SELF-HOMOTOPY EQUIVALENCES RELATED TO COHOMOTOPY GROUPS

HO WON CHOI, KEE YOUNG LEE, AND HYUNG SEOK OH

Abstract. Given a topological space $X$ and a non-negative integer $k$, we study the self-homotopy equivalences of $X$ that do not change maps from $X$ to $n$-sphere $S^n$ homotopically by the composition for all $n \geq k$. We denote by $\mathcal{E}_k^k(X)$ the set of all homotopy classes of such self-homotopy equivalences. This set is a dual concept of $\mathcal{E}_k^k(X)$, which has been studied by several authors. We prove that if $X$ is a finite CW complex, there are at most a finite number of distinguishing homotopy classes $\mathcal{E}_k^k(X)$, whereas $\mathcal{E}_k^k(X)$ may not be finite. Moreover, we obtain concrete computations of $\mathcal{E}_k^k(X)$ to show that the cardinal of $\mathcal{E}_k^k(X)$ is finite when $X$ is either a Moore space or co-Moore space by using the self-closeness numbers.

1. Introduction

Throughout this paper, all topological spaces are based and have the based homotopy type of a CW-complex, and all maps and homotopies preserve base points. For the spaces $X$ and $Y$, we denote by $[X,Y]$ the set of homotopy classes of maps from $X$ to $Y$. No distinction is made between the notation of a map $X \to Y$ and that of its homotopy class in $[X,Y]$. Let $S^n$ be the $n$-sphere. Then, $[S^n,Y]$ is known as the $n$-th homotopy group of space $Y$, denoted by $\pi_n(Y)$ and $[X,S^n]$ is referred to as the $n$-th cohomotopy group of $X$, denoted by $\pi^n(X)$.

Given $X$, we denote by $\mathcal{E}(X)$ the set of all homotopy classes of self-homotopy equivalences of $X$. Then, $\mathcal{E}(X)$ is a subset of $[X,X]$ and has a group structure given by the composition of homotopy classes. $\mathcal{E}(X)$ has been studied extensively by various authors, including Arkowitz [2], Maruyama [3], Lee [7], Rutter [8], Sawashita [9], and Sieradski [10]. Moreover, several subgroups of $\mathcal{E}(X)$ have also been studied, notably the subgroup $\mathcal{E}_k^k(X)$, which consists of the elements of $\mathcal{E}(X)$ that induce the identity homomorphism on homotopy...
groups $\pi_i(X)$ for $i = 0, 1, 2, \ldots, k$. In [3], Arkowitz and Maruyama introduced and determined these subgroups for Moore spaces and co-Moore spaces using the homological method. In [4], the second and third authors used homotopy techniques to calculate these subgroups for the wedge products of Moore spaces.

Given a topological space $X$ and a non-negative integer $k$, consider the self-map $f : X \to X$ such that $g \circ f$ is homotopic to $g$ for each $g : X \to S^n$ and for each $n \geq k$. We denote by $[X, X]_k^\sharp$ the set of all homotopy classes of such self-maps of $X$, that is,

$$[X, X]_k^\sharp = \{ f \in [X, X] \mid g \circ f \sim g \text{ for each } g : X \to S^n, \text{ for all } n \geq k \}.$$  

This set has a monoid structure by composition. We define

$$E_k^\sharp(X) = E(X) \cap [X, X]_k^\sharp.$$  

Then, it is easy to prove that $E_k^\sharp(X)$ is a subgroup of $E(X)$ and has a lower bound, whereas $E_k^\sharp(X)$ has an upper bound.

In this paper, we compute these subgroups of Moore spaces and co-Moore spaces, by first showing that if $X$ is a finite CW complex, then there are at most a finite number of distinguishing subgroups $E_k^\sharp(X)$.

When $G$ is an abelian group, we let $M(G, n)$ denote the Moore space; i.e., the space in which $G$ is a single non-vanishing homology group at the $n$-level. We note that if $n \geq 3$, then $M(G, n)$ is characterized by

$$\tilde{H}^i(M(G, n)) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

Furthermore, we let $C(G, n)$ denote the co-Moore space of type $(G, n)$ defined by

$$\tilde{H}^i(C(G, n)) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

Next, we compute $E_k^\sharp(X)$ for $X = M(Z_p, n)$ or $X = C(Z_p, n)$ to obtain the following tables:

<table>
<thead>
<tr>
<th>$p$ odd</th>
<th>$p \equiv 0 \pmod{4}$</th>
<th>$p \equiv 2 \pmod{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{n-1}^\sharp(M(Z_p, n))$</td>
<td>1</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$E_n^\sharp(M(Z_p, n))$</td>
<td>1</td>
<td>$\mathbb{Z}_2$</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>$p$ odd</th>
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<th>$p \equiv 2 \pmod{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{n+1}^\sharp(M(Z_p, n))$</td>
<td>1</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$E_{n}^\sharp(M(Z_p, n))$</td>
<td>1</td>
<td>$\mathbb{Z}_2$</td>
</tr>
</tbody>
</table>

Henceforth, when a group $G$ is generated by a set $\{a_1, \ldots, a_n\}$, then we denote the group by $G\langle a_1, \ldots, a_2 \rangle$ or $G = \langle a_1, \ldots, a_n \rangle$. Moreover, when $f : X \to Y$ is a map, $f_{\#} : \pi_k(X) \to \pi_k(Y)$ and $f^{\#} : \pi_k(Y) \to \pi_k(X)$ denote the
induced homomorphisms in $k$-th homotopy group and $k$-th cohomotopy group, respectively.

2. Preliminaries

In this section, we review and summarize selected definitions and results provided in [1, 3, 5, 12], knowledge of which would be useful when reading this paper.

First, we summarize the concepts and results introduced in [5]. For any non-negative integer $n$, $A^n_\#(X)$ consists of the homotopy classes of the self-map of $X$ that induce an automorphism from $\pi_i(X)$ to $\pi_i(X)$ for $i = 0, 1, \ldots, n$. $A^k_\#(X)$ is a submonoid of $[X, X]$ and always contains $E(X)$. If $n = \infty$, we briefly denote $A^\infty_\#(X)$ as $A_\#(X)$. If $k < n$, then $A^n_\#(X) \subseteq A^k_\#(X)$; thus, we have the following chain by inclusion:

$E(X) \subseteq A_\#(X) \subseteq \cdots \subseteq A^1_\#(X) \subseteq A^0_\#(X) = [X, X]$.

**Definition 2.1.** Let $X$ be a CW complex. The self-closeness number of $X$ is the minimum number $n$ such that $A^n_\#(X) = E(X)$ and is denoted here by $N\mathcal{E}(X)$. That is,

$$N\mathcal{E}(X) = \min\{n \mid A^n_\#(X) = E(X) \text{ for } n \geq 0\}.$$ 

By [5, Theorem 1], the self-closeness number is a homotopy invariant. Moreover, if $X$ is an $n$-connected space with dimension $m$ and $E(X) \neq [X, X]$, then we have $n < N\mathcal{E}(X) \leq m$ by [5, Lemma 4 and Theorem 2].

**Proposition 2.1 ([3]).** If $X$ is $(k-1)$-connected, $Y$ is $(l-1)$-connected, and further, if $k, l \geq 2$, and $\dim P \leq k + l - 1$, then the projections $X \vee Y \to X$ and $X \vee Y \to Y$ induce a bijection.

$$[P, X \vee Y] \to [P, X] \oplus [P, Y].$$

By [12], the generators of some homotopy groups of spheres can be summarized as follows.

<table>
<thead>
<tr>
<th>$[S^{n+1}, S^n]$</th>
<th>$i &lt; 0$</th>
<th>$i = 0$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4, 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generator</td>
<td>$\iota_n$</td>
<td>$\eta_n$</td>
<td>$\eta^n_p$</td>
<td>$\nu_n$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Here, we note that for the Moore space $M(\mathbb{Z}_p, n) = S^n \cup_p e^{n+1}$, there exists a mapping cone sequence

$$S^n \overset{p}{\to} S^n \overset{i}{\to} S^n \cup_p e^{n+1} \overset{\pi}{\to} S^{n+1} \overset{p}{\to} S^{n+1},$$

where $p$ is a map of degree $p$, $i$ is an inclusion and $\pi$ is a quotient map. In [1], Araki and Toda computed the homotopy groups, and cohomotopy groups of $M(\mathbb{Z}_p, n)$, and set of homotopy classes of self-maps on $M(\mathbb{Z}_p, n)$. The results can be summarized as follows.

1. Homotopy group $\pi_k(M(\mathbb{Z}_p, n))$ for $k = n, n + 1$:

$$\pi_n(M(\mathbb{Z}_p, n)) \cong \mathbb{Z}_p \{i^*(\iota_n)\}$$
and

\[ \pi_{n+1}(M(\mathbb{Z}_p, n)) \cong \begin{cases} 0 & \text{if } p = \text{odd}, \\ \mathbb{Z}_2 \{i_2(\eta_n)\} & \text{if } p = \text{even}. \end{cases} \]

2. Cohomotopy groups \( \pi^{n+i-3}(M(\mathbb{Z}_p, n)) \):

<table>
<thead>
<tr>
<th>( i \leq 0 )</th>
<th>( i = 1 )</th>
<th>( i = 2 )</th>
<th>( i = 3 )</th>
<th>( i = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \equiv 1 \pmod{2} )</td>
<td>( \mathbb{Z}_p )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( \mathbb{Z}_{(p,24)} )</td>
</tr>
<tr>
<td>( p \equiv 0 \pmod{4} )</td>
<td>( \mathbb{Z}_p )</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 \oplus \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}<em>2 \oplus \mathbb{Z}</em>{(p,24)} )</td>
</tr>
<tr>
<td>( p \equiv 2 \pmod{4} )</td>
<td>( \mathbb{Z}_p )</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_4 )</td>
<td>( \mathbb{Z}<em>2 \oplus \mathbb{Z}</em>{(p,24)} )</td>
</tr>
</tbody>
</table>

Generators: \( \iota_n \circ q, \eta_n \circ q, \bar{\eta}_n, \eta_n, \eta_n, \eta_n, \iota_n \circ q \)

3. The set of homotopy classes \([M(\mathbb{Z}_p, n), M(\mathbb{Z}_p, n)]\):

<table>
<thead>
<tr>
<th>( p \equiv 1 \pmod{2} )</th>
<th>( p \equiv 2 \pmod{4} )</th>
<th>( p \equiv 0 \pmod{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([M(\mathbb{Z}_p, n), M(\mathbb{Z}_p, n)])</td>
<td>( \mathbb{Z}_p )</td>
<td>( \mathbb{Z}_{2p} )</td>
</tr>
<tr>
<td>Generators</td>
<td>( 1_X )</td>
<td>( 1_X )</td>
</tr>
</tbody>
</table>

3. Self-homotopy equivalences that induce the identity on cohomotopy groups

In this section, we study the properties of the sets \( \mathcal{E}^n_k(X) \). We recall

\[ \mathcal{E}^n_k(X) = \mathcal{E}(X) \cap [X, X]^n_k, \]

where

\[ [X, X]^n_k = \{ f \in [X, X] \mid g \circ f \sim g \text{ for each } g : X \rightarrow S^n, \text{ for all } n \geq k \}. \]

Equivalently,

\[ \mathcal{E}^n_k(X) = \{ f \in \mathcal{E}(X) \mid f^{I_h} = id_{\pi^n(X)} \text{ on } \pi^n(X) \text{ for } n \geq k \}. \]

This definition indicates that \( \mathcal{E}^n_m(X) \subseteq \mathcal{E}^n_k(X) \) for \( n \geq m \). Hence, we obtain a chain of subsets as follows:

\[ (3.1) \quad \mathcal{E}(X) \supseteq \cdots \supseteq \mathcal{E}^n_k(X) \supseteq \mathcal{E}^n_{n-1}(X) \supseteq \cdots \supseteq \mathcal{E}^n_1(X). \]

**Proposition 3.1.** \( \mathcal{E}^n_k(X) \) is a subgroup of \( \mathcal{E}(X) \).

**Proof.** Let \( f, g \in \mathcal{E}^n_k(X) \) and \( \bar{g} \) be the homotopy inverse map of \( g \). Because \( g \circ g = id_X \) and \( \bar{g} \circ g = id_X \),

\[ id_{\pi^n(X)} = (g \circ \bar{g})^{I_h} = \bar{g}^{I_h} \circ g^{I_h} = \bar{g}^{I_h}. \]

Thus,

\[ (f \circ \bar{g})^{I_h} = \bar{g}^{I_h} \circ f^{I_h} = g^{I_h} = id_{\pi^n(X)}. \]

Hence, \( f \circ \bar{g} \in \mathcal{E}^n_k(X) \). Consequently, \( \mathcal{E}^n_k(X) \) is a subgroup of \( \mathcal{E}(X) \). \qed
Lemma 3.2. If $X$ is a finite CW complex, then there exists a positive integer $N$ such that $[X, S^N] = 0$.

Proof. Let $\dim(X) = m < \infty$. We choose $N$ such that $N > m$. If $f \in [X, S^N]$, then $f(X) \subseteq (S^N)_m$ by the cellular approximation theorem, where $(S^N)_m$ is $m$-skeleton of $S^N$. Because $S^N = e^0 \cup e^N$, $(S^N)_m = e^0$. Thus $f = 0$ and consequently, $[X, S^N] = 0$. □

Theorem 3.3. If $X$ is a finite CW complex, then there are at most a finite number of distinguishing subgroups $\mathcal{E}_k^f(X)$.

Proof. Let $m < \infty$ be the dimension of $X$. By definition of $\mathcal{E}_k^f(X)$, we see that

$$\mathcal{E}(X) \supseteq \cdots \supseteq \mathcal{E}_k^f(X) \supseteq \mathcal{E}_{k-1}^f(X) \supseteq \cdots \supseteq \mathcal{E}_1^f(X).$$

By Lemma 3.2, $[X, S^{n+i}] = 0$ for $i = 1, 2, \ldots$. Hence $\mathcal{E}(X) = \mathcal{E}_{m+1}^f(X)$. Consequently, we have the following finite chain of subsets:

$$\mathcal{E}(X) = \mathcal{E}_{m+1}^f(X) \supseteq \mathcal{E}_m^f(X) \supseteq \cdots \supseteq \mathcal{E}_1^f(X).$$

Next, we consider abelian groups $G_1$ and $G_2$ and Moore spaces $M_1 = M(G_1, n_1)$ and $M_2 = M(G_2, n_2)$. Let $X = M_1 \vee M_2$. We denote by $i_j : M_j \to X$ the inclusion and by $q_j : X \to M_j$ the projection, where $j = 1, 2$. If $f : X \to X$, then we define $f_{jk} : M_k \to M_j$ by $f_{jk} = q_j \circ f \circ i_k$ for $j, k = 1, 2$.

By Proposition 2.1, let $X = M_1 \vee M_2$ then

$$[X, X] = [M_1, M_1] \oplus [M_1, M_2] \oplus [M_2, M_1] \oplus [M_2, M_2].$$

By [3, Proposition 2.6], the function $\theta$ which assigns to each $f \in [X, X]$, the $2 \times 2$ matrix

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{jk} \in [M_k, M_j]$, is a bijection. In addition,

1. $\theta(f + g) = \theta(f) + \theta(g)$, so $\theta$ is an isomorphism $[X, X] \to \bigoplus_{j,k=1,2} [M_k, M_j]$.
2. $\theta(fg) = \theta(f)\theta(g)$, where $fg$ denotes composition in $[X, X]$ and $\theta(f)\theta(g)$ denotes matrix multiplication.

Further, [3] also introduced the forms of the homomorphism induced by $f$ on homotopy, homology, and cohomology groups, respectively.

Now, we determine the form of the homomorphism induced by $f$ on cohomotopy groups.

Proposition 3.4. For any $f \in [X, X]$, we have

$$f_{jk}^k(\gamma_1, \gamma_2) = (f_{11}^k(\gamma_1) + f_{21}^k(\gamma_2), f_{12}^k(\gamma_1) + f_{22}^k(\gamma_2)),$$

where $\gamma_1 \in \pi^k(M_1)$ and $\gamma_2 \in \pi^k(M_2)$. 
Proof. For any $f \in [X, X]$, we identify that

$$f = \left( \begin{array}{cc} f_{11} & f_{12} \\ f_{21} & f_{22} \end{array} \right).$$

Thus, $f$ induces the homomorphism $f^{\sharp k}$ on cohomotopy groups as follows:

$$f^{\sharp k} = \left( \begin{array}{cc} f_{11}^{\sharp k} & f_{12}^{\sharp k} \\ f_{21}^{\sharp k} & f_{22}^{\sharp k} \end{array} \right).$$

Because $\pi^k(X) = \pi^k(M_1) \oplus \pi^k(M_2)$, we are able to identify $\gamma \in \pi^k(X)$ as $\gamma = (\gamma_1, \gamma_2)$, for some $\gamma_i \in \pi^k(M_i)$. Then

$$f^{\sharp k}(\gamma) = \gamma f = (\gamma_1, \gamma_2) \left( \begin{array}{cc} f_{11} & f_{12} \\ f_{21} & f_{22} \end{array} \right) = (f_{11}^{\sharp k}(\gamma_1) + f_{21}^{\sharp k}(\gamma_2), f_{12}^{\sharp k}(\gamma_1) + f_{22}^{\sharp k}(\gamma_2)).$$

□

Proposition 3.5. If $f \in E_1^k(X)$, then

$$f^{\sharp k} = \left( \begin{array}{cc} 1_{\pi^k(M_1)} & 0 \\ 0 & 1_{\pi^k(M_2)} \end{array} \right).$$

Proof. Because $f$ induces the identity on $\pi^k(X)$, $f^{\sharp k} = id_{\pi^k(X)} = 1_X^{\sharp k}$, where $1_X \in [X, X]$ is the identity map. As $1_X = \left( \begin{array}{cc} 1_{M_1} & 0 \\ 0 & 1_{M_2} \end{array} \right)$, we have

$$f^{\sharp k} = 1_X^{\sharp k} = \left( \begin{array}{cc} 1_{\pi^k(M_1)} & 0 \\ 0 & 1_{\pi^k(M_2)} \end{array} \right).$$

□

Here, we review the group of self homotopy equivalences of Moore space. Let $p$ be a positive integer. In [11], Sieradski proved the following result by using the universal coefficient theorem for homotopy:

$$E(M(\mathbb{Z}_p, n)) \cong \begin{cases} \mathbb{Z}_p \rtimes \mathbb{Z}_p^* & n = 2 \\ \mathbb{Z}(2, p) \rtimes \mathbb{Z}_p^* & n \geq 3, \end{cases}$$

where $\mathbb{Z}_p^*$ is the automorphism group of $\mathbb{Z}_p$.

Our computations require us to determine the definite forms of elements in $E(M(\mathbb{Z}_p, n))$ and we use the concept of the self-closeness number introduced in [5] for this purpose. Because the Moore space of type $(G, n)$ has the self closeness number $n$ by [5, Corollary 3], $A^0_p(M(\mathbb{Z}_p, n)) = E(M(\mathbb{Z}_p, n))$ by [5, Definition 2.1 or Theorem 4], where $A^0_p(M(\mathbb{Z}_p, n))$ is the set of homotopy classes of self-maps of $M(\mathbb{Z}_p, n)$ that induce an automorphism of $\pi_i(X)$ for $i = 0, 1, \ldots, n$. To determine the definite forms of elements in $E(M(\mathbb{Z}_p, n))$, we compute $A^0_p(M(\mathbb{Z}_p, n))$ rather than $E(M(\mathbb{Z}_p, n))$.

Consider the mapping cone sequence

$$S^n \xrightarrow{p} S^n \xrightarrow{i} S^n \cup_p e^{n+1} \xrightarrow{\pi} S^{n+1} \xrightarrow{p} S^{n+1},$$
where \( p \) is a map of degree \( p \), \( i \) is the inclusion and \( \pi \) is the quotient map.

**Theorem 3.6.** Let \( X = M(\mathbb{Z}_p, n) \) be a Moore space. Then we have

\[
\mathcal{A}^p_\ell(X) = \begin{cases}
\{ k \cdot 1_X \mid (k, p) = 1 \} & p \equiv 1 \pmod{2}, \\
\{ \ell \cdot i \circ \eta_n \circ \pi + k \cdot 1_X \mid (k, p) = 1 \} & p \equiv 0 \pmod{4}, \\
\{ k \cdot 1_X, (k + p) \cdot 1_X \mid (k, p) = 1 \} & p \equiv 2 \pmod{4}.
\end{cases}
\]

**Proof.** We first note that \( \pi_n(X) \cong \mathbb{Z}_p \{i_\sharp(t_n)\} \).

Suppose that \( p \) is odd. Then \( [X, X] = \mathbb{Z}_p \{1_X\} \). Moreover, we have

\[
1_{X^2}(i_\sharp(t_n)) = 1_X \circ i \circ \tau_n = i_\sharp(t_n).
\]

Thus, \( (k \cdot 1_X)_\sharp(i_\sharp(t_n)) = k \cdot (i_\sharp(t_n)) \). It follows that

\[
\mathcal{A}^p_\ell(X) = \{ k \cdot 1_X \mid (k, p) = 1 \} = \mathbb{Z}_p^*.
\]

Suppose that \( p \equiv 0 \pmod{4} \). In this case,

\[
[X, X] = \mathbb{Z}_2 \oplus \mathbb{Z}_p \{i \circ \eta_n \circ \pi, 1_X\}.
\]

Because \( 1_{X^2}(i_\sharp(t_n)) = i_\sharp(t_n) \) and \( (i \circ \eta_n \circ \pi)_\sharp(i_\sharp(t_n)) = i \circ \eta_n \circ \pi \circ i \circ \tau_n = 0 \), we have

\[
(\ell \cdot (i \circ \eta_n \circ \pi) + k \cdot 1_X)_\sharp(i_\sharp(t_n)) = (k \cdot 1_X)_\sharp(i_\sharp(t_n)) = k \cdot (i_\sharp(t_n))
\]

for \( \ell \in \mathbb{Z}_2 \) and \( k \in \mathbb{Z}_p \). Therefore

\[
\mathcal{A}^p_\ell(X) = \{ \ell \cdot (i \circ \eta_n \circ \pi) + k \cdot 1_X \mid (k, p) = 1 \}.
\]

Suppose that \( p \equiv 2 \pmod{4} \). In this case, we have \( [X, X] = \mathbb{Z}_2 \{1_X\} \). As \( k \cdot 1_X(i_\sharp(t_n)) = k \cdot i_\sharp(t_n) \) for \( 0 < k \leq p \) and \( (p + k) \cdot 1_X(i_\sharp(t_n)) = k \cdot i_\sharp(t_n) \) for \( 0 < k \leq p \), we have

\[
\mathcal{A}^p_\ell(X) = \{ k \cdot 1_X \mid (k, p) = 1, 1 < k \leq p \} \cup \{ (p + k) \cdot 1_X \mid (k, p) = 1, 0 < k < p \}.
\]

\[
[4, \text{Computation of } \mathcal{E}^k_p(M(\mathbb{Z}_p, n))]
\]

In this section, we compute \( \mathcal{E}^k_p(M(\mathbb{Z}_p, n)) \) and determine their generators for \( k = n + 1, n \), and \( n - 1 \). Throughout this section, we let \( X = M(\mathbb{Z}_p, n) \).

**Theorem 4.1.** For \( \mathcal{E}^{n+1}_k(X) \), we have the following table:

<table>
<thead>
<tr>
<th>( \mathcal{E}^{n+1}_k(X) )</th>
<th>( p ) odd</th>
<th>( p \equiv 0 \pmod{4} )</th>
<th>( p \equiv 2 \pmod{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_2 )</td>
<td>( 1_X )</td>
<td>( \ell i \circ \eta_n \circ q \oplus 1_X )</td>
<td>( (\ell + 1)1_X )</td>
</tr>
</tbody>
</table>

**Proof.** Because \( \pi^k(X) = 0 \) for \( k > n + 1 \) by Theorem 3.3, it is sufficient to consider the \((n + 1)\)-th cohomotopy group of \( X \). From Table 1, \( \pi^{n+1}(X) = \mathbb{Z}_p \{i_{n+1} \circ q\} \).

**Case 1.** Let \( p \) be odd.
By Theorem 3.6, for each \( f \in \mathcal{E}(X) \), \( f = k1_X \) for some \( k \) such that \( 0 \leq k \leq p - 1 \) and \( (k, p) = 1 \). Thus, we have

\[
f^{2n+1}(t_{n+1} \circ \pi) = t_{n+1} \circ \pi \circ (k1_X) = k(t_{n+1} \circ \pi \circ 1_X) = k(t_{n+1} \circ \pi).
\]

Therefore, to ensure that \( f^{2n+1} = 1_{\pi_{n+1}} \) holds, \( k \) is require to be 1. Hence \( \mathcal{E}_{n+1}^2(X) \cong 1 \{1_X\} \).

**Case 2.** Let \( p \equiv 0 \) (mod 2).

By Theorem 3.6, for each \( f \in \mathcal{E}(X) \), \( f = \ell i \circ \eta_n \circ \pi \oplus k1_X \), for some \( \ell = 0, 1 \), where \( k \) is an integer such that \( 0 \leq k \leq p - 1 \) and \( (k, p) = 1 \). Thus, we have

\[
f^{2n+1}(t_{n+1} \circ \pi) = t_{n+1} \circ \pi (\ell i \circ \eta_n \circ \pi \oplus k1_X)
= \ell t_{n+1} \circ \pi \circ i \circ \eta_n \circ \pi \oplus k t_{n+1} \circ \pi \circ 1_X
= k t_{n+1} \circ \pi
\]

because \( q \circ i \) is homotopic to the constant map. Thus \( k \) is require to be 1 and \( \ell = 0 \) or 1 to ensure that \( f^{2n+1} = 1_{\pi_{n+1}} \) holds. Hence

\[\mathcal{E}_{n+1}^2(X) \cong \mathbb{Z}_2\{\ell i \circ \eta_n \circ q \oplus 1_X | \ell = 0, 1\} \]

**Case 3.** Let \( p \equiv 2 \) (mod 4).

By Theorem 3.6, for each \( f \in \mathcal{E}(X) \), \( f = (k + \ell)1_X \) for some \( k \) and \( \ell \) such that \( 0 \leq k \leq p - 1 \), \( (k, p) = 1 \) and \( \ell = 0, p \). Thus, we have

\[
f^{2n+1}(t_{n+1} \circ \pi) = (k + \ell)(t_{n+1} \circ \pi \circ 1_X) = (k + \ell)t_{n+1} \circ \pi.
\]

Thus \( k \) is require to be 1 and \( \ell = 0 \) or \( p \) to ensure that \( f^{2n+1} = 1_{\pi_{n+1}} \) holds. Hence

\[\mathcal{E}_{n+1}^2(X) \cong \mathbb{Z}_2\{1_X, \ell i \circ \eta_n \circ \pi, (\ell + 1)1_X | \ell = 0, p\} \]

\[\square\]

**Theorem 4.2.** For \( \mathcal{E}_{n}^2(X) \), we have the following table:

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>( \mathbb{Z}_2{1_X, \ell i \circ \eta_n \circ \pi, (\ell + 1)1_X } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \equiv 0 ) (mod 4)</td>
<td>( \mathcal{E}_{n}^2(X) )</td>
</tr>
<tr>
<td>( p \equiv 2 ) (mod 4)</td>
<td>( \mathcal{E}_{n+1}^2(X) )</td>
</tr>
</tbody>
</table>

| \( \pi^n(X) \) \( (\ell + 1)1_X \) | \( p \equiv 1 \) (mod 2), \( \mathbb{Z}_2\{\eta_n \circ \pi \} \) \( p \equiv 0 \) (mod 2), \( \mathbb{Z}_2\{\eta_n \circ \pi \oplus \ell i \circ \eta_n \circ \pi, (\ell + 1)1_X \} \)

**Proof.** We first note that \( \mathcal{E}_n^2(X) \subseteq \mathcal{E}_{n+1}^2(X) \). From Table 1,

\[
\pi^n(X) = \begin{cases} 
0 & p \equiv 1 \text{ (mod 2),} \\
\mathbb{Z}_2\{\eta_n \circ \pi\} & p \equiv 0 \text{ (mod 2).}
\end{cases}
\]

**Case 1.** Let \( p \) be odd.

By Theorem 4.1, \( \mathcal{E}_n^2(X) \subseteq \mathcal{E}_{n+1}^2(X) \cong 1.

**Case 2.** Let \( p \equiv 0 \) (mod 4).

By Theorem 4.1, for each \( f \in \mathcal{E}(X) \), we have \( f = \ell i \circ \eta_n \circ \pi \oplus 1_X \) for \( \ell = 0, 1 \). Thus, we have

\[
f^{2n}(\eta_n \circ \pi) = \eta_n \circ \pi \circ (\ell i \circ \eta_n \circ \pi \oplus 1_X)
= \ell \eta_n \circ \pi \circ i \circ \eta_n \circ \pi \oplus \eta_n \circ \pi \circ 1_X
= \eta_n \circ \pi
\]
Theorem 4.4. For $X = M(\mathbb{Z}_p, 3)$, we have the following table:
Proof. Based on the Puppe Sequence, we have
\[
\pi^2(X) = \begin{cases} 
0 & p \equiv 1 \pmod{2}, \\
\mathbb{Z}_2 \{\eta^2_2 \circ q\} & p \equiv 0 \pmod{2}.
\end{cases}
\]

Case 1. Let \( p \) be odd.
By Theorem 4.2, \( E^\sharp_{n-1} \mathbb{Z}_2 \{X\} \). Thus we have
\[
f^{* n-1}(1) = (\eta^2_2 \circ q) \circ (k1_X \oplus \ell \circ \eta_3 \circ p) = k\eta^2_2 \circ p \circ 1_X \oplus \ell \eta^2_2 \circ p \circ \eta_3 \circ p = k\eta^2_2 \circ p
\]
because \( \pi \circ i \) is homotopic to the constant map. Thus, for \( f^{* n-1} = 1_{n-1} \) to hold, \( k \) is required to be 1. Hence,
\[
E^\sharp_{n-1} \mathbb{Z}_2 \{X\} \cong \mathbb{Z}_2 \{1_X \oplus \ell \circ \eta_3 \circ q| \ell = 0, 1\}.
\]

Case 2. Let \( p \equiv 0 \pmod{4} \).
By Theorem 3.6, for each \( f \in E(X) \), \( f = k1_X \oplus \ell i \circ \eta_n \circ \pi \) for some \( k \) and \( \ell \) such that \( 0 \leq k \leq p - 1, (k, p) = 1 \) and \( \ell = 0, 1 \). Thus we have
\[
f^{* n-1}(\eta^2_2 \circ p) = (\eta^2_2 \circ p) \circ (k1_X \oplus \ell i \circ \eta_3 \circ p) = k\eta^2_2 \circ p \circ 1_X \oplus \ell \eta^2_2 \circ p \circ \eta_3 \circ p = k\eta^2_2 \circ p
\]
for \( f^{* n-1} = 1_{n-1} \) to hold, \( k \) is required to be 1. Hence,
\[
E^\sharp_{n-1} \mathbb{Z}_2 \{X\} \cong \mathbb{Z}_2 \{(\ell + 1)1_X| \ell = 0, p\}.
\]

5. Computation of \( E^\sharp_k(C(Z_p, n)) \)

In this section, we compute \( E^\sharp_k(C(Z_p, n)) \) and determine their generators for \( k = n, n - 1 \) and \( n - 2 \). Throughout this section, we let \( X = C(Z_p, n) \) and \( M_2 = M(Z_p, n - 1) \).

First of all, we determine the generators of \([X, X]\) and \([X, S^n]\).

It is well known that
\[
C(Z_p, n) = M(Z, n) \vee M(Z_p, n - 1) = S^n \vee M(Z_p, n - 1)
\]
for \( n > 3 \). Thus we have
\[
[X, X] \cong [S^n, X] \oplus [M_2, X]
\]
and by Proposition 2.1,
\[
[X, X] \cong [X, S^n] \oplus [X, M_2].
\]
Consequently,
\[ [X, X] \cong [S^n, S^n] \oplus [M_2, S^n] \oplus [S^n, M_2] \oplus [M_2, M_2]. \]

In [3], we have
\[ \mathcal{E}(X) = \mathcal{E}(S^n) \oplus [M_2, S^n] \oplus [S^n, M_2] \oplus \mathcal{E}(M_2). \]

Consider the mapping cone sequence
\[ S^n \xrightarrow{p} S^n \xrightarrow{i} M(\mathbb{Z}_p, n) \xrightarrow{\pi} S^{n+1} \xrightarrow{p} S^{n+1} \]
and let \( i_1 : S^n \to X \) and \( i_2 : M_2 \to X \) be inclusion maps and \( q_1 : X \to S^n \) and \( q_2 : X \to M_2 \) be projection maps.

Then, from Table 1 and Theorem 3.6, we have the following lemmas.

**Lemma 5.1.** If \( i_1 : S^n \to X \) and \( i_2 : M_2 \to X \) are inclusion maps and \( q_1 : X \to S^n \) and \( q_2 : X \to M_2 \) are projection maps, then we have:

<table>
<thead>
<tr>
<th>( p \equiv 1 \pmod{2} )</th>
<th>( [X, X] )</th>
<th>( \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p )</th>
<th>Generators</th>
<th>( i_1 \circ \iota_1 \circ \eta_1, i_1 \circ \iota_2 \circ \pi \circ \eta_2, i_1 \circ \iota_1 \circ \eta_1 \circ \eta_2 \circ q_2, i_1 \circ \iota_1 \circ \iota_1 \circ \iota_1 \circ \eta_1 \circ \eta_2 \circ q_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \equiv 0 \pmod{4} )</td>
<td>( [X, X] )</td>
<td>( \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_p )</td>
<td>Generators</td>
<td>( i_1 \circ \iota_1 \circ \eta_1, i_1 \circ \iota_2 \circ \pi \circ \eta_2, i_2 \circ \iota_1 \circ \eta_1 \circ \iota_1 \circ \eta_2 \circ q_2, i_2 \circ \iota_1 \circ \iota_1 \circ \iota_1 \circ \eta_1 \circ \eta_2 \circ q_2 )</td>
</tr>
<tr>
<td>( p \equiv 2 \pmod{4} )</td>
<td>( [X, X] )</td>
<td>( \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_p )</td>
<td>Generators</td>
<td>( i_1 \circ \iota_1 \circ \eta_1, i_1 \circ \iota_2 \circ \pi \circ \eta_2, i_2 \circ \iota_1 \circ \eta_1 \circ \iota_1 \circ \eta_2 \circ q_2, i_2 \circ \iota_1 \circ \iota_1 \circ \iota_1 \circ \eta_1 \circ \eta_2 \circ q_2 )</td>
</tr>
</tbody>
</table>

As \( \mathcal{E}(S^n) \cong \mathbb{Z}_2 \), we have the following lemma.

**Lemma 5.2.** \( \mathcal{E}(X) \) is isomorphic to

<table>
<thead>
<tr>
<th>( p \equiv 1 \pmod{2} )</th>
<th>( \mathbb{Z}_2 \oplus \mathbb{Z}_p \oplus (\mathbb{Z}_p^*) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \equiv 0 \pmod{4} )</td>
<td>( \mathbb{Z}_2 \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \times \mathbb{Z}_p^*) )</td>
</tr>
<tr>
<td>( p \equiv 2 \pmod{4} )</td>
<td>( \mathbb{Z}_2 \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \times \mathbb{Z}_p^*) )</td>
</tr>
</tbody>
</table>

By Proposition 2.1,
\[ \pi^r(X) = \pi^r(S^n) \oplus \pi^r(M(\mathbb{Z}_p, n-1)). \]

Thus, we have the following lemma from Table 1,

**Lemma 5.3.** For \( \pi^r(X) \), we have the following table:

<table>
<thead>
<tr>
<th>( r = n )</th>
<th>( \pi^r(X) )</th>
<th>Generators</th>
<th>( \mathbb{Z} \oplus \mathbb{Z}_p \oplus \iota_n \circ \eta_1 \circ \iota_1 \circ \eta_2 \circ q_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = n - 1 )</td>
<td>( \pi^r(X) )</td>
<td>Generators</td>
<td>( \mathbb{Z}_2 \oplus \mathbb{Z}<em>2 \oplus \eta</em>{n-1} \circ \eta_1 \circ \eta_2 \circ q_2 )</td>
</tr>
</tbody>
</table>

| \( r = n - 2 \) | \( \pi^r(X) \) | Generators | \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \eta_{n-2} \circ \eta_1 \circ \eta_2 \circ q_2 \) |

Now, we compute \( \mathcal{E}_n(X) \) and determine their generators for \( k = n, n - 1, \) and \( n - 2 \).

**Theorem 5.4.** For \( \mathcal{E}_n(X) \), we have the following table:
Let $\gamma = 3.4$, each $f \in E(X)$ can be identified as

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$ 

Let $\gamma = (\gamma_1, \gamma_2) = (t_n \circ q_1, t_n \circ \pi \circ q_2)$ be the generator.

**Case 1.** Let $p$ be odd.

By Lemma 5.2, for each $f \in E(X)$, we have

- $f_{11} = s t_1 \circ t_n \circ q_1$,
- $f_{12} = t_1 \circ t_n \circ \pi \circ q_2$,
- $f_{21} = 0$,
- $f_{22} = k t_1 \circ 1_{M_2} \circ q_2$

for some $k, s$ and $t$ such that $0 \leq t \leq p - 1$, $0 \leq k \leq p - 1$ and $(k, p) = 1$, $s = -1, 1$. Thus we have

$$f^{p_n}_{11}(\gamma_1) = t_n \circ q_1 \circ s(i_1 \circ t_n \circ q_1)$$
$$= s t_n \circ q_1 \circ i_1 \circ t_n \circ q_1 = s t_n \circ q_1,$$

$$f^{p_n}_{12}(\gamma_1) = t_n \circ q_1 \circ t(i_1 \circ t_n \circ \pi \circ q_2)$$
$$= t t_n \circ q_1 \circ i_1 \circ t_n \circ \pi \circ q_2 = t t_n \circ \pi \circ q_2,$$

$$f^{p_n}_{22}(\gamma_2) = t_n \circ \pi \circ q_2 \circ (k t_1 \circ 1_{M_2} \circ q_2) = k t_n \circ \pi \circ q_2.$$

By Proposition 3.5, $s = 1$, $t = 0$ and $k = 1$. Hence

$$E^p_n(X) \cong 1 \left\{ \begin{pmatrix} i_1 \circ t_n \circ q_1 & 0 \\ 0 & i_2 \circ 1_{M_2} \circ q_2 \end{pmatrix} \right\}.$$ 

**Case 2.** $p \equiv 0 \pmod{4}$.

By Lemma 5.2, for each $f \in E(X)$, we have

- $f_{11} = s t_1 \circ t_n \circ q_1$,
- $f_{12} = t_1 \circ t_n \circ \pi \circ q_2$,
- $f_{21} = m t_1 \circ i \circ t_n \circ q_1$,
- $f_{22} = \ell t_1 \circ i \circ t_n \circ q_2 \circ k t_2 \circ 1_{M_2} \circ q_2$

for $k, \ell, m, s$ and $t$ such that $0 \leq t \leq p - 1$, $0 \leq k \leq p - 1$, $m, \ell = 0, 1, s = -1, 1$ and $(k, p) = 1$. Thus we have

$$f^{p_n}_{11}(\gamma_1) = t_n \circ q_1 \circ s(i_1 \circ t_n \circ q_1)$$
$$= s t_n \circ q_1 \circ i_1 \circ t_n \circ q_1 = s t_n \circ q_1,$$

$$f^{p_n}_{12}(\gamma_1) = t_n \circ q_1 \circ t(i_1 \circ t_n \circ \pi \circ q_2)$$

$$\begin{array}{c|c|c}
 p \equiv 1 \pmod{2} & p \equiv 0 \pmod{4} & p \equiv 2 \pmod{4} \\
 \mathcal{E}_n^p(X) & 1 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
\end{array}$$
Theorem 5.5. For $E^2_{n-1}(X)$, we have the following table:
Let $f$. By Theorem 5.4, for each $f$

From Lemma 5.3, we have

Proof. From Lemma 5.3, we have

$$
\pi^{n-1}(X) = \begin{cases} 
\mathbb{Z}_2\{\eta_{n-1} \circ q_1\} & p \equiv 1 \pmod{2}, \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2\{\eta_{n-1} \circ q_1, \eta_{n-1} \circ \pi \circ q_2\} & p \equiv 0 \pmod{2}.
\end{cases}
$$

Case 1. Let $p$ be odd.

By Theorem 5.4, we have $E_{n-1}(X) \subseteq E_n(X) \cong 1$.

Case 2. Let $p \equiv 0 \pmod{4}$.

By Theorem 5.4, for each $f \in E(X)$, we have

$$
f_{i1} = i_1 \circ \iota_n \circ q_1, \quad f_{i2} = 0, \quad f_{21} = mi_2 \circ i \circ \eta_{n-1} \circ q_1, \quad f_{22} = \ell i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2 \oplus i_2 \circ 1_{M_2} \circ q_2$$

for $m, \ell = 0, 1$. By Proposition 3.5, it is sufficient to determine $f_{21}$ and $f_{22}$; however, we have

$$
f_{21}^{n-1}(\gamma_2) = \eta_{n-1} \circ \pi \circ q_2 \circ m(i_2 \circ i \circ \eta_{n-1} \circ q_1) \\
= mn \eta_{n-1} \circ \pi \circ q_2 \circ i_2 \circ i \circ \eta_{n-1} \circ q_1 = 0,
$$

$$
f_{22}^{n-1}(\gamma_2) = \eta_{n-1} \circ \pi \circ q_2 \circ (\ell i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2 \oplus i_2 \circ 1_{M_2} \circ q_2) \\
= \ell \eta_{n-1} \circ \pi \circ q_2 \circ i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2 \oplus \eta_{n-1} \circ \pi \circ q_2 \circ i_2 \circ 1_{M_2} \circ q_2 \\
= \eta_{n-1} \circ \pi \circ q_2
$$

because $\pi \circ i$ is homotopic to the constant map.

By Proposition 3.5, $m, \ell = 0, 1$. Hence,

$$
E_{n-1}(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \left\{ \begin{pmatrix} i_1 \circ \iota_n \circ q_1 & 0 \\
mi_2 \circ i \circ \eta_{n-1} \circ q_1 & \ell \alpha \oplus i_2 \circ 1_{M_2} \circ q_2 \end{pmatrix} \right\} \\
\ell = 0, 1 \text{ and } m = 0, 1,
$$

where $\alpha = i_2 \circ i \circ \eta_{n-1} \circ q \circ q_2$.

Case 3. Let $p \equiv 2 \pmod{4}$.

By Theorem 5.4, for each $f \in E(X)$, we have

$$
f_{i1} = i_1 \circ \iota_n \circ q_1, \quad f_{i2} = 0, \quad f_{21} = mi_2 \circ i \circ \eta_{n-1} \circ q_1, \quad f_{22} = (1 + \ell)i_2 \circ 1_{M_2} \circ q_2$$

for $m = 0, 1, \ell = 0, p$. By Proposition 3.5, it is sufficient to determine $f_{21}$ and $f_{22}$; however, we have

$$
f_{21}^{n-1}(\gamma_2) = \eta_{n-1} \circ \pi \circ q_2 \circ m(i_2 \circ i \circ \eta_{n-1} \circ q_1)
$$
Proof. By Theorem 5.5, for each $E$ Theorem 5.6.

Because $q \circ i$ is homotopic to the constant map.

By Proposition 3.5, $\ell = 0, p$ and $m = 0, 1$. Hence

$$
\mathcal{E}_{n-1}^p(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \left\{ \left( \begin{array}{c} i_1 \circ i_n \circ q_1 \\
\ell \circ i \circ \eta_{n-1} \circ q_1 
\end{array} \right) : \left( \begin{array}{c}
\ell = 0, p \text{ and } m = 0, 1
\end{array} \right) \right\}.
$$

\begin{proof}

Case 1. Let $p$ be even. By Theorem 5.5, $\mathcal{E}_{n-2}^p(X) \subseteq \mathcal{E}_{n-1}^p(X) \cong 1$.

Case 2. Let $p \equiv 0 \pmod 4$.

Then, the generator of $\pi_{n-2}(X)$ is

$$
\gamma = (\gamma_1, \gamma_2) = (\eta_{n-2} \circ q_1, \eta_{n-2} \circ \bar{q} \circ \eta \circ q_2).
$$

By Theorem 5.5, for each $f \in \mathcal{E}(X)$, we have

\begin{align*}
f_{11} &= i_1 \circ i_n \circ q_1, \\
f_{12} &= 0, \\
f_{21} &= m i_2 \circ i \circ \eta_{n-1} \circ q_1, \\
f_{22} &= \ell \alpha \oplus i_2 \circ 1_{M_2} \circ q_2
\end{align*}

for $m, \ell = 0, 1$ and $\alpha = i_2 \circ i \circ \eta_{n-1} \circ q \circ q_2$. By Proposition 3.5, it is sufficient to determine $f_{21}$ and $f_{22}$; however, we have

\begin{align*}
f_{21}^{\pi_{n-2}}(\gamma_2) &= \eta_{n-2}^2 \circ \pi \circ q_2 \oplus \bar{q} \circ q_2 \circ m (i_2 \circ i \circ \eta_{n-1} \circ q_1) \\
&= m \eta_{n-2}^2 \circ \pi \circ q_2 \circ i_2 \circ i \circ \eta_{n-1} \circ q_1 \oplus m \bar{q} \circ q_2 \circ i_2 \circ i \circ \eta_{n-1} \circ q_1 \\
&= m \eta_{n} \circ i \circ \eta \circ q_1 = m \eta_{n-2} \eta \circ q_1 \\
&= m \eta_{n-2} \circ q_1, \\
f_{22}^{\pi_{n-2}}(\gamma_2) &= \eta_{n-2}^2 \circ \pi \circ q_2 \oplus \bar{q} \circ q_2 \circ (\ell \alpha \oplus i_2 \circ 1_{M_2} \circ q_2) \\
&= \eta_{n-2}^2 \circ \pi \circ q_2 \circ (\ell \alpha \oplus i_2 \circ 1_{M_2} \circ q_2) + \bar{q} \circ q_2 \circ (\ell \alpha \oplus i_2 \circ 1_{M_2} \circ q_2) \\
&= \ell \eta_{n-2}^2 \circ \pi \circ q_2 \circ \alpha \oplus \eta_{n-2} \circ \pi \circ q_2 \circ i_2 \circ 1_{M_2} \circ q_2 + \ell \bar{q} \circ q_2 \circ \alpha \oplus q_2 \circ q_2 \circ i_2 \circ 1_{M_2} \circ q_2 \\
&= \eta_{n-2}^2 \circ \pi \circ q_2 + \ell \bar{q} \circ q_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2 \oplus \bar{q} \circ q_2
\end{align*}

\end{proof}
\[ f_{21}^{n-2}(\gamma_2) = \bar{\eta}_n \circ q_2 \circ m(i_2 \circ i\eta_{n-1} \circ q_1) = m\bar{\eta}_n \circ q_2 \circ i_2 \circ i\eta_{n-1} \circ q_1 = m\eta_{n-2}^2 \circ q_1, \]

\[ f_{22}^{n-2}(\gamma_2) = \bar{\eta}_n \circ q_2 \circ (1 + \ell)i_2 \circ 1_{M_2} \circ q_2 = (1 + \ell)\bar{\eta}_n \circ q_2 \]

because \( \bar{\eta}_n \circ i\eta_{n-2} = \eta_{n-2} \circ \eta_{n-1} = \eta_{n-2}^2 \).

If \( \ell = p \), then \((1 + p)\bar{\eta}_n \circ q_2 = 3\bar{\eta}_n \circ q_2 \neq \bar{\eta}_n \circ q_2 \) because \( \bar{\eta}_n \circ q_2 \) has order 4.

Thus, by Proposition 3.5, \( \ell = 0 \) and \( m = 0 \). Hence,

\[ \mathcal{E}_{n-2}^z(X) \cong 1 \left\{ \begin{pmatrix} i_1 \circ \iota_n \circ q_1 & 0 \\ 0 & i_2 \circ 1_{M_2} \circ q_2 \end{pmatrix} \right\}. \]

\[ \square \]

References


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