \in S$ is an idempotent, so $Ker(\gamma\alpha) \neq M$ is a direct summand of M. (3) ⇒ (4). It is obvious, hence $Ker(\alpha) \subseteq Ker(\gamma\alpha)$. (4) ⇒ (1). Let $\alpha \in S \setminus J(S)$ and $N \neq M$ be a direct summand of M, $Ker(\alpha) \subseteq N$. Suppose that $e : M \to N$ the projection, then $1 \neq e \in S$ is an idempotent and $Ker(\alpha) \subseteq N = Im(e) = Ker(1-e)$ by Lemma 3.6, $1-e \in S(1-e) \subseteq S\alpha$ and $1-e \in S$ is a nonzero idempotent, so S is semi-potent. □

Theorem 3.9. *Let M_R be a semi-injective module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The ring S is semi-potent and $J(S) = \Delta S$.*
(2) *For every $\alpha \in S$ which $Ker(\alpha)$ is not large in M, $Ker(\alpha)$ contained in a direct summand of $N \neq M$ of M.*

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in S$ with $Ker(\alpha)$ is not large in M. Then $\alpha \notin \Delta S = J(S)$, by assumption $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in S$. Let $e = \beta\alpha$, then $e \in S$ is a nonzero idempotent and $Ker(\alpha) \subseteq Ker(e) = Im(1-e)$. Since $1 - e \neq 1$ is an idempotent, $Im(1-e) \neq M$ is a direct summand of M.
(2) $\Rightarrow$ (1). First we will prove that $J(S) = \Delta S$. By Lemma 3.7 we have $\Delta S \subseteq J(S)$. Let $\alpha \in J(S)$. If $\alpha \notin \Delta S$, $Ker(\alpha)$ is not large in M, by assumption there exists a direct summand $N \neq M$ of M such that $Ker(\alpha) \subseteq N$. Let $e : M \to N$ be the projection. Then $1 \neq e \in S$ is an idempotent and $Ker(\alpha) \subseteq N = Im(e) = Ker(1-e)$ by Lemma 3.6, $1 - e \in S\alpha \subseteq J(S)$, so $1 - e = 0$ a contradiction, thus $\alpha \in \Delta S$ and so $J(S) = \Delta S$. Let $\alpha \in S \setminus J(S)$. Then $Ker(\alpha)$ is not large in M, so there exists a direct summand $N \neq M$ of M, $Ker(\alpha) \subseteq N = Ker(1-g)$ where $g : M \to N$ the projection. Since M is semi-injective $1 - g \in \alpha S$ and $0 \neq 1 - g \in S$ is an idempotent, so S is semi-potent. □

From Theorem 3.9 we conclude the following:

COROLLARY 3.10. *Let M_R be a semi-injective module and $S = End_R(M)$. Then the following are equivalent:*
(1) *The ring S is semi-potent and $J(S) = 0$.*
(2) *For every nonzero $\alpha \in S$, $Ker(\alpha)$ contained in a direct summand $N \neq M$ of M.*

4. Direct-projective (injective) modules.

Recall that a module M_R is *direct-projective* [10] if for every direct summand N of M and every epimorphism $\alpha : M \to N$ there exists $\beta \in End_R(M)$ such that $\alpha\beta = \pi$, where $\pi : M \to N$ the projection.
Following [10], A module M_R is direct-projective if and only if for every direct summand N of M and every epimorphism $\alpha : M \to N$, $Ker(\alpha)$ is a direct summand of M.

LEMMA 4.1. *Let M_R be a direct-projective module and $S = End_R(M)$. Then $\nabla S \subseteq J(S) = \widehat{\nabla} S$.*

Proof. By Lemma 2.4 we have $J(S) \subseteq \widehat{\nabla} S$. Let $\alpha \in \widehat{\nabla} S$, then for every $\beta \in S$ $Im(1-\alpha\beta) = M$. Since M is direct-projective, $(1-\alpha\beta)\lambda = 1_M$ for some $\lambda \in S$, so $\alpha \in J(S)$. □

Theorem 4.2. *Let M_R be a module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The ring S is semi-potent and $J(S) = 0$.*
(2) *The module M is direct-projective and for every $0 \neq \alpha \in S$, $Im(\gamma\alpha)$ is a nonzero direct summand of M for some $\gamma \in S$.*
(3) *The module M is direct-projective and for every $0 \neq \alpha \in S$, $Im(\alpha\beta)$ is a nonzero direct summand of M for some $\beta \in S$.*
(4) *The module M is direct-projective and for every $0 \neq \alpha \in S$, $Im(\alpha)$ contains a nonzero direct summand N of M.*

Proof. (1) $\Rightarrow$ (2). Let $0 \neq \alpha \in S$. By assumption $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in S$. Then $e = \alpha\beta \in S$ is a nonzero idempotent and so $Im(\alpha\beta) \neq 0$ is a direct summand of M. Now we will prove that M is direct-projective. Let N be a direct summand of M and $\lambda : M \to N$ be an epimorphism. If $N = 0$, then $Ker(\lambda) = M$ is a direct summand of M. Assume that $N \neq 0$, then $\lambda \neq 0$ and by assumption $\mu = \mu\lambda\mu$ for some $0 \neq \mu \in S$. Let $e = \lambda\mu$, then $0 \neq e \in S$ is idempotent and $Im(e) \subseteq Im(\lambda) = N$. Suppose that $\pi : M \to N$ be the projection. Since for each $m \in M$, $m = e(m) + (1 - e)(m)$ and $e(m) \in N$, $\pi(m) = e(m)$, thus $\pi = e = \lambda\mu$ and so M is direct-projective. (2) $\Rightarrow$ (3). Let $0 \neq \alpha \in S$. Then by assumption $Im(\gamma\alpha)$ is a nonzero direct summand of M for some $\gamma \in S$. Since M is direct-projective and $\gamma\alpha : M \to Im(\gamma\alpha)$ is an epimorphism, $Ker(\gamma\alpha)$ is a direct summand of M. So by Lemma 2.5 there exists $g \in S$ such that $\gamma\alpha = (\gamma\alpha)g(\gamma\alpha)$. Let $e = g\gamma\alpha$, then $0 \neq e \in S$ is an idempotent and $\alpha e = \alpha e(g\gamma)\alpha e$. Suppose that $\beta = eg\gamma$ we found that $\alpha\beta = \alpha eg\gamma \in S$ is a nonzero idempotent, therefore $Im(\alpha\beta)$ is a nonzero direct summand of M.
(3) $\Rightarrow$ (4). It is clear.
(4) $\Rightarrow$ (1). Let $\alpha \in S$, $\alpha \neq 0$. By assumption there exists a direct summand $N \neq 0$ of M, $N \subseteq Im(\alpha)$. If $\pi : M \to N$ the projection, then $N = Im(\pi) = Im(\pi\alpha)$. Since $\pi\alpha : M \to N$ is an epimorphism and M is direct-projective, $Ker(\pi\alpha) \neq M$ is a direct summand of M. By Lemma 2.5 $\pi\alpha = (\pi\alpha)g(\pi\alpha)$ for some $g \in S$. Let $e = \pi\alpha g$, then $e \in S$ is a nonzero idempotent. If $\alpha \in J(S)$, $e \in J(S)$ a contradiction, so $J(S) = 0$ and $ge\pi = (ge\pi)\alpha(geb)$, for $\mu = ge\pi$, $0 \neq \mu \in S$ and $\mu = \mu\alpha\mu$, so S is semi potent. □

Theorem 4.3. *Let M_R be a module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The ring S is semi-potent and $J(S) = \nabla S$.*

(2) *The module M is direct-projective and for every $\alpha \in S$ which $Im(\alpha)$ is not small in M, $Im(\alpha)$ contains a nonzero direct summand of M.*

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in S$ which $Im(\alpha)$is not small in M, then $\alpha \notin \nabla S = J(S)$, so $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in S$ and $Im(\alpha\beta)$ is a nonzero direct summand of M, $Im(\alpha\beta) \subseteq Im(\alpha)$, hence $0 \neq \alpha\beta$ is idempotent. Similarly as in Theorem 4.2 we can prove that M is direct-projective.
(2) $\Rightarrow$ (1). First we will prove that $\nabla S = J(S)$. Since M is direct-projective, by Lemma 4.1 we have $\nabla S \subseteq J(S)$. Let $\alpha \in J(S)$, if $\alpha \notin \nabla S$, then $Im(\alpha)$ is not small in M and by assumption $Im(\alpha)$ contains direct summand $N \neq 0$ of M. Let $\pi : M \to N$ be the projection, then $N = Im(\pi) = Im(\pi\alpha)$. Since $\pi\alpha : M \to N$ is an epimorphism and M is direct-projective, there exists $\beta \in S$ such that $(\pi\alpha)\beta = \pi$. For $\mu = \alpha\beta\pi$, $0 \neq \mu \in S$ is idempotent and $\mu \in J(S)$, hence $\alpha \in J(S)$ a contradiction, so $\nabla S = J(S)$. By analogous as in Theorem 4.2 we can prove that S is semi-potent. □

Recall a module M_R is *direct-injective* [10] if for every direct summand N of M and every monomorphism $\alpha : N \to M$ there exists $\beta \in End_R(M)$ such that $\beta\alpha = \tau$ where $\tau : N \to M$ the inclusion.
Following [10], a module M_R is direct-injective if and only if every monomorphism $\alpha : N \to M$, $Im(\alpha)$ is a direct summand of M.

Lemma 4.4. *Let M_R be a direct-injective module and $S = End_R(M)$. Then $\Delta S \subseteq J(S) = \widehat{\Delta} S$.*

Proof. By Lemma 2.4 we have $J(S) \subseteq \widehat{\Delta} S$. Let $\alpha \in \widehat{\Delta} S$, then for every $\beta \in S$ $Ker(1 - \beta\alpha) = 0$. Since M is direct-injective, $\lambda(1 - \beta\alpha) = 1_M$ for some $\lambda \in S$, so $\alpha \in J(S)$. □

Theorem 4.5. *Let M_R be a module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The ring S is semi-potent and $J(S) = 0$.*
(2) *The module M is direct-injective and for every $0 \neq \alpha \in S$, $Ker(\alpha\beta) \neq M$ is a direct summand of M for some $\beta \in S$.*
(3) *The module M is direct-injective and for every $0 \neq \alpha \in S$, $Ker(\gamma\alpha) \neq M$ is a direct summand of M for some $\gamma \in S$.*
(4) *The module M is direct-injective and for every $0 \neq \alpha \in S$, $Ker(\alpha)$ is contained in a direct summand $N \neq M$ of M.*

Proof. (1) $\Rightarrow$ (2). Let $0 \neq \alpha \in S$. By assumption $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in S$. Then $e = \alpha\beta \in S$ is a nonzero idempotent and so $Ker(\alpha\beta) \neq M$ is a direct summand of M. Now we will prove that M is direct-injective. Let N be a direct summand of M and $\alpha : N \to M$ be a monomorphism, $\pi : M \to N$ be the projection, then $0 \neq \alpha\pi \in S$. By assumption $\mu = \mu(\alpha\pi)\mu$ for some $0 \neq \mu \in S$. Assume that $e = \pi\mu\alpha$, $e \in S$ is a nonzero idempotent and $Im(e) \subseteq Im(\pi) = N$. Since for each $m \in M$, $m = e(m) + (1-e)(m)$ implies that $\pi(m) = e(m)$, so for every $y \in N$, $y = \pi(y) = e(y) = \pi\mu\alpha(y)$. Let $\pi\mu = \beta$, then $\beta\alpha = \tau$ where $\tau : N \to M$ the inclusion, thus M is direct-injective.
(2) $\Rightarrow$ (3). Let $0 \neq \alpha \in S$. Then by assumption $Ker(\alpha\beta) \neq M$ is a direct summand of M for some $\beta \in S$, so $Ker(\alpha\beta) = Im(e)$ where $1 \neq e \in S$ is an idempotent. Assume that $(\alpha\beta)_0 : Im(1-e) \to M$ the restriction of $\alpha\beta$ on $Im(1-e)$, then $(\alpha\beta)_0$ is a monomorphism. Since M is direct-injective, there exists $\lambda \in S$ such that $\lambda(\alpha\beta)_0 = \tau$, where $\tau : Im(1-e) \to M$ the inclusion. Let $\pi : M \to Im(1-e)$ be the projection. Then for every $m \in M$,

$$\lambda(\alpha\beta)\pi(m) = \lambda(\alpha\beta)_0(\pi(m)) = \tau(\pi(m)) = \pi(m)$$

so $\lambda\alpha\beta\pi = \pi$ and $(\beta\pi\lambda)\alpha(\beta\pi\lambda) = \beta\pi\lambda$. Suppose that $\mu = \beta\pi\lambda$, we found that $0 \neq \mu \in S$ such that $\mu = \mu\alpha\mu$, thus $0 \neq \mu\alpha \in S$ is an idempotent and so $Ker(\mu\alpha) \neq M$ is a direct summand of M. (3) $\Rightarrow$ (4). It is clear, hence $Ker(\alpha) \subseteq Ker(\gamma\alpha)$.
(4) $\Rightarrow$ (1). Let $0 \neq \alpha \in S$, then $Ker(\alpha) \neq M$ by assumption $Ker(\alpha) \subseteq N$ where $N \neq M$ is a direct summand of M. So $M = N \oplus K$ for some submodule $K \neq 0$ of M. Suppose that $\alpha_0 : K \to M$ the restriction of α on K, then α_0 is monomorphism. Since M is direct injective, $\beta\alpha_0 = \tau$ where $\tau : K \to M$ the inclusion. Let $\pi : M \to K$ be the projection, then for every $m \in M$, $\pi(m) \in K$ and so $\beta\alpha\pi(m) = \beta\alpha_0(\pi(m)) = \tau(\pi(m)) = \pi(m)$, thus $\beta\alpha\pi = \pi$. Let $\mu = \pi\beta$, then $0 \neq \mu \in S$ such that $\mu = \mu\alpha\mu$, so $\alpha\mu \in S$ is a nonzero idempotent. If $\alpha \in J(S)$ a contradiction. Thus $J(S) = 0$ and S is semi-potent. □

THEOREM 4.6. *Let M_R be a module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The ring S is semi-potent and $J(S) = \Delta S$.*
(2) *The module M is direct-injective and for every $\alpha \in S$, which $Ker(\alpha)$ is not large in M, $Ker(\alpha)$ is contained in a direct summand $N \neq M$ of M.*

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in S$, $Ker(\alpha)$ be not large in M. Then by assumption $\alpha \notin \Delta S = J(S)$, by assumption $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in S$, so $\beta\alpha \in S$ is a nonzero idempotent and so $Ker(\beta\alpha) \neq M$ is a direct summand of M such that $Ker(\alpha) \subseteq Ker(\beta\alpha)$. Now we will prove that M is direct-injective. Let N be a direct summand of M, $\alpha : N \to M$ be a monomorphism and $\pi : M \to N$ be the projection, then $\alpha\pi \in S$.
- If $Ker(\alpha\pi)$ is a large submodule in M, then $Ker(\pi)$ is large in M. Because for any $x \in Ker(\alpha\pi)$, $\alpha\pi(x) = 0$ and so $\pi(x) = 0$, hence α is monomorphism. Therefore $\pi \in \Delta S = J(S)$, so $\pi = 0$, hence $\pi^2 = \pi$. Thus $\alpha = 0$, hence $N = Im(\pi) = 0$ and so $Im(\alpha) = 0$ is a direct summand in M.
- Suppose that $Ker(\alpha\pi)$ is not large in M, then $\alpha\pi \notin \Delta S = J(S)$. Since S is semi-potent, $\mu = \mu(\alpha\pi)\mu$ for some $0 \neq \mu \in S$. Let $e = \pi\mu\alpha\pi$, then $e \in S$ is a nonzero idempotent and $Im(e) \subseteq Im(\pi) = N$. Since for any $x \in M$, $e(x) \in N$ we found that $\pi(x) = e(x)$ and so $\pi = e$. Thus for every $y \in N$, $y = \pi(y) = e(y) = \pi\mu\alpha\pi(y) = \pi\mu\alpha(y)$. Suppose that $\beta = \pi\mu \in S$, then follows that $\beta\alpha = \tau$ where $\tau : N \to M$ the inclusion, this shows that M is direct-injective.
(2) $\Rightarrow$ (1). First we will prove that $\Delta S = J(S)$. Since M is direct-injective, by Lemma 4.4 we have $\Delta S \subseteq J(S)$. Let $\alpha \in J(S)$. If $\alpha \notin \Delta S$, then $Ker(\alpha)$ is not large in M, by assumption $Ker(\alpha)$ contained in a direct summand $N \neq M$ of M, so $M = N \oplus K$ for some submodule $K \neq 0$ of M. Let $\pi : M \to K$ be the projection, then $Ker(\alpha) \subseteq Ker(\pi)$ and so $S\pi \subseteq S\alpha$ by Lemma 4.4, hence M is direct-injective. Thus $\pi = \lambda\alpha$ for some $\lambda \in S$ and so $\pi\lambda = \pi\lambda\alpha\pi\lambda$. Thus $\alpha\pi\lambda \in S$ is a nonzero idempotent and $\alpha\pi\lambda \in J(S)$ a contradiction, thus $\Delta S = J(S)$. By analogous as in Theorem 4.5 we can prove that S is semi-potent. □

From Theorems 4.3 and 4.6 we conclude the following:

Corollary 4.7. *Let M_R be a module and $S = End_R(M)$, if $J(S) = \nabla S = \Delta S$. Then the following statements are equivalent:*
(1) *The module M is direct-projective and for every $\alpha \in S$ which $Im(\alpha)$ is not small in M, $Im(\alpha)$ contains a nonzero direct summand of M.*
(2) *The ring S is semi-potent.*
(3) *The module M is direct-injective and for every $\alpha \in S$ which $Ker(\alpha)$ is not large in M, $Ker(\alpha)$ is contained in a direct summand $N \neq M$ of M.*

Also, from Theorems 4.2 and 4.5 we conclude the following:

COROLLARY 4.8. *Let M_R be a module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The module M is direct-projective and for every $0 \neq \alpha \in S$, $Im(\gamma\alpha)$ is a nonzero direct summand of M for some $\gamma \in S$.*
(2) *The module M is direct-projective and for every $0 \neq \alpha \in S$, $Im(\alpha\beta)$ is a nonzero direct summand of M for some $\beta \in S$.*
(3) *The module M is direct-projective and for every $0 \neq \alpha \in S$, $Im(\alpha)$ contains a nonzero direct summand N of M.*
(4) *The ring S is semi-potent and $J(S) = 0$.*
(5) *The module M is direct-injective and for every $0 \neq \alpha \in S$, $Ker(\alpha)$ is contained in a direct summand $N \neq M$ of M.*
(6) *The module M is direct-injective and for every $0 \neq \alpha \in S$, $Ker(\gamma\alpha) \neq M$ is a direct summand of M for some $\gamma \in S$.*
(7) *The module M is direct-injective and for every $0 \neq \alpha \in S$, $Ker(\alpha\beta) \neq M$ is a direct summand of M for some $\beta \in S$.*

5. (Co)semi-potent modules.

For every submodule N of a module M_R we use the notation $\widehat{N} = Hom_R(M, N)$ which is a right ideal of $S = End_R(M)$.

Recall that a module M_R is *retractable* [3], if for every nonzero submodule N of M, $\widehat{N} \neq 0$. It is clear that every free module and every projective module P with $J(P) = 0$ are retractable modules.

LEMMA 5.1. *Let M_R be a semi-projective retractable module. Then for every $\alpha \in S = End_R(M)$ the following are equivalent:*
(1) *The right ideal αS is large in S.*
(2) *The submodule $Im(\alpha)$ is large in M.*

Proof. (1) $\Rightarrow$ (2). Let U be a submodule of M such that $Im(\alpha) \cap U = 0$. If $U \neq 0$, $\widehat{U} \neq 0$ hence M is retractable. It is easy to see that $\widehat{U} \cap \alpha S = 0$. Since αS is large in S, $\widehat{U} = 0$ a contradiction. So $Im(\alpha)$ is large in M.
(2) $\Rightarrow$ (1). Let I be a right ideal of S such that $\alpha S \cap I = 0$. Suppose that $I \neq 0$, then $Im(\beta) \neq 0$ for some $0 \neq \beta \in I$ and $\widehat{Im(\beta)} \neq 0$ hence M is retractable. Since M is semi-projective,

$Hom_R(M, Im(\alpha) \cap Im(\beta)) = Hom_R(M, Im(\alpha)) \cap Hom_R(M, Im(\beta)) =$

$$= \alpha S \cap \beta S \subseteq \alpha S \cap I = 0$$

So $Im(\alpha) \cap Im(\beta) = 0$. Since $Im(\alpha)$ is large in M, $Im(\beta) = 0$ and so $\beta = 0$ a contradiction, thus $I = 0$. □

LEMMA 5.2. *Let M_R be a semi-projective retractable module and $S = End_R(M)$. Then the following are equivalent:*
(1) *For every $\alpha \in S$ with αS is not large in S, αS is contained in a direct summand $K \neq S$ of S.*
(2) *For every $\alpha \in S$ with $Im(\alpha)$ is not large in M, $Im(\alpha)$ is contained in a direct summand $N \neq M$ of M.*

Proof. It is clear by Lemma 5.1. □

Recall that a module M_R is *semi-potent* or I_0-*module* [4], if for every submodule $A \nsubseteq J(M)$ of M contains a nonzero direct summand of M.

THEOREM 5.3. *Let M_R be a semi-projective module with $J(M) = 0$ and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The module M is semi-potent.*
(2) *The module M is retractable and for every $0 \neq \alpha \in S$, $Im(\alpha)$ contains a nonzero direct summand of M.*
(3) *The module M is retractable and S is a semi-potent ring with $J(S) = 0$.*

Proof. (1) ⇒ (2). Let $A \neq 0$ be a submodule of M. Since $A \nsubseteq J(M)$, A contains a direct summand $N \neq 0$ of M. If $e : M \to N$ is the projection, $0 \neq e \in S$ is idempotent and $e \in \widehat{A}$, so M is retractable. Let $0 \neq \alpha \in S$, then $Im(\alpha) \nsubseteq J(M)$, so $Im(\alpha)$ contains a nonzero direct summand of M.
(2) ⇒ (3). By corollary 3.5.
(3) ⇒ (1). Let A be a submodule of M and $A \nsubseteq J(M) = 0$. Since M is retractable, $\widehat{A} \neq 0$ is a right ideal of S. So there exists idempotent $0 \neq e \in S$ and $e \in \widehat{A}$ hence S is semi-potent and $J(S) = 0$. Thus, $Im(e) \neq 0$ is a direct summand of M and $Im(e) \subseteq A$, so M is semi-potent. □

Recall that a module M_R is $e-$*retractable* [3], if for every nonzero submodule N of M there exists epimorphism $\alpha : M \to N$. It is clear that every $e-$retractable module is retractable.

THEOREM 5.4. *Let M_R be a semi-projective $e-$retractable module with $J(M)$ is small in M and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The module M is semi-potent.*
(2) *For every $\alpha \in S$ with $Im(\alpha)$ not small in M, $Im(\alpha)$ contains a nonzero direct summand of M.*
(3) *The ring S is semi-potent and $J(S) = \nabla S$.*

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in S$, $Im(\alpha)$ is not small in M. Since $J(M) \ll M$, $Im(\alpha) \nsubseteq J(M)$ by assumption $Im(\alpha)$ contains a nonzero direct summand of M.
(2) $\Rightarrow$ (3). By Theorem 3.4.
(3) $\Rightarrow$ (1). Let $A \nsubseteq J(M)$ be a submodule of M, then $A \neq 0$ and $\widehat{A} \neq 0$ hence M is retractable. Also, the right ideal $\widehat{A} \nsubseteq J(S)$. Because if $\widehat{A} \subseteq J(S)$ and hence M is $e-$retractable there is an epimorphism $\lambda : M \to A$ of M, so $\lambda \in \widehat{A} \subseteq J(S) = \nabla S$, thus $A = Im(\lambda) \subseteq J(M)$ a contradiction. Since S is semi-potent there is idempotent $0 \neq e \in S$ such that $e \in \widehat{A}$, so $Im(e) \neq 0$ is a direct summand of M and $Im(e) \subseteq A$, thus M is semi-potent. □

Recall that a module M is *co-semi-potent* or I^*-*module* [1], if every not large submodule A of M is contained in a direct summand $N \neq M$ of M. Note that if for a module M, $J(M)$ is small in M, then the concept of I^*-module is dual of I_0-module.

LEMMA 5.5. *Let M_R be a nonzero $e-$retractable module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *M is an I^*-module.*
(2) *For every $\alpha \in S$ with $Im(\alpha)$ not large in M, $Im(\alpha)$ is contained in a direct summand $N \neq M$ of M.*

Proof. (1) $\Rightarrow$ (2). Obvious. (2) $\Rightarrow$ (1). Let A be a not large submodule of M. If $A = 0$, then A is a direct summand of M. Suppose that $A \neq 0$, since M is $e-$retractable, there is an epimorphism $\lambda : M \to A$. On the other hand, $\widehat{A}$ is not large in S_S, hence if $\widehat{A}$ is large follows that A is large in M. So by assumption $A = Im(\lambda)$ is contained in a direct summand $N \neq M$ of M. □

THEOREM 5.6. *Let M_R be a semi-projective $e-$retractable module and $S = End_R(M)$. Then the following statements are equivalent:*
(1) *The module M is I^*- module.*

(2) *For every $\alpha \in S$ with $Im(\alpha)$ not large in M, $Im(\alpha)$ contained in a direct summand $N \neq M$ of M.*
(3) *For every $\alpha \in S$ with αS not large in S, αS contained in a direct summand $I \neq S$ of S.*

Proof. (1) $\Rightarrow$ (2). Obvious. (2) $\Rightarrow$ (3). By Lemma 5.2. (3) $\Rightarrow$ (1). By Lemma 5.5 and Lemma 5.1 □

Recall that a module M_R is *co-retractable* [2], if for every submodule $N \neq M$ of M, $\ell_S(N) \neq 0$.

Lemma 5.7. *Let M_R be a semi-injective co-retractable module. Then for every $\alpha \in S = End_R(M)$ the following are equivalent:*
(1) *The left ideal $S\alpha$ is large in S.*
(2) *The submodule $Ker(\alpha)$ is small in M.*

Proof. (1) $\Rightarrow$ (2). Suppose that $Ker(\alpha)$ is not small in M, then $M = Ker(\alpha)+K$ for some submodule $K \neq M$ of M. Since M is co-retractable $\ell_S(K) \neq 0$. Let $\lambda \in S\alpha \cap \ell_S(K)$, then $\lambda = \mu\alpha$ for some $\mu \in S$ and $\lambda(K) = \mu\alpha(K) = 0$. So $\lambda(M) = \lambda(Ker(\alpha) + K) = \mu\alpha(Ker(\alpha)) + \mu\alpha(K) = 0$. Thus $S\alpha \cap \ell_S(K) = 0$. Since $S\alpha$ is large in S implies $\ell_S(K) = 0$ a contradiction.
(2) $\Rightarrow$ (1). If $Ker(\alpha) = 0$, then $S\alpha = \ell_S(Ker(\alpha)) = S$ hence M is semi-injective, and so $S\alpha$ is large in S. Suppose that $Ker(a) \neq 0$. Let I be a left ideal of S such that $S\alpha \cap I = 0$. Suppose that $I \neq 0$, then there is $0 \neq \lambda \in I$ and $Ker(\lambda) \neq 0$, hence if $Ker(\lambda) = 0$ implies that $S\lambda = \ell_S(Ker(\lambda)) = S$ because M is semi-injective. Thus, $S = S\lambda \subseteq I \subseteq S$, so $S = I$ and so $S\alpha = S\alpha \cap S = S\alpha \cap I = 0$ a contradiction hence $S\alpha$ is large in S. Since M is semi-injective

$$S\alpha \cap S\lambda = \ell_S(Ker(\alpha) + Ker(\lambda)) = 0$$

Since M is co-retractable implies that $Ker(\alpha) + Ker(\lambda) = 0$ and so $Ker(\alpha) = 0$ a contradiction, thus $S\alpha$ is large in S. □

Theorem 5.8. *Let M_R be a semi-injective co-retractable module and $J(S) = 0$. Then the following are equivalent:*
(1) *M is an I^*-module.*
(2) *For every $0 \neq \alpha \in S$, $Ker(\alpha)$ contained in a direct summand $N \neq M$ of M.*
(3) *The ring S is semi-potent.*

Proof. (1) $\Rightarrow$ (2). Since M is semi-injective, by Lemma 3.7 $\Delta S \subseteq J(S) = 0$, so $\Delta S = 0$. If $0 \neq \alpha \in S$, then $\alpha \notin \Delta S$ and so $Ker(\alpha)$ is not large in M, by assumption $Ker(\alpha)$ contained in a direct summand $N \neq M$ of M.
(2) $\Rightarrow$ (3). By Corollary 3.10. (3) $\Rightarrow$ (1). Let A be not large submodule of M, then $A \neq M$. If $A = 0$ prove is completed. Suppose that $A \neq 0$, since M is co-retractable, $\ell_S(A) \neq 0$ so $\ell_S(A) \nsubseteq J(S)$. By assumption there exists an idempotent $0 \neq e \in S$, $e \in \ell_S(A)$, thus $A \subseteq Ker(\alpha)$ and $Ker(\alpha) \neq M$ is a direct summand of M. □

THEOREM 5.9. *Let M_R be a semi-injective module and $Soc(M) = M$. Then the following are equivalent:*
(1) *M is an I^*-module.*
(2) *The module M is co-retractable and for every $0 \neq \alpha \in S$, $Ker(\alpha)$ contained in a direct summand $N \neq M$ of M.*
(3) *The module M is co-retractable with $J(S) = 0$ and S is a semi-potent ring.*

Proof. (1) $\Rightarrow$ (2). Let $A \neq M$ be a submodule of M, then $A \nsubseteq Soc(M)$ so A is not large in M. By assumption $A \subseteq N$ for some direct summand $N \neq M$ of M. Thus $M = N \oplus K$ for some submodule $K \neq 0$ of M. Let $e : M \to K$ be the projection, then $0 \neq e \in S$ is an idempotent and $e(A) = 0$ hence $A \subseteq N$, so $e \in \ell_S(A)$, and hence M is co-retractable. Let $0 \neq \alpha \in S$, then $Ker(\alpha) \neq M$ so $Soc(M) \nsubseteq Ker(\alpha)$ therefore $Ker(\alpha)$ is not large in M by assumption $Ker(\alpha)$ contained in a direct summand $D \neq M$ of M. (2) $\Rightarrow$ (3). First we will prove that $J(S) = 0$. Assume that $J(S) \neq 0$. Let $0 \neq \alpha \in J(S)$, then by assumption $Ker(\alpha) \subseteq N$ for some direct summand $N \neq M$ of M. Let $e : M \to N$ be the projection, then $1 \neq e \in S$ is an idempotent, thus $Ker(\alpha) \subseteq N = Im(e) = Ker(1 - e)$. Since M is semi-injective, by Lemma 3.6, $S(1 - e) \subseteq S\alpha \subseteq J(S)$ so $1 - e = 0$ a contradiction. Since M is semi-injective co-retractable and $J(S) = 0$, semi-potency of S implies from Theorem 5.8. (3) $\Rightarrow$ (1). By Theorem 5.8. □

THEOREM 5.10. *Let M_R be a semi-injective co-retractable module and $Soc(M) = M$. Then the following are equivalent:*
(1) *M is an I^*-module.*
(2) *For every $0 \neq \alpha \in S$, $Ker(\alpha)$ contained in a direct summand $N \neq M$ of M.*
(3) *$J(S) = \Delta S$ and S is a semi-potent ring.*

Proof. (1) ⇒ (2). By Theorem 5.9. (2) ⇒ (3). First we will prove that $J(S) = \Delta S$. Since M is semi-injective, by Lemma 3.7 $\Delta S \subseteq J(S)$. Let $\alpha \in J(S)$. Assume that $\alpha \notin \Delta S$, then $Ker(\alpha)$ is not large in M by assumption $Ker(\alpha) \subseteq N$ for some direct summand $N \neq M$ of M. Let $e : M \to N$ be the projection, then $1 \neq e \in S$ is an idempotent, thus $Ker(\alpha) \subseteq N = Im(e) = Ker(1-e)$. Since M is semi-injective, by Lemma 3.6, $S(1-e) \subseteq S\alpha \subseteq J(S)$ so $1-e = 0$ a contradiction, thus $J(S) = \Delta S$. Since M is semi-injective co-retractable and $Soc(M) = M$, semi-potency of S implies from Theorem 5.9. (3) ⇒ (1). Let $A \neq 0$ be a not large submodule of M, then $A \neq M$. Since M is co-retractable, $\ell_S(A) \neq 0$, so there exists $0 \neq \alpha \in S$, $\alpha \in \ell_S(A)$ and so $A \subseteq Ker(\alpha)$. Assume that $\alpha \in J(S) = \Delta S$, then $Ker(\alpha)$ is large in M. Since $Soc(M) = M$, $M = Ker(\alpha)$ so $\alpha = 0$ a contradiction. Therefore $\alpha \notin J(S)$, by assumption $\beta = \beta\alpha\beta$ for some $0 \neq \beta \in S$. For $g = \beta\alpha$ follows that $0 \neq g \in S$ is an idempotent and $A \subseteq Ker(\alpha) \subseteq Ker(g)$ where $Ker(g) \neq M$ is a direct summand of M, So M is an I^*-module. □

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