SPARSE GRID STOCHASTIC COLLOCATION METHOD FOR
STOCHASTIC BURGERS EQUATION

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Abstract. We investigate an efficient approximation of solution to stochastic Burgers equation driven by an additive space-time noise. We discuss existence and uniqueness of a solution through the Orstein-Uhlenbeck (OU) process. To approximate the OU process, we introduce the Karhunen-Lo`eve expansion, and sparse grid stochastic collocation method. About spatial discretization of Burgers equation, two separate finite element approximations are presented: the conventional Galerkin method and Galerkin-conservation method. Numerical experiments are provided to demonstrate the efficacy of schemes mentioned above.

1. Introduction

In this paper, we consider the stochastic sparse grid collocation method to the time-dependent Burgers equation driven by an additive space-time noise having the form ([5, 9, 10])

$$\begin{cases}
\frac{\partial}{\partial t} u - \nu \Delta u - \frac{1}{2} \frac{\partial}{\partial x} u^2 = \sigma \xi(t, x) & \text{in } (0, T] \times (0, 1), \\
u |_{(0, T] \times \{0\}} = u_0(x), & u |_{(0, T] \times \{1\}} = 0,
\end{cases}$$

where $\nu$ is a viscosity, which is positive, $(0, T]$ denotes a time interval, $(0, 1)$ is a spatial domain and $u_0(x)$ denotes deterministic data functions. On a force term, $\sigma$ denotes a constant and $\xi(t, x)$ is a space-time random field representing a stochastic perturbation. Because of expressing nonlinear term, we call the equation (1) the conservation form.

Especially, we are interested in the following form of noise

$$\xi(t, x) = \sum_{j=1}^{\infty} w_j \Phi_j(x) \dot{\zeta}_j(t),$$

where $\{\dot{\zeta}_j(t)\}_{j=1}^{\infty}$ denotes a set of independent and identically distributed time-dependent white noise, which are interpreted as the derivative of Browonian
motion $\zeta_j(t)$ only formally; see [7]. Here $\Phi_j(x)$ are basis functions in $L^2(0,1)$. Thus, if $\{w_j\}_{j=1}^{\infty}$ satisfy some restrictions, we have that (2) is a white noise process in time and in general, has some spatial regularity whose character, e.g., spatial smoothness, is determined by the properties of the coefficients $w_j$.

The existence and uniqueness of solutions of the stochastic Burgers equation (1) with the random noise (2) was studied in D. Blömker, A. Jentzen [5]. Here, we focus on efficient numerical methods for (1). Instead of discretizing white noises and approximating SPDEs directly, as shown for linear problems in [1, 12], first, we derive the Ornstein-Uhlenbeck (OU) process from the stochastic parabolic equation. Using the Karhunen-Loève (KL) expansion, we develop a good low-dimensional approximation to the random field. Then we express the solution of the stochastic Burgers equation in terms of the OU process and other random field, which transformations of (1) to the new equations having a colored noise in a nonlinear term. On the other hand, for spatial discretization, we introduce two separate finite element approximation schemes: the conventional Galerkin method and the Galerkin-conservation method.

When dealing with stochastic problems, the curse of dimensionality arises in a random parameter space; see [21]. For a problem to obtain statistical moments, the sparse grid stochastic collocation method provides quadrature rule in terms of minimal number of sample points in multi-dimensional random parameter space, but the error is not much different from the full-tensor grids.

2. Ornstein-Uhlenbeck process and stochastic Burgers equation

Let $A$ denote the operator $-\nu \Delta$ with zero Dirichlet boundary condition for $u \in H^2(D) \cap H^1_0(D)$, i.e., $Au := -\nu \Delta u$. Since $-\Delta$ with zero Dirichlet boundary condition is self-adjoint and positive definite and $\nu$ is also positive, corresponding to the operator $-\nu \Delta$ with zero Dirichlet boundary condition, we have the eigenpairs $\{\lambda_j, \Phi_j\}_{j=1}^{\infty}$, which are

$$-\nu \Delta \Phi_j = \lambda_j \Phi_j, \quad \Phi_j(0) = \Phi_j(1) = 0.$$  \hspace{1cm} (3)

Furthermore, $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with $\lim_{j \to \infty} \lambda_j = \infty$, and $\{\Phi_j\}_{j=1}^{\infty}$ forms a complete orthonormal basis of $L^2(D)$; see [13, 23]. The eigenvalues and eigenfunctions are known to be explicitly, for $j = 1, 2, \ldots$,

$$\lambda_j = \nu j^2 \pi^2 \quad \text{and} \quad \Phi_j(x) = \sqrt{2} \sin(j\pi x).$$

Thus, given $\{w_j\}_{j=1}^{\infty}$, we represent the noise (2) as

$$\xi(t,x) = \sum_{j=1}^{\infty} w_j \Phi_j(x) \zeta_j(t) = \sqrt{2} \sum_{j=1}^{\infty} w_j \sin(j\pi x) \zeta_j(t).$$  \hspace{1cm} (4)
2.1. Stochastic parabolic equation

From now on, we formally solves the following linear problem ([11])

\[
\begin{align*}
\frac{d}{dt}\eta &= -A\eta + \sigma \xi(t,x) \quad \text{in } (0,T] \times (0,1), \\
\eta|_{t=0} &= u_0(x), \quad \eta|_{t=T} = \eta|_{t=0} = 0.
\end{align*}
\]

It is not difficult to show that the solution of (5) is given by

\[
\eta(t,x) = \sigma \int_0^t e^{-(t-s)A}\xi(s,x) \, ds.
\]

Let \((\cdot, \cdot)\) denote an inner product of \(L^2(D)\)-space. By definition ([5, 16]), for all \(t > 0\) and a proper \(u(x)\),

\[
e^{-tA}u(x) := \sum_{j=1}^{\infty} e^{-\lambda_j t} (u, \Phi_j) \Phi_j(x)
\]

and using the form of \(\xi\) as (2) and (4), we obtain

\[
\eta(t,x) = \sigma \sum_{j=1}^{\infty} w_j \Phi_j(x) \int_0^t e^{-(t-s)\lambda_j} \dot{\zeta}_j(s) \, ds,
\]

which is called the Ornstein-Uhlenbeck (OU) process. The next theorem guarantees the existence and uniqueness of the OU process; see [5].

**Proposition 2.1.** Let \(V = C[0,1]\) with \(||v||_V = ||v||_{C[0,1]}\) for every \(v \in V\). Suppose there is \(\rho \in (0,\infty)\) such that

\[
\sum_{j=1}^{\infty} \lambda_j^{(\rho-1)} w_j^2 < \infty,
\]

where \(-\nu \Delta \Phi_j = \lambda_j \Phi_j\) with \(\Phi_j|_{\partial D} = 0\), and \(\xi(t,x) = \sum_{j=1}^{\infty} w_j \Phi_j(x) \dot{\zeta}_j(t)\) with a real sequence \(\{w_j\}_{j=1}^{\infty}\) and a family of i.i.d. standard Brownian motions \(\{\zeta_j(t)\}_{j=1}^{\infty}\). Then there exists the stochastic process \(\eta : \Omega \times [0,T] \to V\) with continuous sample paths, which satisfies (7) \(\mathbb{P}\)-a.s. for every \(t \in (0,T]\) and \(E[\eta] = 0\). In particular, \(\eta(t,x)\) is the unique solution of (5).

The parameter \(\rho\) in Proposition 2.1 actually tells some information about the regularity of the OU process. According to the orthonormality of the basis and Ito calculus, we have, for any nonnegative integer \(l\),

\[
E[(A^l \eta, A^l \eta)] = \sum_{j=1}^{\infty} w_j^2 \lambda_j^l E\left[ \int_0^t e^{-\lambda_j(t-s)} \dot{\zeta}_j(s) \, ds \right]
\]

where \(\dot{\zeta}_j(t) = \sum_{k=1}^{\infty} w_k \lambda_k^l \Phi_k \int_0^t e^{-\lambda_k(t-s)} \dot{\zeta}_k(s) \, ds\)
\[
= \sigma^2 \sum_{j=1}^{\infty} w_j^2 \lambda_j^{2l-1} (1 - e^{-2\lambda_j}).
\]

If for some \(l\),
\[
\sum_{j=1}^{\infty} w_j^2 \lambda_j^{2l-1} < \infty,
\]
then \(\mathbb{E} \left[ \|A\eta\|_{L^2(D)}^2 \right] < \infty\). From (8) and (9), we observe that \(2l = \rho\). If we let \(l = 0\), and hence \(\sum_{j=1}^{\infty} w_j^2 / \lambda_j < \infty\), then \(\mathbb{E} \left[ \|\eta\|_{L^2(0,1)}^2 \right] < \infty\).

From the definition of covariance function, we calculate the covariance of the OU process \(\eta(t, x)\) at a given point \(x \in [0, 1]\), and it is given by
\[
\text{Cov} [\eta](t, s) := \mathbb{E}[(\eta(t, x) - \mathbb{E}[\eta(t, x)]) (\eta(s, x) - \mathbb{E}[\eta(t, x)])]
\]
\[
= \sigma^2 \sum_{j=1}^{\infty} \frac{w_j^2}{2\lambda_j} \Phi_j^2(x) (e^{-\lambda_j|t-s|} - e^{-\lambda_j(t+s)}).
\]

Setting \(t = s\), we have the variance of the OU process \(\eta(t, x)\) given by
\[
\text{Var} [\eta](t, x) = \sigma^2 \sum_{j=1}^{\infty} \frac{w_j^2}{2\lambda_j} \Phi_j^2(x) (1 - e^{-2\lambda_j t}).
\]

Note that the OU process is a stationary Gaussian process, which means that for any point \((t, x)\), \(\eta(t, x)\) has a normal distribution and its covariance function depends on a difference \(t - s\).

### 2.2. Stochastic Burgers equation

Let \(T \in (0, \infty)\), let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and let \((V, \|\cdot\|_V)\) and \((W, \|\cdot\|_W)\) be two Banach spaces of \(\mathbb{R}\)-valued functions. Moreover, let \(P_t : V \to V\), \(t \in \mathbb{N}\) be a sequence of bounded linear operators from \(V\) to \(V\).

**Assumption 1** (semigroup \(S_t\)). Let \(\gamma \in [0, 1)\) and \(\alpha \in (0, \infty)\) be real numbers and let \(S_t : (0, T] \to \mathcal{L}(W, V)\) be a strongly continuous mapping which satisfies
\[
\sup_{t \in (0, T]} (t^{\gamma} \|S_t\|_{\mathcal{L}(W, V)}) < \infty \quad \text{and} \quad \sup_{t \in (0, T]} (t^\alpha \|S_t - P_t S_t\|_{\mathcal{L}(W, V)}) < \infty.
\]

**Assumption 2** (nonlinearity \(F\)). Let \(F : V \to W\) be a mapping which satisfies
\[
\sup_{\|v\|_V \leq r, v \neq w} \frac{\|F(v) - F(w)\|_W}{\|v - w\|_W} < \infty, \quad \forall r \in (0, \infty).
\]

**Assumption 3** (stochastic process \(O\)). Let \(O : [0, T] \times \Omega \to V\) be a stochastic process with continuous sample paths and \(\sup_{d \in \mathbb{N}} \sup_{0 \leq t \leq T} d^\gamma \|O_t(\omega) - P_d(O_t(\omega))\|_V < \infty\) for every \(\omega \in \Omega\), where \(\gamma \in (0, \infty)\) is given in Assumption 1.
**Assumption 4** (existence of solutions). Let \( X^d : [0, T] \times \Omega \to V, \ d \in \mathbb{N} \) be a sequence of stochastic process with continuous sample paths and with

\[
X^d_t(\omega) = \int_0^t P_d S_{t-s} F(X^d_s(\omega)) ds + P_d(O_t(\omega)) \quad \text{and sup} \sup_{n \in \mathbb{N}} \sup_{0 \leq s \leq T} \|X^d_s(\omega)\|_V < \infty
\]

for every \( t \in [0, T], \ \omega \in \Omega, \) and every \( d \in \mathbb{N}. \)

Note that if \( Y : [0, T] \times \Omega \to V \) is a stochastic process with continuous sample paths, then Assumptions 1 and 2 ensure that for every \( \omega \in \Omega, \ t \in [0, T], \) and every \( d \in \mathbb{N}, \) the mapping \( s \mapsto P_d S_{t-s} F(Y_s(\omega)) \in V, \ 0 < s < t, \) is continuous, and therefore we obtain that for every \( \omega \in \Omega, \ t \in [0, T], \) and every \( d \in \mathbb{N}, \) the \( V\)-valued Bochner integral \( \int_0^t P_d S_{t-s} F(Y_s(\omega)) ds \in V \) is well defined. One can find the following theorem in [5].

**Proposition 2.2.** Let assumptions 1-4 be fulfilled. Then there exists a unique stochastic process \( X : [0, T] \times \Omega \to V \) with continuous sample paths which fulfills

\[
X_t(\omega) = \int_0^t S_{t-s} F(X_s(\omega)) ds + O_t(\omega)
\]

for every \( t \in [0, T] \) and every \( \omega \in \Omega. \) Moreover, there exists an \( \mathcal{F}/\mathcal{B}([0, \infty))\)-measurable mapping \( C : \Omega \to [0, \infty) \) such that

\[
\sup_{0 \leq t \leq T} \|X_t(\omega) - X^d_t(\omega)\|_V \leq C(\omega)d^{-\gamma}
\]

for every \( d \in \mathbb{N} \) and every \( \omega \in \Omega, \) where \( \gamma \in (0, \infty) \) is given in Assumption 1.

In our problem, we assume that the initial condition \( u_0(x) \) is deterministic and belonging to \( C[0,1], \) Then existence and uniqueness of solutions to the stochastic Burgers equation is presented by the following theorem.

**Theorem 2.3.** There exists a unique stochastic process \( u(t, \cdot) : \Omega \times [0, T] \to V \) satisfying (1) with continuous sample paths, which fulfills

\[
u(t, x) = \int_0^t e^{-(t-s)A} \frac{1}{2} \frac{\partial}{\partial x} u^2 ds + e^{-tA} u_0(x) + \eta(t, x)
\]

for every \( \omega \in \Omega, \) and every \( t \in [0, T]. \) Here, \( \eta \) is the OU process from Theorem 2.1 and \( \int_0^t e^{-(t-s)A} \frac{\partial}{\partial x} u^2 ds \) is interpreted as \( V\)-valued Bochner integral.

**Proof.** Let \( V = C[0,1] \) with \( \|v\|_V = \|v\|_{C[0,1]} \) for all \( v \in V \) and \( W = H^{-1}(0,1) \) with \( \|v\|_W = \|v\|_{H^{-1}} \) for all \( v \in W. \) We define the operator \( e^{-tA} : (0, T] \to \mathcal{L}(W, V) \)

\[
e^{-tA} v(x) := \sum_{j=1}^{\infty} e^{-tj\pi}(v, \sin(j\pi x)) \sin(j\pi x)
\]

for every \( x \in [0,1], \) \( v \in W \) and every \( t \in (0, T]. \) Whenever \( v \in L^2(0,1), \) the definition (15) coincides with (6). We also define \( \frac{\partial}{\partial x} : L^2(0,1) \to W \)
as the distributional derivative in \( L^2(0, 1) \), which is \( (\frac{\partial}{\partial x} v)(\varphi) := -(v, \varphi') \) for every \( \varphi \in H^1_0(0, 1) \) and every \( v \in L^2(0, 1) \). We use the projection operators \( P_d : V \to V, \ d \in \mathbb{N} \), defined by
\[
(P_d(v))(x) := \sum_{j=1}^{d} \int_{0}^{1} \sin(j\pi x)s v(s) ds \cdot \sin(j\pi x)
\]
for every \( x \in [0, 1], \ v \in V, \) and every \( d \in \mathbb{N} \). Then, according to Lemma 4.6 in [5], the operator (15) is well defined and satisfies Assumption 1 for every \( \gamma \in (0, 1/2) \). Let \( c \in (-\infty, \infty) \) be a real number. Then the mapping \( F : V \to W \) given by
\[
F(v) = c\frac{\partial}{\partial x}(G + \eta)^2
\]
for every \( v \in V \) satisfies Assumption 2 (Lemma 4.7 in [5]), so that we take \( c = 1/2 \). The above Proposition 2.1 and Proposition 4.2 in [5] say that the stochastic process \( \eta \) satisfies Assumption 3 for every \( \gamma \in (0, \rho) \). Finally, referring to Lemma 4.8 in [5], we can show that Assumption 4 is fulfilled. Therefore, the desired result follows Proposition 2.2 and the proof is completed.

According to the theorem, we present the solution of the stochastic Burgers equation (1) to be related to the OU process. By setting \( u = G + \eta \),
\[
G(t, x) = e^{-tA}u_0(x) + \int_{0}^{t} e^{-(t-s)A} \frac{\partial}{\partial x} (G + \eta)^2 ds,
\]
and hence, we transform (1) to the equivalent SPDE
\[
\begin{align*}
\frac{\partial}{\partial t} G - \nu \Delta G - \frac{1}{2} \frac{\partial}{\partial x} (G + \eta)^2 &= 0 \quad \text{in } (0, T] \times (0, 1), \\
G_{|[0]} = u_0(x), \quad G_{|[0,T] \times \{0\}} = G_{|[0,T] \times \{1\}} &= 0.
\end{align*}
\]
We see that (1) is driven by \( \xi \), whereas (17) has the colored noise \( \eta \) in a nonlinear term.

3. Approximation of the Ornstein-Uhlenbeck process and sparse grid stochastic collocation

3.1. Karhunen-Loève exapansion

We specialize results that hold for more general random processes having continuous, symmetric, and positive definite covariance functions to the case of the OU process. We consider determining pairs \( \{\tau_i(x), \psi_i(t, x)\}_{i=1}^{\infty} \) such that for any \( T > 0 \),
\[
\int_{0}^{T} \text{Cov} [\eta] (t, s; x) \psi_i(s, x) ds = \tau_i(x) \psi_i(s, x), \quad i = 1, 2, \ldots.
\]
From (10), \( \text{Cov} [\eta] (t, s; x) \) is positive definite and symmetric with respect to \( t \) and \( s \), so that \( \{\tau_i(x), \psi_i(t, x)\}_{i=1}^{\infty} \) exists. Moreover, they satisfy
\[
\int_{0}^{T} \psi_i(s, x) \psi_j(s, x) ds = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}
\]
and \( \tau_1(x) \geq \tau_2(x) \geq \cdots > 0 \). Since the \( \text{Cov} [\eta](t, s; x) \) is continuous in \( t \) for any given point \( x \in [0, 1] \), applying Mercer’s theorem ([8]) with respect to \( t \) and \( s \) gives

\[
\text{Cov} [\eta](t, s; x) = \sum_{i=1}^{\infty} \tau_i(x) \psi_i(t, x) \psi_i(s, x).
\]

Setting \( t = s \) again, we have \( \text{Var} [\eta](t, s; x) = \sum_{i=1}^{\infty} \tau_i(x) \psi_i^2(t, x) \). It follows that the OU process \( \eta(t, x) \) has \( \text{Karhunen-Loève (KL)} \) expansion

\[
\eta(t, x) = \sum_{i=1}^{\infty} \sqrt{\tau_i(x)} \psi_i(t, x) Y_i,
\]

where \( \{Y_i\}_{i=1}^{\infty} \) are i.i.d. standard normal random variables since \( \eta(t, x) \) is a Gaussian process at each \( x \in [0, 1] \). This brings us an another way to express the OU process, which is a decomposition of \( \eta(t, x) \) in terms of the eigenpairs \( \{\tau_i(x), \psi_i(t, x)\}_{i=1}^{\infty} \).

**Remark 3.1.** The variables \( Y_i \) actually depend on \( x \), i.e., \( Y_i = Y_i(x) \), however \( Y_i \) have same distributions of \( N(0, 1) \) for any given point \( x \in D \). Moreover, \( Y_i \) are uncorrelated to \( Y_j \) with different indices, which implies \( Y_i \) are independent each other. Thus, we can say that \( Y_i \) are independent and identically distributed random variables; see p. 144 in [17].

A possible approximation to the OU process \( \eta \) can be given by truncating its KL expansion (19).

\[
\eta_{N_{KL}}(t, x) = \sum_{i=1}^{N_{KL}} \sqrt{\tau_i(x)} \psi_i(t, x) Y_i.
\]

This kind of approximation of the OU process was introduced in [18]. It is known that the Galerkin projection of \( \eta(t, x) \) onto the space spanned by the basis function \( \{\psi_i\}_{i=1}^{N_{KL}} \) is, in the Fourier sense, a best approximation; see [15].

The effectiveness of (20) surely depends on the rate of decay of the KL eigenvalues. If eigenvalues decay very fast, then the finite term KL expansion would be good approximations the stochastic process. In this case, we only need to keep the leading order terms in the finite KL expansion and still capture the most of the energy of the OU process \( \eta \). Hence, when determining the number of truncated terms \( N_{KL} \), we use energy ratio defined by

\[
e_{N_{KL}} = \frac{\sum_{i=1}^{N_{KL}} ||\tau_i||_{L^2(0,1)}^2}{\sum_{i=1}^{\infty} ||\tau_i||_{L^2(0,1)}^2}.
\]

Of course, the KL eigenpairs can also only be determined approximately. We solve the integral equation (18) by quadrature rule, or equivalently saying, collocation method; see [2].
3.2. Stochastic collocation and Smolyak formula

In practice, we are usually interested in the $r$th statistical moment. After we determined the finite-dimensional approximation of random noise, the left problem is multi-dimensional integration. For example, the $r$th statistical moment of truncated KL expansion of $\eta_{N_{KL}}(t,x)$ is

$$E[\eta_{N_{KL}}^r(t,x)] = \int_{\mathbb{R}^{N_{KL}}} \eta_{N_{KL}}^r(t,x) d\mu(y_1) \cdots d\mu(y_{N_{KL}}),$$

where $d\mu$ refers to the standard normal distribution. Monte Carlo (MC) method is very simple and easy to implement, but it shows very slow convergences. We rather use a quadrature rule with tensor product in multi-dimensional space. Smolyak shows an impressive algorithm, which provides much less number of points than full tensor product formula in multi-dimensional space. The detailed explanation of stochastic collocation idea and the Smolyak approximation can be viewed in [3, 4, 19, 20, 21, 22].

The idea of the collocation method is to approximate the function $u(y,t,x)$ for all $y \in \Gamma$ and for all $(t,x) \in (0,T] \times D$, where $\Gamma = \prod_{n=1}^{N} \Gamma_n \subset \mathbb{R}^N$ with $\Gamma_n = Y_n(\Omega) \subset \mathbb{R}$ for the random variable $Y_n$, $n = 1, 2, \ldots, N$. Let $P_p(\Gamma) \subset L^2(\Gamma, d\mu)$ denote the span of tensor product polynomials with degree at most $p = (p_1 \ldots p_N)$, i.e.,

$$P_p(\Gamma) = \bigotimes_{n=1}^{N} P_{p_n}(\Gamma_n) \text{ with } P_{p_n}(\Gamma_n) = \text{span}(y_m^n, m = 0, \ldots, p_n),$$

where $n = 1, \ldots, N$.

Then $P_p(\Gamma) \subset L^2(\Gamma, d\mu)$, where $L^2(\Gamma, d\mu) = \{ v : \int_{\Gamma} |v(y)|^2 \mu(y) dy < \infty \}$.

Stochastic collocation entails the sampling of the solution $u(y_k)$ on a suitable set of points $y_k \in \Gamma$ and a global interpolation of the $u$

$$u^p(y, \cdot, \cdot) = \sum_k u(y_k, \cdot, \cdot) L_k^p(y),$$

where the functions $L_k^p$ can be Lagrange polynomials. Then approximated the $r$th moment is

$$E[u^r](t,x) \approx \sum_k u^r(y_k, t, x) \int_{\Gamma} L_k^p(y) \mu(y) dy.$$

In practice, we use the fully discrete solution by some numerical schemes rather than the exact solution.

We introduce an index $i \in \mathbb{N}_+$. Then, for each value of $i$, let $\{y_{i1}^1, \ldots, y_{im}^i\} \subset \mathbb{R}$ be a sequence of abscissas for Lagrange interpolation on $\mathbb{R}$. Let $W := W(0,T; D)$ be a Banach space of functions $v : (0,T] \times D \to \mathbb{R}$. For $u \in C^0(\Gamma_n; W)$ with $N = 1$, we introduce a sequence of one-dimensional Lagrange interpolation operators $\mathcal{L}_i : C^0(\Gamma; W) \to V_m(\Gamma; W)$ such that

$$\mathcal{L}_i^i(u)(y) = \sum_{j=1}^{m_i} u(y_j^i) L_j^i(y) \quad \forall u \in C^0(\Gamma; W),$$

(22)
where \( L_k \in \mathcal{P}_{m_i - 1}(\Gamma) \) are Lagrange polynomials of degree \( p_i = m_i - 1 \), i.e.,

\[
L_k^p(y) = \prod_{j=1, j \neq k}^{m_i} \frac{(y - y_k^i)}{(y_j^i - y_k^i)}
\]

and

\[
V_{m_i}(\Gamma; W) = \left\{ v \in C^0(\Gamma_n; W) : v(y, t, x) = \sum_{k=1}^{m_i} \tilde{v}_k(t, x) L_k^p(y), \{ \tilde{v}_k \}^{m_i}_{k=1} \in W \right\}.
\]

The formula (22) reproduces exactly all polynomials of degree less than \( m_i \).

Now, in the multivariate case \( N > 1 \), for each \( u \in C^0(\Gamma_n; W) \) and the multi-index \( i = (i_1, \ldots, i_N) \in \mathbb{N}_+^N \), we define the full tensor product interpolation formulas

\[
I_N^i u(y) = (\mathcal{Y}^{i_1} \otimes \cdots \otimes \mathcal{Y}^{i_N})(u)(y)
= \sum_{k_1=1}^{m_{i_1}} \cdots \sum_{k_N=1}^{m_{i_N}} u(y_{k_1} \cdots y_{k_N}) \left( L_{k_1}^{p_{i_1}} \otimes \cdots \otimes L_{k_N}^{p_{i_N}} \right).
\]

Clearly, the above product needs \( \prod_{n=1}^N m_{i_n} \) function evaluations. These formula will also be used as the building blocks for the Smolyak formula, described in the next.

Here, we describe the Smolyak isotropic formulas \( \mathcal{A}(q, N) \) ([6, 24, 25]), which means that all directions are treated equally. With \( \mathcal{Y}^0 = 0 \) and for \( i \in \mathbb{N}_+ \) define

\[
\Delta^i = \mathcal{Y}^i - \mathcal{Y}^{i-1}.
\]

Moreover, given an integer \( q \in \mathbb{N}_+ \), hereafter called the *level*, and for \( i \in \mathbb{N}_+^N \) with \( |i| = i_1 + \cdots + i_N \), the Smolyak algorithm is given by

\[
\mathcal{A}(q, N) = \sum_{|i| \leq q} (\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_N}).
\]

To compute \( \mathcal{A}(q, N)(u) \), we only needs to know function values on the sparse grids

\[
\mathcal{H}(q, N) = \bigcup_{q-N+1 \leq |i| \leq q} (\mathcal{Y}^{i_1} \otimes \cdots \otimes \mathcal{Y}^{i_N}),
\]

where \( \mathcal{Y}^i = \{ y_1^i, \ldots, y_{m_i}^i \} \subset \mathbb{R} \) denote the set of nodes used by \( \mathcal{Y}^i \).

We look into the Smolyak algorithm based on polynomial interpolation at the zeros of the *Hermite polynomials*, which are called the *Gauss-Hermite abcissas*. It is well-known by [8] that the Hermite polynomials are a complete orthonormal basis in the Hilbert space \( L^2(\mathbb{R}, d\mu) \) where \( d\mu = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \). Hence, it is a natural choice for the standard normal distribution.
4. Two finite element methods for the Burgers equation

4.1. The Galerkin method

For a while, we consider the deterministic analogue of the stochastic Burgers equation (1). We also represent (1) as non-conservation form, which is mathematically equivalent.

\begin{equation}
\begin{cases}
\frac{\partial}{\partial t} u - \nu \Delta u - u \frac{\partial}{\partial x} u = f & \text{in } (0, T] \times (0, 1), \\
u u|_{(0, T] \times \{0\}} = u_0(x) & \text{and } u|_{(0, T] \times \{1\}} = 0,
\end{cases}
\end{equation}

where \( f(t, x) \) is a deterministic function. We may think that \( f(t, x) \) is one realization of \( \xi(t, x) \) in (1).

The conventional finite element approximation follows. Let \( V^h \) be a finite-dimensional subspace of \( H^1_0(0, 1) \). Then the semi-discretization in space domain leads to the following problem: Find \( u^h(t, \cdot) \in V^h, \ t \in (0, T] \) such that

\begin{equation}
\begin{cases}
(u^h_t, \nu \Delta u^h - u^h \frac{\partial}{\partial x} u^h) + (u^h u^h_x, v^h) = (f, v^h) & \text{for all } v^h \in V^h, \\
(u^h|_{(0, T] \times \{0\}}, v^h) = (u_0^h(x), v^h)
\end{cases}
\end{equation}

where \( u_0^h(x) \in V^h \) is a projection of \( u_0(x) \) to the space \( V^h \).

Let us write (24) by the representation

\[ u^h(t, x) = \sum_{j=1}^{J} \alpha_j(t) \phi_j(x), \quad t \in (0, T] \]

with time-dependent coefficients \( \alpha_j(t) \). Using this representation and taking \( v^h = \phi_j, j = 1, \ldots, J \), which are finite element basis functions, we have the nonlinear system of ordinary differential equations that determine the coefficient.
functions \( \{\alpha_j(t)\}_{j=1}^J \)

\[
\begin{cases}
\mathbf{M} \frac{d}{dt} \vec{\alpha}(t) + \mathbf{S} \vec{\alpha}(t) + (\vec{\alpha}(t))^T \mathbf{N} \vec{\alpha}(t) = \vec{f}(t) \\
\mathbf{M} \vec{\alpha}(0) = \vec{\alpha}_0,
\end{cases}
\]

where the mass matrix \( \mathbf{M} \), stiffness matrix \( \mathbf{S} \), convection tensor \( \mathbf{N} \), and solution vector \( \vec{\alpha}(t) \) are given by

\[
\mathbf{M}_{jk} = (\phi_j, \phi_k), \quad \mathbf{S}_{jk} = (\phi_j', \phi_k'), \quad \mathbf{N}_{jlk} = (\phi_j \phi_l', \phi_j), \quad \text{and} \quad (\vec{\alpha}(t))_k = \alpha_k(t)
\]

for \( j, k, l = 1, \ldots, J \) respectively. The force vector \( \vec{f}(t) \) and the initial condition vector \( \vec{\alpha}_0 \) are given by

\[
(\vec{f}(t))_k = (f(t, \cdot), \phi_k) \quad \text{and} \quad (\vec{\alpha}_0)_k = (u_0, \phi_k)
\]

for \( k = 1, \ldots, J \).

### 4.2. The Galerkin-conservation method

The nonlinear term \( (\vec{\alpha}(t))^T \mathbf{N} \vec{\alpha}(t) \) of (25) can complicate computations for large value \( J \), which is the number of elements. For this reason, we seek an alternative treatment of the nonlinear term, which is introduced in [14]. Here, we write the Burgers equation in the conservation form (1) again and approximate nonlinear term \( u^2(t, x) \) to

\[
(u^h(t, x))^2 = \sum_{j=1}^J \alpha_j^2(t) \phi_j(x).
\]

When the Galerkin method is applied to that approximation subject to the deterministic force term \( f(t, x) \) same as (23), the system of nonlinear ordinary differential equations is given by

\[
\begin{cases}
\mathbf{M} \frac{d}{dt} \vec{\alpha}(t) + \mathbf{S} \vec{\alpha}(t) + \mathbf{N}_c \vec{\alpha}^2(t) = \vec{f}(t) \\
\mathbf{M} \vec{\alpha}(0) = \vec{\alpha}_0,
\end{cases}
\]

where the all terms are same as (26) except nonlinear part \( \mathbf{N}_c \vec{\alpha}^2(t) \), which are

\[
(\mathbf{N}_c)_{jk} = (\phi_j', \phi_k) \quad \text{and} \quad (\vec{\alpha}^2(t))_k = \alpha_k^2(t)
\]

for \( j, k = 1, \ldots, J \). The force vector \( \vec{f}(t) \) and the initial condition vector \( \vec{\alpha}_0 \) are also same as (27). Discretization of (25) or (28) is completed by choosing a time marching method such as the backward-Euler scheme. We can simulate (1) by the Galerkin method and the Galerkin-conservation method with the noise approximations (30) for both equations. On the other hand, with the OU processes and its KL approximations, we can also compute (17) by the Galerkin-conservation method and the Galerkin method respectively.

We will refer to this method as the Galerkin-conservation method, which have nonlinear term simpler than (25) as shown in (28), therefore the Galerkin-conservation method is faster that the Galerkin method for fine grids. Based
on our numerical experiments accuracy, however, the accuracy of Galerkin-conservation method and the Galerkin method shows no difference indeed.

5. Numerical experiments

To test the accuracy of the present method, three numerical examples are given in this section. All examples are implemented for the same spatial discretization with grid size $h = 1/128$ for the finite element method using continuous piecewise linear functions and temporal discretization with $\Delta t = 5 \times 10^{-3}$ for the backward Euler method.

Example 5.1. Numerical solution of the stochastic Burgers equation with a viscosity $\nu = 0.01$, noise parameters $\sigma = 0.25$, $w_j = 1/j$, final time $T = 1$, and an initial condition $u|_{t=0} = \sin(\pi x)$.

To simulate (1) directly, let $\{t_k = k\Delta t\}_{k=0}^M$ be a partition of $[0,T]$ with $\Delta t = T/M$. Then for each $\tilde{\zeta}_j(t)$, a piecewise constant approximation of white noise is given by [1, 12]

$$\hat{\zeta}_{j,M}(t) := \frac{1}{\sqrt{\Delta t}} \sum_{k=1}^{M} Y_{jk} \chi_{jk}(t) = \sqrt{\frac{M}{T}} \sum_{k=1}^{M} Y_{jk} \chi_{jk}(t),$$

where $\{\chi_k(t)\}_{k=1}^{M}$ denotes the set of characteristic functions and $Y_{jk} \sim N(0, 1)$ denote i.i.d. standard normal variables, which satisfy $\sqrt{\Delta t} Y_{jk} = \int_{t_{k-1}}^{t_k} \tilde{\zeta}_j(s) ds$ for $k = 1, \ldots, M$.

With the finite-dimensional white noise (29), we approximate $\xi(t, x)$ as

$$\hat{\xi}_{N,M}(t, x) := \sum_{j=1}^{N} w_j \Phi_j(x) \hat{\zeta}_{j,M}(t) = \sqrt{\frac{2M}{T}} \sum_{j=1}^{N} w_j \sin(j \pi x) \sum_{k=1}^{M} Y_{jk} \chi_{jk}(t).$$

To perform realizations, each of them requires sampling of the $MN$ standard normal random variables $Y_{jk}$. This provides a direct approach toward approximating the solution of Example 5.1.

We also use the indirect approach. As stated in Section 2.2, setting $u = G + \eta$ with the OU process $\eta$, we obtain the new equivalent equation like (17) for the conservation form. In the same way, we can derive the non-conservation form. Keep in mind that the OU process is approximated by truncated KL expansion.

Figure 2 and Figure 3 show realizations of the solution $u$ and its time snapshots at $T = 0$, 0.2, 0.5, and 1 by the direct simulation and the indirect approach using the OU process respectively. For the direct simulation we use sampling $MN = 128 \times 200 = 25600$ from i.i.d. standard normal random variables, meanwhile for the indirect approach, we approximate the OU process for 10-dimensional KL expansion. One can observe that Figure 3 looks smoother than Figure 2.
**Figure 2.** Realizations of the solution to Example 5.1 and its time snapshots at $T = 0$, $0.2$, $0.5$, and $1$ through the direct simulation with the Galerkin method (the first column) and the Galerkin-conservation method (the second column).

**Example 5.2.** Numerical solution of the stochastic Burgers equation with a viscosity $\nu = 0.01$, noise parameters $\sigma = 0.1$, $w_j = 1/j^{3/2}$, final time $T = 1$, and an initial condition

$$
|\{0\} \times [0,1] = \begin{cases} 
1 - 3|x - \frac{1}{2}| & \text{if } \frac{1}{6} \leq x \leq \frac{5}{6} \\
0 & \text{otherwise.}
\end{cases}
$$

We use the sparse grid stochastic collocation method for approximating the $r$th statistical moments in Example 5.2. To determine the dimension of random parameter space to approximate the OU process by KL expansion, we make computations of the eigenvalues corresponding to $\text{Cov} \{\eta\}(t,s;x)$ in (10) for given parameters in Example 5.2 by the $L^2(0,1)$-norm and the related energy ratios stated in (21). Table 1 shows that the first 12 eigenvalues of KL expansion of the OU process. The denominator of the energy ratios $e_{N_{KL}}$ are approximated over 1000 terms.
Figure 3. Realizations of the solution to Example 5.1 and its time snapshots at $T = 0, 0.2, 0.5,$ and 1 through the indirect approach using the OU process with the Galerkin method (the first column) and the Galerkin-conservation method (the second column).

Table 1. The $L^2(0,1)$-norm of the first 12 KL eigenvalues and the energy ratios for $w_j = 1/j^{3/2}$.

<table>
<thead>
<tr>
<th>#</th>
<th>Eigenvalue</th>
<th>Energy ratio</th>
<th>#</th>
<th>Eigenvalue</th>
<th>Energy ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1499</td>
<td>0.8059</td>
<td>7</td>
<td>0.0009</td>
<td>0.9704</td>
</tr>
<tr>
<td>2</td>
<td>0.0171</td>
<td>0.8981</td>
<td>8</td>
<td>0.0007</td>
<td>0.9741</td>
</tr>
<tr>
<td>3</td>
<td>0.0062</td>
<td>0.9313</td>
<td>9</td>
<td>0.0005</td>
<td>0.9770</td>
</tr>
<tr>
<td>4</td>
<td>0.0032</td>
<td>0.9483</td>
<td>10</td>
<td>0.0004</td>
<td>0.9793</td>
</tr>
<tr>
<td>5</td>
<td>0.0019</td>
<td>0.9586</td>
<td>11</td>
<td>0.0004</td>
<td>0.9812</td>
</tr>
<tr>
<td>6</td>
<td>0.0013</td>
<td>0.9654</td>
<td>12</td>
<td>0.0003</td>
<td>0.9827</td>
</tr>
</tbody>
</table>

Note that from the first and second moments, the mean of solution is obtained by $u_m = \mathbb{E}[u]$ directly, and the standard deviation of solution is computed by $u_{sd} = \sqrt{\mathbb{E}[u^2] - u_m^2}$ easily. Recall that each scheme is divided into two finite element approximation for space discretization again: the Galerkin method and the Galerkin-conservation method.
Let \( Q_{N_{KL}}^k \) denote the Smolyak quadrature rule for \( N_{KL} \) dimension with the level \( k \). Figure 4 and Figure 5 present the means and variances of solutions using the MC method with 12000 realizations (the first row) and the indirect method with KL approximation and the Smolyak rule of \( Q_{210}^{10} \) (the second row) by the conventional Galerkin method and the Gerlarkin-conservation method respectively. We emphasize that the Smolyak quadrature \( Q_{210}^{10} \) uses 217 points, which is much less than 12000 realizations of the MC method while the results are similar.

We also compare the \( r \)th statistical moments obtained by the MC method and KL-S expansion method. To achieve them, we define the following error measures

\[
E_{MC,KL}^{T,r} := \int_0^T \left\| E\left[u_{MC}^{r}\right] - E\left[u_{KL}^{r}\right] \right\|_{L^2(0,1)} dt,
\]

where \( E[u_{MC}^{r}] \) and \( E[u_{KL}^{r}] \) denote the \( r \)th statistical moments obtained by the MC method and KL-S approximation respectively. Since we cannot find the explicit exact moments of solution, we set the solution with Smolyak quadrature
Figure 5. The means and standard deviations of the MC with \( S = 12000 \) (the first row) and the KL approximation with the Smolyak rule of \( Q_{10}^2 \) (217 points) (the second row) by the Gerlarkin-conservation method.

\( Q_{11}^2 \) approximation as the benchmark solutions. We implemented four cases: the MC method and KL-S approximation by the Galerkin methods, and the MC method and KL-S approximation by the Galerkin-conservation method.

Now we check the convergent rates of spatial discretization for the Galerkin method and the Galerkin-conservation method to approximate the Burgers equation in stochastic settings. We solve the equations as before with the Smolyak quadrature rule \( Q_{11}^2 \), varying space discretizations for \( h = 1/16, 1/32, 1/64, 1/128, 1/256 \). It is known that in the deterministic Burgers equation case, the \( L^2 \)-error at time \( t \) converges with the rate \( O(h^{1/2}) \) in both the Galerkin and the Galerkin-conservation methods theoretically, which can be found in [14].

To verify those facts in the stochastic cases, we define, at the final time \( T = 1 \),

\[
\epsilon^{T,r}_{GC,GC} := \left\| \left( \mathbb{E} \left[ u_{GC}^T(T, \cdot) \right] \right)^{1/r} - \left( \mathbb{E} \left[ u^T(T, \cdot) \right] \right)^{1/r} \right\|_{L^2(0,1)},
\]

where \( \mathbb{E}[u^T(t, x)] \) and \( \mathbb{E}[u_{GC}^T(t, x)] \) are the \( r \)th statistical moments given by the Galerkin method and the Galerkin-conservation method respectively, and
\( E[u^r(t, x)] \) is the exact \( r \)th moments of solution. When we compute \( e^{T,r}_{G\rightarrow GC} \) using approximated solutions, the convergent rates are measured by \( \text{Rate} = \frac{e^{T,r}_{G\rightarrow GC}(h)}{e^{T,r}_{G\rightarrow GC}(h/2)} \) for given \( r \) and \( h \). In our numerical example, we do not know explicit exact moments of solution, so that the solutions with \( h = 1/128 \) becomes the benchmark solutions as \( E[u^r(t, x)] \) in the both methods.

Meanwhile, we measure the CPU times of each method for total computations over \( Q_5^2 \) points. The programs were written in MATLAB Version R2014a and executed on a 3.0 GHz INTEL Xeon processor.

**Table 2.** The errors of \( r \)th moments of the solutions over \( Q_5^2 \) for \( r = 1, 2, 3, 4 \) by the Galerkin method.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( e^1_i )</th>
<th>Rate</th>
<th>( e^2_i )</th>
<th>Rate</th>
<th>( e^3_i )</th>
<th>Rate</th>
<th>( e^4_i )</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
<td>0.6308</td>
<td></td>
<td>0.6322</td>
<td></td>
<td>0.6344</td>
<td></td>
<td>0.6351</td>
<td></td>
</tr>
<tr>
<td>1/16</td>
<td>0.3996</td>
<td>1.5786</td>
<td>0.3980</td>
<td>1.5884</td>
<td>0.3985</td>
<td>1.5920</td>
<td>0.3987</td>
<td>1.5929</td>
</tr>
<tr>
<td>1/32</td>
<td>0.2406</td>
<td>1.6608</td>
<td>0.2385</td>
<td>1.6688</td>
<td>0.2385</td>
<td>1.6709</td>
<td>0.2386</td>
<td>1.6710</td>
</tr>
<tr>
<td>1/64</td>
<td>0.1120</td>
<td>2.1833</td>
<td>0.1120</td>
<td>2.1295</td>
<td>0.1120</td>
<td>2.1295</td>
<td>0.1121</td>
<td>2.1285</td>
</tr>
</tbody>
</table>

**Table 3.** The errors of \( r \)th moments of the solutions over \( Q_5^2 \) for \( r = 1, 2, 3, 4 \) by the Galerkin-conservation method.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( e^1_i )</th>
<th>Rate</th>
<th>( e^2_i )</th>
<th>Rate</th>
<th>( e^3_i )</th>
<th>Rate</th>
<th>( e^4_i )</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
<td>0.5401</td>
<td></td>
<td>0.5411</td>
<td></td>
<td>0.5419</td>
<td></td>
<td>0.5433</td>
<td></td>
</tr>
<tr>
<td>1/16</td>
<td>0.3636</td>
<td>1.4854</td>
<td>0.3632</td>
<td>1.4898</td>
<td>0.3632</td>
<td>1.4920</td>
<td>0.3631</td>
<td>1.4963</td>
</tr>
<tr>
<td>1/32</td>
<td>0.2250</td>
<td>1.6100</td>
<td>0.2255</td>
<td>1.6106</td>
<td>0.2255</td>
<td>1.6106</td>
<td>0.2255</td>
<td>1.6102</td>
</tr>
<tr>
<td>1/64</td>
<td>0.1087</td>
<td>2.0699</td>
<td>0.1086</td>
<td>2.0764</td>
<td>0.1085</td>
<td>2.0783</td>
<td>0.1087</td>
<td>2.0745</td>
</tr>
</tbody>
</table>

Table 2 presents the \( L^2 \)-errors of each statistical moments at the final time and convergent rates by the Galerkin method. The same items under the Galerkin-conservation method are given in Table 3. We see that in the both methods, the convergent rates of our numerical moments are slightly better than the theoretical rate \( h^{1/2} \). However, the Galerkin-conservation method is computationally faster than the Galerkin method, especially when the spatial grids are fine, i.e., \( h \) is small. Table 4 confirms the former statement. It shows the CPU seconds for \( h = 1/8, 1/16, 1/32, 1/64, 1/128 \) for the Galerkin method and the Galerkin-conservation method respectively. For example, as \( h \) decreases to \( 1/32 \) and \( 1/64 \) the CPU times of the both methods dramatically increase but the Galerkin method consumes twice bigger time than the Galerkin-conservation method.

In our experiments, relatively low level and dimension for Smolyak quadrature rule are used. Therefore, the results suggests that the Galerkin-conservation method can be more effective in approximating the statistical moments of solutions of the stochastic Burgers equation if higher level and dimension are
Table 4. The CPU seconds of the Galerkin method and the Galerkin-conservation method.

<table>
<thead>
<tr>
<th>$h$</th>
<th>Galerkin</th>
<th>Galerkin-conservation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
<td>9.12</td>
<td>6.21</td>
</tr>
<tr>
<td>1/16</td>
<td>14.59</td>
<td>8.32</td>
</tr>
<tr>
<td>1/32</td>
<td>23.77</td>
<td>18.87</td>
</tr>
<tr>
<td>1/64</td>
<td>368.49</td>
<td>221.11</td>
</tr>
<tr>
<td>1/128</td>
<td>1065.04</td>
<td>533.46</td>
</tr>
</tbody>
</table>

needed. It is worthwhile to note that when we need to exploit finer grids for physical domains (time and space), the Galerkin-conservation method is very attractive for stochastic settings since we have to compute the deterministic equations repeatedly. The Galerkin-conservation method consumes less time but performs similar accuracy among given sample points.

It is well known that the MC moments of solutions tend to the exact stochastic moments of numerical solutions as the sample size increases and the convergent rate is $O(N^{-1/2})$ for the sample size $N$. Figure 6 confirms that the convergent rate of the MC method again. Here, in Galerkin and Galerkin-conservation method, we use KL-S approximated solutions over $Q_{11}^{2}$ (309 points) as the benchmark solutions for both MC and KL-S cases.

From this convergence comparison between MC method and KL-S approximation, in the figures, we can also deduce that the KL-S approximations converges faster than the MC method as the sample sizes increase. Therefore, the effectiveness of the low-dimensional approximation in random parameter space, based on KL expansion, is verified for higher-order statistical moments as well.

6. Conclusion

It spends expensive costs to obtain precise statistics about solutions of the stochastic Burgers’ equation driven by space-time additive noise in the form of (2) due to the great number of dimensions in random parameter space, which causes the curse of dimensionality inevitably. Recall that the OU process transforms the space-time additive noise to the stationary Gaussian colored noise, which is a more regularized one.

If one settles on the OU process, then one can take advantage of the KL expansion. Regardless of discussing with the stochastic jargons, the truncated KL expansion is essentially in common use for reduced-order modeling (ROM) of deterministic systems and develops a low-dimensional approximation to stochastic systems. Sparse grid collocation technique provides a very powerful tool to overcome the curse of dimensionality of stochastic systems and we combined it with the truncated KL expansion in this paper.

The results given in Section 5 suggest that the numerical solution based on the OU process with the KL approximation can be very effective in computation
for the statistical moments of solutions to our nonlinear equation. Sparse grid collocation works still efficiently in our problem.

When approximating the Burgers’ equation by the FEM, the Galerkin-conservation method saves computing times greater than the conventional Galerkin method. We remark that stochastic problems require repeated computations of deterministic equation in many cases. Therefore, it is worthwhile to note that a little improvement for an efficacy of deterministic techniques begets much strong performance in stochastic settings.

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