U-FLATNESS AND NON-EXPANSIVE MAPPINGS
IN BANACH SPACES

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Abstract. In this paper, we define the modulus of $n$-dimensional $U$-flatness as the determinant of an $(n+1) \times (n+1)$ matrix. The properties of the modulus are investigated and the relationships between this modulus and other geometric parameters of Banach spaces are studied. Some results on fixed point theory for non-expansive mappings and normal structure in Banach spaces are obtained.

1. Introduction

Let $X$ be a real Banach space with the dual space $X^*$. Denote by $B_X$ and $S_X$ the closed unit ball and the unit sphere of $X$, respectively. Recall that $\nabla_x \subset S_{X^*}$ denotes the set of norm 1 supporting functionals of $x \in S_X$.

Brodskiǐ and Mil'man [2] introduced the following geometric concepts in 1948:

Definition 1.1. Let $X$ be a Banach space. A nonempty bounded and convex subset $K$ of $X$ is said to have normal structure if for every convex subset $C$ of $K$ that contains more than one point there is a point $x_0 \in C$ such that

$$\sup \{ \| x_0 - y \| : y \in C \} < \text{diam} C.$$ 

A Banach space $X$ is said to have

- normal structure if every bounded convex subset of $X$ has normal structure;
- weak normal structure if every weakly compact convex set $K$ of $X$ has normal structure;
- uniform normal structure if there exists $0 < c < 1$ such that for every bounded closed convex subset $C$ of $K$ that contains more than one point there is a point $x_0 \in C$ such that

$$\sup \{ \| x_0 - y \| : y \in C \} < c \cdot \text{diam} C.$$
Remark 1.2. The following facts are known.

- uniform normal structure $\Rightarrow$ normal structure $\Rightarrow$ weak normal structure.
- In the setting of reflexive spaces, normal structure $\iff$ weak normal structure.

Kirk [9] proved that if a Banach space $X$ has weak normal structure, then it has weak fixed point property, that is, every non-expansive mapping from a weakly compact and convex subset of $X$ into itself has a fixed point.

Let $N$ be the set of all natural numbers and $n \in N$.

For two sets of vectors $\{x_i\}_{i=1}^{n+1} \subseteq X$ and $\{f_i\}_{i=2}^{n+1} \subseteq X^*$, the following $(n+1) \times (n+1)$ matrix

$$
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\langle x_1, f_2 \rangle & \langle x_2, f_2 \rangle & \cdots & \langle x_{n+1}, f_2 \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle x_1, f_{n+1} \rangle & \langle x_2, f_{n+1} \rangle & \cdots & \langle x_{n+1}, f_{n+1} \rangle
\end{pmatrix}
$$

is denoted by $m(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1})$ [6].

Gao and Saejung [6] introduced the concept of volume by the convex hull of $x_1, x_2, \ldots, x_{n+1}$ in $X$ of

$$
v(x_1, x_2, \ldots, x_{n+1}) := \sup \{ \det m(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1}) \},
$$

where the supremum is taken over all $f_i \in \nabla x_i$, where $i = 2, 3, \ldots, n+1$.

Definition 1.3 ([6]). Let $\nu^X_n = \sup \{ v(x_1, x_2, \ldots, x_{n+1}) : x_1, x_2, \ldots, x_{n+1} \in S_X \}$ be the upper bound of all $n$-dimensional volume in $X$.

Definition 1.4 ([6]). Let $X$ be a Banach space. Define

$$
U^n_X(\varepsilon) = \inf \left\{ 1 - \frac{1}{n+1} \| x_1 + x_2 + \cdots + x_{n+1} \| : x_1, x_2, \ldots, x_{n+1} \in S_X, \ v(x_1, x_2, \ldots, x_{n+1}) \geq \varepsilon \right\},
$$

where $0 \leq \varepsilon \leq \nu^X_n$ to be the modulus of $n$-dimensional $U$-convexity of $X$.

The following results were proved [6]:

Proposition 1.5. For a Banach space $X$ with $\dim(X) > n$, we have $\nu^X_n \geq 2$.

Lemma 1.6. $U^n_X(\varepsilon)$ is a continuous function in $[0, \nu^X_n]$.

Theorem 1.7. If $X$ is a Banach space with $U^n_X(1) > 0$ for some $n \in N$, then $X$ is reflexive.

Theorem 1.8. If $X$ is a Banach space with $U^n_X(1) > 0$ for some $n \in N$, then $X$ has normal structure.
2. Main results

We introduce the concept of the modulus of \( n \)-dimensional flatness as follows:

**Definition 2.1.** Let \( X \) be a Banach space and \( 0 \leq \varepsilon \leq \nu_X^U \). Then the modulus of \( n \)-dimensional \( U \)-flatness of \( X \) is defined as follows:

\[
W_X^n(\varepsilon) = \sup \left\{ 1 - \frac{1}{n+1} \| x_1 + x_2 + \cdots + x_{n+1} \| \right\},
\]

where the supremum is taken over all \( \{ x_i \}_{i=1}^{n+1} \subseteq S_X \) such that there exist \( \{ f_i \}_{i=1}^{n+1} \subseteq S_X^* \) with \( f_i \in \nabla x_i \) for all \( i = 2, \ldots, n+1 \) and \( \det(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1}) \leq \varepsilon \).

**Remark 2.2.** \( W_X^n(\varepsilon) \) is an increasing and continuous function on \([0,\nu_X^U]\).

**Proof.** The proof is the same as that of Corollary 5 of [10]. \( \square \)

**Remark 2.3.** The name of the modulus, \( U \)-flatness, is defined by comparing with Definition 1.4.

**Lemma 2.4** (Bishop-Phelps-Bollobás [1]). Let \( X \) be a Banach space, and let \( 0 < \varepsilon < 1 \). Given \( z \in B_X \) and \( h \in S_X^* \) with \( 1 - (z,h) < \frac{\varepsilon^2}{2} \), then there exist \( y \in S_X \) and \( g \in \nabla y \) such that \( \|y-z\| < \varepsilon \) and \( \|g-h\| < \varepsilon \).

**Lemma 2.5.** Let \( A_{n \times n} \) be the following \( n \times n \) matrix

\[
A_{n \times n} := \begin{bmatrix}
1 & -1 & 1 & \cdots & (-1)^{n-1} & (-1)^{n-2} & (-1)^{n-1} \\
-\frac{1}{2} & 1 & -1 & \cdots & (-1)^{n-1} & (-1)^{n-2} & (-1)^{n-1} \\
0 & -\frac{1}{2} & 1 & \cdots & (-1)^{n-1} & (-1)^n & (-1)^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1 & 1 \\
0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & -1 \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \\
\end{bmatrix}
\]

Then \( \det(A_{n \times n}) = \frac{1}{2^{n-1}} \).

**Proof.** It follows from mathematical induction:

By repeatedly using add \( \frac{1}{2} \) times the first row to second row, then use the first row to estimate the determinant, we get the result. \( \square \)

**Lemma 2.6.** Let \( B_{(n+1) \times (n+1)} \) be the following \( (n+1) \times (n+1) \) matrix

\[
B_{(n+1) \times (n+1)} := \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
-\frac{1}{2} & 1 & -1 & \cdots & (-1)^{n-1} & (-1)^n & (-1)^{n+1} \\
0 & -\frac{1}{2} & 1 & \cdots & (-1)^n & (-1)^{n+1} & (-1)^{n+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1 & 1 \\
0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & -1 \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \\
\end{bmatrix}
\]
Then \(\det(B_{(n+1)\times(n+1)}) = \frac{2n+1}{2^n}\).

**Proof.** It follows from mathematical induction and the preceding lemma:

Let \(n = 1\), \(B_{2\times2} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & 1 \end{bmatrix}\), \(\det(B_{2\times2}) = \frac{3}{4}\).

If for \(n\), \(\det(B_{n\times n}) = \frac{2n-1}{2^n}\), then for \(n + 1\), by using the first column to estimate the matrix, we have

\[
\det(B_{(n+1)\times(n+1)}) = \det(A_{n\times n}) + \frac{1}{2} \det(B_{n\times n})
\]

\[
= \frac{1}{2^n-1} + \frac{2n-1}{2^n} = \frac{2n+1}{2^n}. \quad \Box
\]

**Theorem 2.7 ([7]).** Let \(X\) be a Banach space. Then \(X\) is not reflexive if and only if for any \(0 < \delta < 1\) there is a sequence \(\{x_n\} \subseteq S_X\) and a sequence \(\{f_n\} \subseteq S_{X^*}\) such that

(a) \(\langle x_m, f_n \rangle = \delta\) whenever \(n \leq m\); and

(b) \(\langle x_m, f_n \rangle = 0\) whenever \(n > m\).

**Theorem 2.8.** If \(X\) is a Banach space with \(W_X^\delta(2^{n+1}) < 1 - \frac{1}{n+1}\) for some \(n \in \mathbb{N}\), then \(X\) is reflexive.

**Proof.** Suppose that \(X\) is not reflexive. Let \(0 < \delta < 1\) be given. Let \(\{x_i\} \subseteq S_X\) and \(\{f_i\} \subseteq S_{X^*}\) be two sequences satisfying the two conditions in Theorem 2.7.

Let \(n \in \mathbb{N}\) be given. Let \(y_i = (-1)^{i+1}\frac{\bar{x}_i + x_{i+1}}{2}\) for \(i = 1, \ldots, n + 1\) and \(g_i = (-1)^{i+1}f_i \in S_{X^*}\) for \(i = 2, \ldots, n + 1\). Then, we have

\[
\delta \leq \langle y_i, g_i \rangle = \left\langle (-1)^{i+1}\frac{x_i + x_{i+1}}{2}, (-1)^{i+1}f_i \right\rangle \leq \frac{1}{2}\|x_i + x_{i+1}\| = \|y_i\| \leq 1,
\]

and

\[
\det m(y_1, y_2, y_3, \ldots, y_{n-1}, y_n, y_{n+1}; g_2, g_3, g_4, \ldots, g_n, g_{n+1})
\]

\[
= \det
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
\langle y_1, g_2 \rangle & \langle y_2, g_2 \rangle & \langle y_3, g_2 \rangle & \cdots & \langle y_{n-1}, g_2 \rangle & \langle y_n, g_2 \rangle & \langle y_{n+1}, g_2 \rangle \\
\langle y_1, g_3 \rangle & \langle y_2, g_3 \rangle & \langle y_3, g_3 \rangle & \cdots & \langle y_{n-1}, g_3 \rangle & \langle y_n, g_3 \rangle & \langle y_{n+1}, g_3 \rangle \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
\langle y_1, g_{n-1} \rangle & \langle y_2, g_{n-1} \rangle & \langle y_3, g_{n-1} \rangle & \cdots & \langle y_{n-1}, g_{n-1} \rangle & \langle y_n, g_{n-1} \rangle & \langle y_{n+1}, g_{n-1} \rangle \\
\langle y_1, g_n \rangle & \langle y_2, g_n \rangle & \langle y_3, g_n \rangle & \cdots & \langle y_{n-1}, g_n \rangle & \langle y_n, g_n \rangle & \langle y_{n+1}, g_n \rangle \\
\langle y_1, g_{n+1} \rangle & \langle y_2, g_{n+1} \rangle & \langle y_3, g_{n+1} \rangle & \cdots & \langle y_{n-1}, g_{n+1} \rangle & \langle y_n, g_{n+1} \rangle & \langle y_{n+1}, g_{n+1} \rangle
\end{bmatrix}
\]

\[
= \det
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \delta & -\delta & \cdots & (1)^{n-1}\delta & -(1)^{n}\delta & -(1)^{n+1}\delta \\
0 & -\frac{\delta}{2} & \delta & \cdots & (1)^{n-2}\delta & -(1)^{n-1}\delta & -(1)^n\delta \\
0 & 0 & 0 & \cdots & -\frac{\delta}{2} & \delta & -\delta \\
0 & 0 & 0 & \cdots & 0 & -\frac{\delta}{2} & \delta
\end{bmatrix}
\]
By Lemmas 2.5 and 2.6, we have

\[
\det m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1}) = \delta^n \frac{2n+1}{2^n}.
\]

On the other hand, since

\[
\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} = \frac{\|(-1)^{n+2}x_{n+2} + x_1\|}{2(n+1)} \leq \frac{1}{n+1},
\]

we have

\[
1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \geq 1 - \frac{1}{n+1}.
\]

Since \(\delta\) can be chosen arbitrarily closed to 1, let \(\delta = 1 - \varepsilon^2/4\) where \(\varepsilon\) can be chosen arbitrarily closed to 0.

Let \(z_1 = y_1\). Next, let \(i = 2, 3, \ldots, n + 1\). From Bishop-Phelps-Bollobás result (Lemma 2.4), there exist \(z_i \in S_X\) and \(h_i \in \nabla z_i\) such that \(\|y_i - z_i\| < \varepsilon\) and \(\|g_i - h_i\| < \varepsilon\).

This implies that

\[
|\langle z_i, h_j \rangle - \langle y_i, g_j \rangle| \leq |\langle z_i - y_i, g_j \rangle| + |\langle y_i, h_j - g_j \rangle| + |\langle z_i - y_i, h_j - g_j \rangle| \leq 3\varepsilon.
\]

It follows then that

\[
\det m(z_1, z_2, \ldots, z_{n+1}; h_2, h_3, \ldots, h_{n+1}) = \left(1 - \frac{\varepsilon^2}{4}\right)^n \frac{2n+1}{2^n} + c\varepsilon,
\]

where \(c\) is a bounded constant. Moreover,

\[
1 - \frac{\|z_1 + z_2 + \cdots + z_{n+1}\|}{n+1} \geq 1 - \frac{1 + \varepsilon}{n+1}.
\]

From the definition of \(W^n_X(\varepsilon)\), we have

\[
W^n_X \left(\left(1 - \frac{\varepsilon^2}{4}\right)^n \frac{2n+1}{2^n} + c\varepsilon\right) \geq 1 - \frac{1 + \varepsilon}{n+1}.
\]

Since \(\varepsilon\) can be arbitrarily close to 0, the theorem is proved. \(\square\)
Let $C_{(n+1) \times (n+1)}$ be the following $(n + 1) \times (n + 1)$ matrix:

$$
C_{(n+1) \times (n+1)} := \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
\frac{-2}{3} & 1 & -1 & \cdots & (-1)^{n-1} & (-1)^n & (-1)^{n+1} \\
\frac{1}{3} & \frac{-2}{3} & 1 & \cdots & (-1)^n & (-1)^{n+1} & (-1)^{n+2} \\
0 & \frac{1}{3} & \frac{-2}{3} & 1 & \cdots & (-1)^{n+1} & (-1)^{n+2} & (-1)^{n+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & \cdots & -\frac{2}{3} & 1 & -1 \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{3} & -\frac{2}{3} & 1
\end{bmatrix}
$$

Then $\det(C_{2 \times 2}) = \frac{4}{3}$, and $\det(C_{3 \times 3}) = \frac{7}{3}$.

**Theorem 2.9.** If $X$ is a Banach space with $W_X(\det(C_{(n+1) \times (n+1)}) < \frac{2}{3}$ for some $n \in \mathbb{N}$, then $X$ is reflexive. In particular, for $n = 1$ we have if $W_X(\frac{2}{3}) < \frac{2}{3}$, then $X$ is reflexive; and for $n = 2$ we have if $W_X(\frac{7}{3}) < \frac{2}{3}$, then $X$ is reflexive.

**Proof.** Suppose that $X$ is not reflexive. Let $0 < \delta < 1$ be given. Let $\{x_i\} \subseteq S_X$ and $\{g_i\} \subseteq S_X$ be two sequences satisfying the two conditions in Theorem 2.7.

Let $n \in \mathbb{N}$ be given. Let $y_i = (-1)^{i+1} x_i + x_{i+2}$ for $i = 1, \ldots, n + 1$ and $g_i = (-1)^{i+1} f_i \in S_X$ for $i = 2, \ldots, n + 1$. Then, we have

$$
\delta \leq \langle y_i, g_i \rangle = \left(\frac{(-1)^{i+1} x_i + x_{i+1} + x_{i+2}}{3}, (-1)^{i+1} f_i \right) \\
\leq \frac{1}{3} \|x_i + x_{i+1} + x_{i+2}\| = \|y_i\| \leq 1,
$$

and

$$
m(y_1, y_2, y_3, \ldots, y_{n-1}, y_n, y_{n+1}; g_2, g_3, g_4, \ldots, g_{n-1}, g_n, g_{n+1}) \\
= \left| \begin{array}{cccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
\langle y_1, g_2 \rangle & \langle y_2, g_3 \rangle & \langle y_3, g_4 \rangle & \cdots & \langle y_{n-1}, g_n \rangle & \langle y_n, g_{n+1} \rangle & \langle y_{n+1}, g_1 \rangle \\
\langle y_1, g_3 \rangle & \langle y_2, g_4 \rangle & \langle y_3, g_5 \rangle & \cdots & \langle y_{n-1}, g_{n+1} \rangle & \langle y_n, g_1 \rangle & \langle y_{n+1}, g_2 \rangle \\
\langle y_1, g_4 \rangle & \langle y_2, g_5 \rangle & \langle y_3, g_6 \rangle & \cdots & \langle y_{n-1}, g_2 \rangle & \langle y_n, g_3 \rangle & \langle y_{n+1}, g_3 \rangle \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\langle y_1, g_{n+1} \rangle & \langle y_2, g_1 \rangle & \langle y_3, g_2 \rangle & \cdots & \langle y_{n-1}, g_1 \rangle & \langle y_n, g_4 \rangle & \langle y_{n+1}, g_4 \rangle \\
\langle y_1, g_n \rangle & \langle y_2, g_{n+1} \rangle & \langle y_3, g_{n+1} \rangle & \cdots & \langle y_{n-1}, g_3 \rangle & \langle y_n, g_5 \rangle & \langle y_{n+1}, g_5 \rangle
\end{array} \right| \\
= \delta^n
$$

$$
= \left| \begin{array}{cccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
-\frac{2}{3} & 1 & -1 & \cdots & (-1)^{n-1} & (-1)^n & (-1)^{n+1} \\
\frac{1}{3} & -\frac{2}{3} & 1 & -1 & \cdots & (-1)^n & (-1)^{n+1} & (-1)^{n+2} \\
0 & \frac{1}{3} & -\frac{2}{3} & 1 & \cdots & (-1)^{n+1} & (-1)^{n+2} & (-1)^{n+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & \cdots & -\frac{2}{3} & 1 & -1 \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{3} & -\frac{2}{3} & 1
\end{array} \right|
$$
We have
\[
\det m(y_1, y_2, y_3, y_4, \ldots, y_{n-1}, y_n, y_{n+1}; g_2, g_3, g_4, \ldots, g_{n-1}, g_n, g_{n+1})
\]
\[= \delta^n \det C_{(n+1) \times (n+1)}.\]

On the other hand, for \(n \geq 2\),
\[
\|y_1 + y_2 + \cdots + y_{n+1}\|_{n+1} \leq \frac{n+1}{2(n+1)} \delta = \frac{1}{3} \delta,
\]
and for \(n = 1\),
\[
\|y_1 + y_2 + \cdots + y_{n+1}\|_{n+1} = \|x_1 - x_4\|_6 \leq \frac{1}{3} \delta.
\]
We have
\[
1 - \|y_1 + y_2 + \cdots + y_{n+1}\|_{n+1} \geq 1 - \frac{1}{3} \delta \geq \frac{2}{3} \delta
\]
for all \(n \in \mathbb{N}\).

The theorem can be proved by using the Bishop-Phelps-Bollobás result (Lemma 2.4), and same idea in the proof of Theorem 2.8. □

We consider \(n = 1\).

**Theorem 2.10.** If \(X\) is a Banach space with \(W^1_X(\frac{2m+1}{m+1}) < \frac{m}{m+1}\) for some \(m \in \mathbb{N}\), then \(X\) is reflexive. In particular, for \(m = 2\) we have if \(W^1_X(\frac{5}{3}) < \frac{2}{3}\), then \(X\) is reflexive.

**Proof.** Suppose that \(X\) is not reflexive. Let \(0 < \delta < 1\) be given. Let \(\{x_i\} \subseteq S_X\) and \(\{f_i\} \subseteq S_{X^*}\) be two sequences satisfying the two conditions in Theorem 2.7.

Let \(m \in \mathbb{N}\) be given. Let
\[
y_1 = \frac{x_1 + x_2 + \cdots + x_m + x_{m+1}}{m+1}, y_2 = \frac{x_2 + x_3 + \cdots + x_{m+1} + x_{m+2}}{m+1}
\]
and \(g_2 = -f_2 \in S_{X^*}\).

Consider the 2-dimensional subspace of \(X\) spanned by \(y_1\) and \(y_2\).

We have
\[
\det m(y_1, y_2; g_2) = \det \begin{bmatrix} 1 & 1 \\ \langle y_1, g_2 \rangle & \langle y_2, g_2 \rangle \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ -\frac{m}{m+1} & 1 \end{bmatrix} \delta = \frac{2m+1}{m+1} \delta,
\]
and
\[
\|\frac{y_1 + y_2}{2}\| = \|\frac{x_1 - x_{m+2}}{2(m+1)}\| \leq \frac{1}{m+1} \delta.
\]
This is
\[
1 - \|\frac{y_1 + y_2}{2}\| \geq \frac{m}{m+1} \delta.
\]
Similar to the proof of Theorem 2.8 we have
\[
W^1_X\left(\frac{2m+1}{m+1}\right) \geq \frac{m}{m+1}.
\]
This completes the proof.

In 2008, Saegung proved the following result:

**Lemma 2.11** ([11]). If $X$ is a Banach space with $B_{X^*}$ is weak* sequentially compact and it fails to have weak normal structure, then for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there are \{x_1, x_2, \ldots, x_n\} $\subseteq S_X$ and \{f_1, f_2, \ldots, f_n\} $\subseteq S_{X^*}$ such that

(a) $\|x_i - x_j\| - 1 < \varepsilon$ for all $i \neq j$;
(b) $\langle x_i, f_i \rangle = 1$ for all $1 \leq i \leq n$; and
(c) $\|\langle x_i, f_j \rangle\| < \varepsilon$ for all $i \neq j$.

**Theorem 2.12.** If $X$ is a Banach space with $B_{X^*}$ is weak* sequentially compact and $W_X^n(1) < 1 - \frac{1}{n+1}$ for some $n \in \mathbb{N}$, then $X$ has weak normal structure.

**Proof.** Suppose that $X$ does not have weak normal structure. Let $0 < \varepsilon < 1$ be given. Then there are \{x_i\}_{i=1}^{n+1} $\subseteq S_X$ and \{f_i\}_{i=1}^{n+1} $\subseteq S_{X^*}$ satisfying the three conditions in Lemma 2.11.

For convenience, let $\|\langle x_i, f_j \rangle\| = \varepsilon_{i,j}$. Then $\varepsilon_{i,j} \leq \varepsilon$ for all $i \neq j$.

Let $y_i = \frac{x_{i+1}-x_i}{\|x_{i+1}-x_i\|} \in S_X$ for $i = 1, \ldots, n+1$ and $g_i = f_{i+1} \in S_{X^*}$ for $i = 2, \ldots, n+1$. Then

$$\|y_i - (x_{i+1} - x_i)\| \leq \varepsilon$$

for $i = 1, \ldots, n+1$. Moreover,

$$\|y_1 + y_2 + \cdots + y_n + y_{n+1}\|
\leq \|(x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{i+1} - x_i) + \cdots + (x_{n+2} - x_{n+1})\| + (n+1)\varepsilon
= \|x_{n+2} - x_1\| + (n+1)\varepsilon.$$

Next, we consider the following matrix:

$$m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1})$$

$$= \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
\langle y_1, g_2 \rangle & \langle y_2, g_2 \rangle & \langle y_3, g_2 \rangle & \cdots & \langle y_n, g_2 \rangle & \langle y_{n+1}, g_2 \rangle \\
\langle y_1, g_3 \rangle & \langle y_2, g_3 \rangle & \langle y_3, g_3 \rangle & \cdots & \langle y_n, g_3 \rangle & \langle y_{n+1}, g_3 \rangle \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\langle y_1, g_{n+1} \rangle & \langle y_2, g_{n+1} \rangle & \langle y_3, g_{n+1} \rangle & \cdots & \langle y_n, g_{n+1} \rangle & \langle y_{n+1}, g_{n+1} \rangle \\
\end{pmatrix}$$

$$= \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
\frac{\varepsilon_2}{\varepsilon_{2,1}} & \frac{\varepsilon_3}{\varepsilon_{3,2}} & \frac{\varepsilon_4}{\varepsilon_{4,3}} & \cdots & \frac{\varepsilon_{n+1}}{\varepsilon_{n+1,n}} & \frac{\varepsilon_{n+2}}{\varepsilon_{n+2,n+1}} \\
\frac{\varepsilon_3}{\varepsilon_{3,2}} & \frac{\varepsilon_4}{\varepsilon_{4,3}} & \frac{\varepsilon_5}{\varepsilon_{5,4}} & \cdots & \frac{\varepsilon_{n+2}}{\varepsilon_{n+2,n+1}} & \frac{\varepsilon_{n+3}}{\varepsilon_{n+3,n+2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\varepsilon_{n+1}}{\varepsilon_{n+1,n}} & \frac{\varepsilon_{n+2}}{\varepsilon_{n+2,n+1}} & \frac{\varepsilon_{n+3}}{\varepsilon_{n+3,n+2}} & \cdots & \frac{\varepsilon_{n+2}}{\varepsilon_{n+2,n+1}} & \frac{\varepsilon_{n+3}}{\varepsilon_{n+3,n+2}} \\
\frac{\varepsilon_{n+2}}{\varepsilon_{n+2,n+1}} & \frac{\varepsilon_{n+3}}{\varepsilon_{n+3,n+2}} & \frac{\varepsilon_{n+4}}{\varepsilon_{n+4,n+3}} & \cdots & \frac{\varepsilon_{n+2}}{\varepsilon_{n+2,n+1}} & \frac{\varepsilon_{n+4}}{\varepsilon_{n+4,n+3}} \\
\end{pmatrix}$$

It follows then that

$$\det m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1}) = 1 + c\varepsilon,$$
where \( c \) is a bounded constant.

On the other hand, since
\[
\left\| \frac{y_1 + y_2 + \cdots + y_{n+1}}{n+1} \right\| \leq \left\| \frac{x_{n+2} - x_1}{n+1} \right\| + \varepsilon \leq \frac{1 + \varepsilon}{n+1} + \varepsilon,
\]
we have
\[
1 - \left( \frac{y_1 + y_2 + \cdots + y_{n+1}}{n+1} \right) \geq 1 - \frac{1 + \varepsilon}{n+1} - \varepsilon.
\]

Let \( z_1 = y_1 \). Next, let \( i = 2, 3, \ldots, n + 1 \).

From Bishop-Phelps-Bollobás result (Lemma 2.4), there exist \( z_i \in S_X \) and \( h_i \in \nabla z_i \), such that
\[
\|y_i - z_i\| < \varepsilon \quad \text{and} \quad \|y_i - h_i\| < \varepsilon.
\]
In particular,
\[
\langle z_i, h_j \rangle - \langle y_i, g_j \rangle \leq \|z_i - y_i\| + \|y_i - h_j - g_j\| + \|z_i - y_i, h_j - g_j\| \leq 3\varepsilon.
\]
This implies that
\[
\det m(z_1, z_2, \ldots, z_{n+1}, h_1, h_2, \ldots, h_{n+1}) = \det \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
\langle z_1, h_2 \rangle & \langle z_2, h_2 \rangle & \cdots & \langle z_n, h_2 \rangle & \langle z_{n+1}, h_2 \rangle \\
\langle z_1, h_3 \rangle & \langle z_2, h_3 \rangle & \cdots & \langle z_n, h_3 \rangle & \langle z_{n+1}, h_3 \rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\langle z_1, h_{n+1} \rangle & \langle z_2, h_{n+1} \rangle & \cdots & \langle z_n, h_{n+1} \rangle & \langle z_{n+1}, h_{n+1} \rangle \\
\end{bmatrix}
\]
\[
= 1 + d\varepsilon,
\]
where \( d \) is a bounded constant. Hence
\[
1 - \frac{\|z_1 + z_2 + \cdots + z_{n+1}\|}{n+1} \geq 1 - \frac{1 + \varepsilon}{n+1} - 2\varepsilon.
\]

Since \( \varepsilon \) can be arbitrarily small, it follows from the definition of \( W_X^n(\cdot) \) that
\[
W_X^n(1) \geq 1 - \frac{1}{n+1}.
\]
This completes the proof. \( \square \)

**Theorem 2.13.** If \( X \) is a Banach space satisfying one of the following two conditions:

1. \( W_X^n(1) < 1 - \frac{1}{n+1} \) for some \( n \in \mathbb{N} \) with \( n \geq 2 \); or
2. \( W_X^1(1) < \frac{1}{2} \) and \( W_X^n(\frac{2}{3}) < \frac{2}{3} \) for \( n = 1 \).

Then \( X \) has normal structure.

**Proof.** Since \( X \) is reflexive, it follows that \( B_X^* \) is weak* sequentially compact. Moreover, \( \frac{2n+1}{3} < 1 \) for \( n \in \mathbb{N} \) and \( n \geq 3 \), and \( \frac{2}{3} < 1 \) for \( n = 2 \). The first result is a direct consequence of Theorems 2.8, 2.9 and 2.12. The second result is a direct consequence of Theorems 2.10 and 2.12. \( \square \)
Definition 2.14 ([4, 5]). Let $X$ and $Y$ be Banach spaces. We say that $Y$ is finitely representable in $X$ if for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$ there is an isomorphism $T : N \to X$ such that for any $y \in N$,

$$(1 - \varepsilon)\|y\| \leq \|Ty\| \leq (1 + \varepsilon)\|y\|.$$ 

We say that $X$ is super-reflexive if any space $Y$ which is finitely representable in $X$ is reflexive.

Theorem 2.15. If $X$ is a Banach space with $W_X^p\left(\frac{2n+1}{2n}\right) < 1 - \frac{1}{n+1}$ for some $n \in \mathbb{N}$ and $n \geq 2$, or $W_X^3\left(\frac{2m+1}{2m}\right) < \frac{m}{m+1}$ for $n = 1$ and some $m \in \mathbb{N}$, then $X$ is super-reflexive. In particular, for $m = 2$ we have if $W_X^1\left(\frac{2}{3}\right) < \frac{2}{3}$, then $X$ is super-reflexive.

Proof. We only prove the first part (for $n \geq 2$). The proof of second part (for $n = 1$) is same.

The proof is similar to that of Theorem 2.12 in [6]. Suppose that $X$ is not super-reflexive. Then there exists a nonreflexive Banach space $Y$ such that $Y$ can be finitely representable. From Remark 2.2 and Theorem 2.8, for each $n$ there exists some positive function $f(\varepsilon)$ which goes to 0 as $\varepsilon$ goes to 0, satisfies $W_Y^p\left(\frac{2n+1}{2n}\right) > 1 - \frac{1}{n+1} - f(\varepsilon)$. Therefore, there exist $\{y_i\}_{i=1}^{n+1} \subseteq S_Y$ and $\{g_i\} \in \nabla y_i \subseteq S_{Y^*}$ for $i = 2, \ldots, n + 1$ such that

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle y_1, g_1 \rangle & \langle y_2, g_1 \rangle & \cdots & \langle y_n, g_1 \rangle & \langle y_{n+1}, g_1 \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle y_1, g_{n+1} \rangle & \langle y_2, g_{n+1} \rangle & \cdots & \langle y_n, g_{n+1} \rangle & \langle y_{n+1}, g_{n+1} \rangle \end{bmatrix} \geq \frac{2n+1}{2n} - 1,$$

and

$$1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} > 1 - \frac{1}{n+1} - f(\varepsilon).$$

Let $N = \text{span}\{y_1, y_2, \ldots, y_{n+1}\}$, and $T : N \to M \subseteq X$ be an isomorphism with range $M$.

Let us consider the conjugate mapping $T^*$ of $T$. Let $g_{i|N}$ be the restricting $g_i$ on $N$. Then $\langle Ty_j, (T^*)^{-1}g_{i|N} \rangle = \langle y_j, g_i \rangle$ for $1 \leq i, j \leq n + 1$.

We have

$$1 - \varepsilon \leq \|T\| \leq 1 + \varepsilon,$$

$$1 - \varepsilon \leq \|T^*\| \leq 1 + \varepsilon,$$

and

$$1 - \varepsilon \leq \|(T^*)^{-1}\| \leq 1 + \varepsilon.$$

Let $x_i = Ty_i$ and $f_i = (T^*)^{-1}g_{i|N}$ for $i = 1, \ldots, n + 1$. Then

$$\langle x_j, f_i \rangle = \langle Ty_j, (T^*)^{-1}g_{i|N} \rangle = \langle y_j, g_i \rangle.$$
If \( i = j \), then \( \langle x_i, f_i \rangle = \langle y_i, g_i \rangle = 1 \), so \( f_i \in \nabla x_i \) and we have

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
\langle x_1, f_2 \rangle & \langle x_2, f_2 \rangle & \cdots & \langle x_n, f_2 \rangle & \langle x_{n+1}, f_2 \rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\langle x_1, f_{n+1} \rangle & \langle x_2, f_{n+1} \rangle & \cdots & \langle x_n, f_{n+1} \rangle & \langle x_{n+1}, f_{n+1} \rangle
\end{vmatrix}
= \begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
\langle y_1, g_2 \rangle & \langle y_2, g_2 \rangle & \cdots & \langle y_n, g_2 \rangle & \langle y_{n+1}, g_2 \rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\langle y_1, g_{n+1} \rangle & \langle y_2, g_{n+1} \rangle & \cdots & \langle y_n, g_{n+1} \rangle & \langle y_{n+1}, g_{n+1} \rangle
\end{vmatrix}
\leq \frac{2n+1}{2^n} - \varepsilon.
\]

On the other hand,

\[
\frac{\|x_1 + x_2 + \cdots + x_{n+1}\|}{n+1} = \frac{\|T(y_1 + y_2 + \cdots + y_{n+1})\|}{n+1}
\leq (1+\varepsilon)\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1}
\leq \frac{1+\varepsilon}{n+1} + (1+\varepsilon)f(\varepsilon).
\]

This implies that

\[
1 - \frac{\|x_1 + x_2 + \cdots + x_{n+1}\|}{n+1} \geq 1 - \frac{1+\varepsilon}{n+1} - (1+\varepsilon)f(\varepsilon).
\]

Since \( f(\varepsilon) \) can be arbitrarily small, we have

\[
W_n^X\left(\frac{2n+1}{2^n}\right) \geq 1 - \frac{1}{n+1}.
\]

This completes the proof. \(\square\)

We consider the uniform normal structure. To discuss this result, let us recall the concept of the “ultra”-technique.

Let \( F \) be a filter of an index set \( I \), and let \( \{x_i\}_{i \in I} \) be a subset in a Hausdorff topological space \( X \), \( \{x_i\}_{i \in I} \) is said to converge to \( x \) with respect to \( F \), denoted by \( \lim_F x_i = x \), if for each neighborhood \( V \) of \( x \), \( \{i \in I : x_i \in V\} \in F \).

A filter \( U \) on \( I \) is called an ultrafilter if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called trivial if it is of the form \( \{A : A \subseteq I, i_0 \in A\} \) for some \( i_0 \in I \). We will use the fact that if \( U \) is an ultrafilter, then

(i) for any \( A \subseteq I \), either \( A \subseteq U \) or \( I - A \subseteq U \);

(ii) if \( \{x_i\}_{i \in I} \) has a cluster point \( x \), then \( \lim_U x_i \) exists and equals to \( x \).

Let \( \{X_i\}_{i \in I} \) be a family of Banach spaces and let \( l_\infty(I, X_i) \) denote the subspace of the product space equipped with the norm \( \|\langle x_i \rangle\| = \sup_{i \in I} \|x_i\| < \infty \).
Definition 2.16 ([3, 12]). Let $\mathcal{U}$ be an ultrafilter on $I$ and let $N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_i \|x_i\| = 0\}$. The ultra-product of $\{X_i\}_{i \in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote the element of the ultra-product. It follows from remark (ii) above, and the definition of quotient norm that

\begin{equation}
\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.
\end{equation}

In the following we will restrict our index set $I$ to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbb{N}$ for some Banach space $X$. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the ultra-product. Note that if $\mathcal{U}$ is nontrivial, then $X$ can be embedded into $X_{\mathcal{U}}$ isometrically.

Lemma 2.17 ([12]). Suppose that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and $X$ is a Banach space. Then $(X^*)_{\mathcal{U}} \cong (X_{\mathcal{U}})^*$ if and only if $X$ is super-reflexive; and in this case, the mapping $J$ defined by

\[ \langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}}) \rangle = \lim_{\mathcal{U}} \langle x_i, f_i \rangle \quad \text{for all} \quad (x_i)_{\mathcal{U}} \in X_{\mathcal{U}} \]

is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

Theorem 2.18. Let $X$ be a super-reflexive Banach space. Then for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and for all $n \in \mathbb{N}$ and $\varepsilon > 0$, we have $W^n_{X_{\mathcal{U}}}(\varepsilon) = W^n_X(\varepsilon)$.

Proof. Since $X$ can be embedded into $X_{\mathcal{U}}$ isometrically, we may consider $X$ as a subspace of $X_{\mathcal{U}}$. From the definition of $W^n_X(\varepsilon)$, we have $W^n_{X_{\mathcal{U}}}(\varepsilon) \geq W^n_X(\varepsilon)$.

We prove the reverse inequality.

For any very small $\eta > 0$, from the definition of $W^n_{X_{\mathcal{U}}}(\varepsilon)$, let $(x^1)_{\mathcal{U}}, (x^2)_{\mathcal{U}}, \ldots, (x^n)_{\mathcal{U}}, (x^{n+1})_{\mathcal{U}}$ belong to $S_{X_{\mathcal{U}}}$, and let $(f^1)_{\mathcal{U}} \in \nabla(x^1)_{\mathcal{U}}, (f^2)_{\mathcal{U}} \in \nabla(x^2)_{\mathcal{U}}, \ldots, (f^n)_{\mathcal{U}} \in \nabla(x^n)_{\mathcal{U}}, (f^{n+1})_{\mathcal{U}} \in \nabla(x^{n+1})_{\mathcal{U}}$ be such that

\[ m((x^1)_{\mathcal{U}}, (x^2)_{\mathcal{U}}, \ldots, (x^n)_{\mathcal{U}}, (x^{n+1})_{\mathcal{U}}; (f^1)_{\mathcal{U}}, (f^2)_{\mathcal{U}}, \ldots, (f^n)_{\mathcal{U}}, (f^{n+1})_{\mathcal{U}}) \leq \varepsilon, \]

and

\[ 1 - \frac{\|(x^1)_{\mathcal{U}} + (x^2)_{\mathcal{U}} + \cdots + (x^n)_{\mathcal{U}} + (x^{n+1})_{\mathcal{U}}\|}{n + 1} > W^n_{X_{\mathcal{U}}}(\varepsilon) - \eta. \]

Without loss of generality, we may assume by (2.1) that

\[ 1 - \eta < \|(x^j)_{\mathcal{U}}\| < 1 + \eta \quad \text{for} \quad j = 1, \ldots, n + 1, \]

\[ 1 - \eta < \|(f^j)_{\mathcal{U}}\| < 1 + \eta \quad \text{for} \quad j = 2, \ldots, n + 1, \]

and

\[ 1 - \eta < \langle (x^j)_{\mathcal{U}}, (f^j)_{\mathcal{U}} \rangle < 1 + \eta \quad \text{for} \quad j = 2, \ldots, n + 1. \]

From the property of ultra-product, we know the subsets

\[ P = \{ i : m((x^1)_{\mathcal{U}}, (x^2)_{\mathcal{U}}, \ldots, (x^n)_{\mathcal{U}}, (x^{n+1})_{\mathcal{U}}; (f^1)_{\mathcal{U}}, (f^2)_{\mathcal{U}}, \ldots, (f^n)_{\mathcal{U}}, (f^{n+1})_{\mathcal{U}}) \leq \varepsilon \} \]
and

\[ Q = \left\{ i : 1 - \frac{\| (x_1^1)_U + (x_2^2)_U + \cdots + (x_n^n)_{U_U} + (x_{n+1}^{n+1})_{U_U} \|}{n+1} > W^n_{X^U}(\varepsilon) - \eta \right\} \]

are all in \( U \). So the intersection \( P \cap Q \) is in \( U \) too, and hence is not empty.

Let \( i \in P \cap Q \) be fixed. Then

\[ 1 - \eta < \| x_i^1 \| < 1 + \eta \quad \text{for} \quad j = 1, \ldots, n+1; \]

\[ 1 - \eta < \| f_i^2 \| < 1 + \eta \quad \text{for} \quad j = 2, \ldots, n+1; \]

\[ 1 - \eta < (x_i^1, f_i^2) < 1 + \eta \quad \text{for} \quad j = 2, \ldots, n+1; \]

\[ m(x_i^1, x_i^2, \ldots, x_i^n, x_i^{n+1}; f_i^2, f_i^3, \ldots, f_i^n, f_i^{n+1}) \leq \varepsilon; \]

and

\[ 1 - \frac{\| x_i^1 + x_i^2 + \cdots + x_i^n + x_i^{n+1} \|}{n+1} > W^n_{X^U}(\varepsilon) - \eta. \]

From Lemma 2.4, for \( 0 < \eta < 1 \) (since \( \eta \) can be arbitrarily small, if necessary we can normalize vectors \( x_i^1 \) and \( f_i^2 \) to use Lemma 2.4) there are \( \{y_j \}_{j=1}^{n+1} \subseteq S_X \) and \( \{y_j \}_{j=2}^{n+1} \subseteq S_{X^*} \) such that

\[ g_j \in \nabla y_j \quad \text{for} \quad j = 2, \ldots, n+1; \]

\[ \| x_i^1 - y_j \| < \eta \quad \text{for} \quad j = 1, \ldots, n+1; \]

\[ \| f_i^2 - g_j \| < \eta \quad \text{for} \quad j = 2, \ldots, n+1. \]

Similar to the proof of Theorem 2.8, we have

\[ \det m(y_1, y_2, \ldots, y_n, y_{n+1}; g_2, g_3, \ldots, g_n, g_{n+1}) \leq c + d\eta, \]

and

\[ 1 - \frac{\| y_1 + y_2 + \cdots + y_n + y_{n+1} \|}{n+1} > W^n_{X^U}(\varepsilon) - \eta, \]

where \( c \) and \( d \) are constants.

Since \( \eta > 0 \) can be arbitrarily small, we have \( W^n_X(\varepsilon) \geq W^n_{X^U}(\varepsilon). \)

**Lemma 2.19 ([8])**. If \( X \) is a super-reflexive Banach space, then \( X \) has uniform normal structure if and only if \( X^U \) has normal structure.

**Theorem 2.20.** Suppose that \( X \) is a Banach space satisfying one of the following conditions:

\[ \bullet \quad W_X^1(1) < 1 - \frac{1}{n+1} \quad \text{for some} \quad n \in \mathbb{N} \quad \text{with} \quad n \geq 2; \]

\[ \bullet \quad W_X^1(1) < \frac{1}{n} \quad \text{and} \quad W_X^2\left( \frac{n}{3} \right) < \frac{1}{4} \quad \text{for} \quad n = 1. \]

Then \( X \) has uniform normal structure.

**Proof.** It follows directly from Theorems 2.13, 2.15, 2.18 and Lemma 2.19. \( \square \)

**Example.** Let \( H \) be a Hilbert space. We have \( W_H^1(\varepsilon) = \frac{2\sqrt{2}\sqrt{1 - \varepsilon}}{2} \) for \( 0 \leq \varepsilon \leq 2. \)

Since \( W_H^1(1) = \frac{2\sqrt{2}}{2} = 0.29289 \cdots < \frac{1}{4} \), and \( W_H^2\left( \frac{n}{3} \right) = \frac{2\sqrt{2}}{2} = 0.59175 \cdots < \frac{1}{4} \), from Theorem 2.20, \( H \) has uniform normal structure.

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