ON THE WEIERSTRASS THEOREM OF A MAXIMAL SPACELIKE SURFACE

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ABSTRACT. The purpose of this paper is to show how to represent a maximal spacelike surface in \( L^n \) in terms of its generalized Gauss map.

1. Introduction

It is well known that in the theory of minimal surfaces in \( R^3 \), the classical Weierstrass representation formula has played a major role [9]. The formula shows that a minimal surface in \( R^3 \) can be represented by real parts of complex integrations of holomorphic functions. The classical result is extended to a minimal surface in \( R^n \) by Hoffman and Osserman. Since a maximal surface in \( L^n \) is a counterpart of a minimal surface in \( R^n \), it is quite natural to ask if similar representation formula of a maximal surface in \( L^n \) can be obtained. The purpose of this paper is to show how to represent a maximal spacelike surface in \( L^n \) in terms of its generalized Gauss map.

2. The main result

We begin with fixing our terminology and notation. Let \( L^n = (R^n, g) \) denote Lorentzian n-space with the flat Lorentzian metric \( g \) of index 1. Let \( M \) be a connected smooth orientable 2 manifold, and \( X : M \to L^n \) be a smooth imbedding of \( M \) into \( L^n \). Throughout this paper, we assume that \( X \) is a spacelike imbedding or \( M \) is a spacelike surface in \( L^n \), that is, the pull back \( X^*g \) of the Lorentzian metric \( g \) via \( X \) is a positive definite metric on \( M \).

Let \( M = (M, \tilde{g}) \) be a spacelike surface in \( L^n \) with the induced metric \( \tilde{g} = X^*g \) so that \( X : M \to L^n \) is an isometric imbedding. By \( (u_1, u_2) \) we always denote isothermal coordinates compatible with the orientation on \( M \). Then the metric \( \tilde{g} \) is expressed locally as

\[
\tilde{g} = \lambda^2((du_1)^2 + (du_2)^2), \quad \lambda > 0.
\]
It is well known that \((u_1, u_2)\) is defined around each point of \(M\), and we may regard \(M\) as a Riemann surface by introducing a complex local coordinate \(z = u_1 + iu_2\).

We shall define the generalized Gauss map using local coordinates. Let \(M\) be a spacelike surface in \(L^n\), or a Riemann surface. Locally, if \(u_1\) and \(u_2\) are isothermal parameters in a neighborhood of \(p\) on \(M\), then \(M\) is defined near \(p\) by a map \(X(z) = (x_1(z), \ldots, x_n(z)) \in L^n\), where \(z = u_1 + iu_2\). Define the generalized Gauss map \(\Psi\) by

\[
\Psi(z) = \frac{\partial X}{\partial u_1} + i \frac{\partial X}{\partial u_2},
\]

where \(\Psi(z) \in CP^{n-1}_+ = \{Z = (z_1, \ldots, z_n) \in CP^{n-1} \mid g_c(Z, Z) > 0\}\). Here \(g_c\) denotes the flat Hermitian metric in \(C^n\) with the signature \((- +, \ldots, +)\). Let us think of the effect of choosing another isothermal parameters \(\tilde{u}_1, \tilde{u}_2\), and \(\tilde{z} = \tilde{u}_1 + i\tilde{u}_2\). Since the change of coordinates on a Riemann surface is analytic, we know that

\[
\frac{\partial X}{\partial u_1} + i \frac{\partial X}{\partial u_2} = \frac{\partial X}{\partial \tilde{u}_1} + i \frac{\partial X}{\partial \tilde{u}_2} \frac{(\partial u_1}{\partial \tilde{u}_1} - i \frac{(\partial u_1}{\partial \tilde{u}_2}) ,
\]

which implies \(\Psi(z) = \Psi(\tilde{z})\) in \(CP^{n-1}_+\). Since the pair of vectors \(\frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2}\) are orthogonal and equal in length in \(L^n\), it follows that

\[
\frac{\partial X}{\partial u_1} + i \frac{\partial X}{\partial u_2} \in Q^{n-2}_+ ,
\]

where \(Q^{n-2}_+ = \{(z_1, \ldots, z_n) \in CP^{n-1}_+ \mid -z_1^2 + z_2^2 + \ldots + z_n^2 = 0\}\). Consequently, the generalized Gauss map \(\Psi\) is given locally by

\[
(u_1, u_2) \rightarrow \frac{\partial X}{\partial u_1} + i \frac{\partial X}{\partial u_2} \in Q^{n-2}_+ \subset CP^{n-1}_+ .
\]

We may represent the Gauss map locally by

\[
\Psi(z) = (\phi_1(z), \ldots, \phi_n(z)) ,
\]

where \(\phi_k = 2 \frac{\partial x_k}{\partial z} = \frac{\partial x_k}{\partial u_1} - i \frac{\partial x_k}{\partial u_2}\). Denote \((\phi_1, \ldots, \phi_n)\) by \(\Phi\). Then \(\Psi\) is holomorphic when \(\Phi\) is antiholomorphic and \(\Psi\) is antiholomorphic when \(\Phi\) is holomorphic. We will consider \(\Phi\) as the Gauss map instead of \(\Psi\).

**Theorem 2.1.** Let \(M\) be a spacelike surface in \(L^n\), and \(\Phi\) the Gauss map on \(M\). Then \(\Phi\) is holomorphic if and only if \(M\) is maximal.

**Proof.** For a maximal surface \(M\) in \(L^n\) defined by an isometric imbedding \(X : M \rightarrow L^n\), we know that

\[
0 = 2\lambda^2 H = \frac{\partial^2 X}{(\partial u_1)^2} + \frac{\partial^2 X}{(\partial u_2)^2} ,
\]

where \(\lambda^2 = g(\frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2})\) and \(u = (u_1, u_2)\) is an isothermal coordinate on \(M\). Therefore each \(x_k\) is harmonic and \(\phi_k = \frac{\partial x_k}{\partial u_1} - i \frac{\partial x_k}{\partial u_2}\) is analytic. For any \(k\) and
\(j, \frac{\phi_j}{\phi_k}\) is analytic around \(p\) when \(\phi_k(p) \neq 0\) and thus
\[
\Phi : z \rightarrow (\phi_1, \ldots, \phi_n) \in CP^{n-1}_+
\]
is holomorphic.

Conversely, suppose \(\Phi\) is holomorphic. In other words, \(\frac{\phi_k}{\phi_j}\) is always holomorphic whenever the denominators do not vanish. Since \(\frac{\partial X}{\partial u_1}\) and \(\frac{\partial X}{\partial u_2}\) are linearly independent at any point \(z_0\), not all \(\phi_j(z_0)\) cannot vanish. Say \(\phi_j(z_0) \neq 0\). Then \(\gamma_k(z) = \phi_k(z)/\phi_j(z)\) is analytic near \(z_0\). Set \(\mu(z) = 1/\phi_j(z)\). Then
\[
0 = \frac{\partial \mu}{\partial \bar{z}} \phi_k + \mu \frac{\partial \phi_k}{\partial \bar{z}}.
\]
But then
\[
\frac{\partial^2 x_k}{(\partial u_1)^2} + \frac{\partial^2 x_k}{(\partial u_2)^2} = 4 \frac{\partial^2 x_k}{\partial z \partial \bar{z}} = 2 \frac{\partial}{\partial \bar{z}} (2 \frac{\partial x_k}{\partial z}) = 2 \frac{\partial}{\partial \bar{z}} (\phi_k) = -2 \frac{1}{\mu} \frac{\partial \mu}{\partial \bar{z}} \phi_k.
\]
Let
\[
-2 \frac{\partial \mu}{\mu} \frac{\partial \bar{z}} = f(z) + ig(z),
\]
where \(f\) and \(g\) are real. Since
\[
\frac{\partial^2 x_k}{(\partial u_1)^2} + \frac{\partial^2 x_k}{(\partial u_2)^2} = -2 \frac{\partial \mu}{\mu} \frac{\partial \phi_k}{\partial \bar{z}}
\]
is real, imaginary part of
\[
\frac{\partial^2 x_k}{(\partial u_1)^2} + \frac{\partial^2 x_k}{(\partial u_2)^2} = (f(z) + ig(z)) \phi_k
\]
must vanish. Hence
\[
\frac{\partial^2 X}{(\partial u_1)^2} + \frac{\partial^2 X}{(\partial u_2)^2} = f \frac{\partial X}{\partial u_1} + g \frac{\partial X}{\partial u_2}.
\]
Note that \(f \frac{\partial X}{\partial u_1} + g \frac{\partial X}{\partial u_2} \in T_{z_0} M\). Since
\[
\frac{\partial^2 X}{(\partial u_1)^2} + \frac{\partial^2 X}{(\partial u_2)^2} = 2 \lambda^2 H \in T_{z_0} M \cap T_{z_0}^\perp M
\]
and \(T_{z_0} M\) is nondegenerate, \(H\) at the given point is zero.

We now turn to the representation of a maximal surface in terms of its Gauss map. We start with the special case of simply connected surfaces.

Let \(X : M \rightarrow L^n\) be simply connected maximal surface defined by an imbedding. Since \(M\) is simply connected Riemann surface, by the uniformization theorem, we may view \(M\) as the unit disk, the unit sphere, or the complex plane. When \(M \cong S^2\), \(X\) is constant since each \(x_k\) is harmonic on the compact Riemann surface. Therefore every simply connected maximal surface is
considered to be an imbedded submanifold of a simply connected domain in the complex plane.

**Theorem 2.2.** Let $D$ be a simply connected domain in the complex plane. Define an 1-1 smooth map

$$X : D \rightarrow L^n \ (n > 3)$$

in one of the following ways:

**Case 1.** $X$ is the direct sum into $L^2$ and $R^{n-2}$, where $(x_3, \ldots, x_n)$ defines a (immersed) minimal surface in isothermal parameters in $D$, and $x_1, x_2$ are harmonic functions such that $x_1 - x_2$ is constant in $D$.

**Case 2.** Let $\psi$ be an arbitrary holomorphic functions in $D$, $\psi \neq 0$, and let $g_1, \ldots, g_{n-2}$ be arbitrary meromorphic functions in $D$ such that at any $p$ in $D$, the maximum order of pole at $p$ of $g_1, \ldots, g_{n-2}$ is greater than or equal to the order of pole of $\sum_{k=1}^{n-2} g_k^2$ and the same as the order of zero of $\psi$ at $p$. Furthermore,

$$\sum_{k=1}^{n-2} |g_k - \overline{g_k}(p) > 0$$

wherever $\psi(p) \neq 0$.

Set

$$\Phi = (\phi_1, \ldots, \phi_n)$$

$$= \frac{\psi}{2} (\sum_{k=1}^{n-2} g_k^2 + 1, \sum_{k=1}^{n-2} g_k^2 - 1, 2g_1, \ldots, 2g_{n-2}) \ (3)$$

and let

$$x_k = \text{Re} \int \phi_k \ , \ k = 1, \ldots, n \ . \ (4)$$

Then the map $X : D \rightarrow L^n$ defines a maximal surface in terms of isothermal parameters in $D$.

Conversely, every simply connected maximal surface in $L^n$ is obtained by the above construction.

**Remark 1.** The two cases of the theorem are mutually exclusive. Let $\psi = \phi_1 - \phi_2$, $\phi_3 = g_1 \psi$, $\ldots$, $\phi_n = g_{n-2} \psi$. If $x_1 - x_2$ is constant, then $\phi_1 \equiv \phi_2$, i.e. $\psi \equiv 0$. Hence the assumption $\psi \neq 0$ guarantees $x_1 - x_2$ is not constant.

**Proof.** Let $u = (u_1, u_2)$ be a coordinate in $D$. We begin with case 1. Since $x_1 - x_2 \equiv$ constant,

$$g(\frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_i}) = \sum_{k=3}^{n} (\frac{\partial x_k}{\partial u_i})^2 > 0 \ (5)$$

by the regularity of $(x_3, \ldots, x_n)$. Since $(x_3, \ldots, x_n) : D \rightarrow R^{n-2}$ defines a minimal surface in isothermal parameters $u_1, u_2$ in $D$, $\phi_3, \ldots, \phi_n$ are analytic functions such that

$$\phi_3^2 + \ldots + \phi_n^2 = 0 \ . \ (6)$$
Since $x_1$, $x_2$ are harmonic functions such that $x_1 - x_2 \equiv \text{constant}$, we have analytic functions $\phi_1, \ldots, \phi_k$ such that $\phi_1 \equiv \phi_2$ and

$$-\phi_1^2 + \phi_2^2 + \phi_3^2 + \ldots + \phi_n^2 = 0. \quad (7)$$

Furthermore,

$$g_c(\Phi, \Phi) = \sum_{k=3}^{n} |\phi_k|^2$$

$$= \sum_{k=3}^{n} \left( (\frac{\partial x_k}{\partial u_1})^2 + (\frac{\partial x_k}{\partial u_2})^2 \right) > 0. \quad (8)$$

Therefore $X : D \rightarrow L^n$ defines a maximal surface in an isothermal coordinate in $D$ and its Gauss map is $\Phi$.

As for the case 2, we know $\phi_k$’s are analytic everywhere and

$$\phi_k = \frac{\partial x_k}{\partial u_1} - i \frac{\partial x_k}{\partial u_2} \quad (9)$$

from (4), where $u = (u_1, u_2)$ is a coordinate in $D$. Direct computation using (3) shows $\Phi$ lies in the quadric $Q^{n-2} \subset CP^{n-1}$. This means

$$g(\frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_1}) = g(\frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_2}) = 0. \quad (10)$$

We also want to show $g_c(\Phi, \Phi) > 0$. When $\psi(p) \neq 0$, all $g_1, \ldots, g_{n-2}$ are analytic and $g_c(\Phi, \Phi) = \frac{|\psi|^2}{2} (\sum_{k=1}^{n-2} |g_k - \overline{g_k}|^2) > 0$ near $p$. When $\psi(p) = 0$, $g_c(\Phi, \Phi) \geq \sum_{k=1}^{n-2} |\psi g_k|^2$ and at least one $(\psi g_k)(p) \neq 0$. Therefore $g_c(\Phi, \Phi) > 0$ everywhere. From this fact we obtain

$$g(\frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_i}) > 0, \quad g(\frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2}) = 0,$$

which implies that $\frac{\partial X}{\partial u_1}$ and $\frac{\partial X}{\partial u_2}$ are linearly independent spacelike vectors. Hence $X : D \rightarrow L^n$ defines a maximal surface in $L^n$ in terms of an isothermal coordinate in $D$.

For the converse, given a simply connected maximal surface in $L^n$, by introducing isothermal parameters, the surface may be represented by an imbedding $X : D \rightarrow L^n$, where $D$ is a simply connected domain in the complex plane. The function defined by (9) will be analytic and satisfy $-\phi_1^2 + \phi_2^2 + \ldots + \phi_n^2 = 0$ and $g_c(\Phi, \Phi) > 0$. There are two possibilities:

1. If $\phi_1 \equiv \phi_2$, then $\sum_{k=3}^{n} \phi_k^2 \equiv 0$. Since $g_c(\Phi, \Phi) = \sum_{k=3}^{n} |\phi_k|^2 > 0$, the nonconstant map $(x_3, \ldots, x_n) : D \rightarrow R^{n-2}$ defines an imbedded minimal surface in isothermal parameters in $D$. Futhermore, $x_1$ and $x_2$ are harmonic maps such that $x_1 - x_2 \equiv \text{constant}$. This is just the case 1.

2. If $\phi_1 \neq \phi_2$, then the map $\psi = \phi_1 - \phi_2$ ia an analytic map with only isolated zeros. Define

$$g_k = \frac{\phi_{k+2}}{\psi}, \quad k = 1, \ldots, n-2.$$

(10)
The function $g_k$'s are meromorphic and can only have poles where $\psi$ vanishes. At a point $p$ where $\psi(p) \neq 0$, it follows that

$$\Phi = (\phi_1, \ldots, \phi_n) = \frac{\psi}{2} \left( \sum_{k=1}^{n-2} g_k^2 + 1, \sum_{k=1}^{n-2} g_k - 1, 2g_1, \ldots, 2g_{n-2} \right),$$

lies on the subset of $Q^{n-2}$, and $\frac{|\psi|}{2} \sum_{k=1}^{n-2} |g_k - \overline{g_k}| > 0$ at $p$. Since (3) holds everywhere except some isolated points, by continuity, it must hold at those isolated points where $\psi$ vanishes. Finally, we will show that $\psi$ and $g_1, \ldots, g_{n-2}$ satisfy all the hypotheses. Suppose $\psi(p) = 0$. From the definition of $\psi$ and $g_k$'s, it is clear that the order of zero of $\psi$ at $p$ is greater than or equal to the maximum order of pole of $g_1, \ldots, g_{n-2}$ at $p$. If the order of zero of $\psi$ at $p$ was greater than the maximum order of pole of $g_1, \ldots, g_{n-2}$ at $p$, then all $\psi g_k$'s would be zero at $p$, and $g_c(\Phi, \Phi) = 0$, a contradiction. Hence the order of zero of $\psi$ at $p$ is exactly the same as the maximum order of pole of $g_1, \ldots, g_{n-2}$ at $p$. If the order of pole of $\sum_{k=1}^{n-2} g_k^2$ was greater than the maximum order of pole of $g_1, \ldots, g_{n-2}$ at $p$, then $\psi(\sum_{k=1}^{n-2} g_k^2) = 2\phi_1 - \psi$ could not be analytic at $p$, a contradiction. Hence the order of pole of $\sum_{k=1}^{n-2} g_k^2$ at $p$ is less than or equal to the maximum order of pole of $g_1, \ldots, g_{n-2}$. This completes the proof. □

We next modify the theorem to give a representation formula of arbitrary maximal surfaces. We begin with a Riemann surface $S_o$ and define an 1-1 map $X : S_o \rightarrow L^n$ in one of two ways. Case 1 is exactly as in Theorem 2. In case 2, we again choose $n-2$ arbitrary meromorphic functions $g_k$ on $S_o$, but in place of the function $\psi$ we choose an analytic differential $\alpha$ on $S_o$ which is locally of the form $\alpha = \psi(z) dz$ in terms of a complex parameter $z$ on $S_o$. If we then define $\phi_k$ locally by (3), we will obtain global differentials $\alpha_k = \phi_k(z) dz$ on $S_o$ and may then set

$$x_k = Re \int \alpha_k,$$

where the integral is taken along a path from a fixed point to a variable point in $S_o$. We must add the condition

$$Re \int_C \alpha_k = 0$$

for any closed curve $C$ on $S_o$, so that (12) defines a single-valued map $X : S_o \rightarrow L^n$. This map will then define a maximal surface in $L^n$ provided the hypotheses in case 2 of Theorem 2 are satisfied. Conversely, every maximal surface in $L^n$ is represented in one of these two forms. The proof is modeled exactly on that of Theorem 2.

References


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