CHARACTERIZATIONS OF CENTRALIZERS AND DERIVATIONS ON SOME ALGEBRAS

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Abstract. A linear mapping $\phi$ on an algebra $A$ is called a centralizable mapping at $G \in A$ if $\phi(AB) = \phi(A)B = A\phi(B)$ for each $A$ and $B$ in $A$ with $AB = G$, and $\phi$ is called a derivable mapping at $G \in A$ if $\phi(AB) = \phi(A)B + A\phi(B)$ for each $A$ and $B$ in $A$ with $AB = G$. A point $G$ in $A$ is called a full-centralizable point (resp. full-derivable point) if every centralizable (resp. derivable) mapping at $G$ is a centralizer (resp. derivation). We prove that every point in a von Neumann algebra or a triangular algebra is a full-centralizable point. We also prove that a point in a von Neumann algebra is a full-derivable point if and only if its central carrier is the unit.

1. Introduction

Let $A$ be an associative algebra over the complex field $\mathbb{C}$, and $\phi$ be a linear mapping from $A$ into itself. $\phi$ is called a centralizer if $\phi(AB) = \phi(A)B = A\phi(B)$ for each $A$ and $B$ in $A$. Obviously, if $A$ is an algebra with unit $I$, then $\phi$ is a centralizer if and only if $\phi(A) = \phi(I)A = A\phi(I)$ for every $A$ in $A$. $\phi$ is called a derivation if $\phi(AB) = \phi(A)B + A\phi(B)$ for each $A$ and $B$ in $A$.

A linear mapping $\phi : A \to A$ is called a centralizable mapping at $G \in A$ if $\phi(AB) = \phi(A)B = A\phi(B)$ for each $A$ and $B$ in $A$ with $AB = G$, and $\phi$ is called a derivable mapping at $G \in A$ if $\phi(AB) = \phi(A)B + A\phi(B)$ for each $A$ and $B$ in $A$ with $AB = G$. An element $G$ in $A$ is called a full-centralizable point (resp. full-derivable point) if every centralizable (resp. derivable) mapping at $G$ is a centralizer (resp. derivation).

In [3], Brešar proves that if $R$ is a prime ring with a nontrivial idempotent, then $0$ is a full-centralizable point. In [18], X. Qi and J. Hou characterize centralizable and derivable mappings at $0$ in triangular algebras. In [17], X. Qi proves that every nontrivial idempotent in a prime ring is a full-centralizable point. In [19], W. Xu, R. An and J. Hou prove that every element in $B(H)$ is...
a full-centralizable point, where \( \mathcal{H} \) is a Hilbert space. For more information on centralizable and derivable mappings, we refer to [2, 7, 11, 12, 14, 20].

For a von Neumann algebra \( \mathcal{A} \), the central carrier \( C(A) \) of an element \( A \) in \( \mathcal{A} \) is the projection \( I - P \), where \( P \) is the union of all central projections \( P_\alpha \) in \( \mathcal{A} \) such that \( P_\alpha A = 0 \).

This paper is organized as follows. In Section 2, by using the techniques about central carriers, we show that every element in a von Neumann algebra is a full-centralizable point.

Let \( A \) and \( B \) be two unital algebras over the complex field \( \mathbb{C} \), and \( M \) be a unital \( (\mathcal{A}, \mathcal{B}) \)-bimodule which is faithful both as a left \( \mathcal{A} \)-module and a right \( \mathcal{B} \)-module. The algebra

\[
\text{Tri}(\mathcal{A}, M, \mathcal{B}) = \left\{ \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} : A \in \mathcal{A}, B \in \mathcal{B}, M \in M \right\}
\]

under the usual matrix addition and matrix multiplication is called a triangular algebra.

In Section 3, we show that if \( A \) and \( B \) are two unital Banach algebras, then every element in \( \text{Tri}(\mathcal{A}, M, \mathcal{B}) \) is a full-centralizable point.

In Section 4, we show that for every point \( G \) in a von Neumann algebra \( \mathcal{A} \), if \( \Delta \) is a derivable mapping at \( G \), then \( \Delta = D + \phi \), where \( D : \mathcal{A} \to \mathcal{A} \) is a derivation and \( \phi : \mathcal{A} \to \mathcal{A} \) is a centralizer. Moreover, we prove that \( G \) is a full-derivable point if and only if \( C(G) = I \).

2. Centralizers on von Neumann algebras

In this section, \( \mathcal{A} \) denotes a unital algebra and \( \phi : \mathcal{A} \to \mathcal{A} \) is a centralizable mapping at a given point \( G \in \mathcal{A} \). The main result is the following theorem.

**Theorem 2.1.** Let \( \mathcal{A} \) be a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \). Then every element \( G \) in \( \mathcal{A} \) is a full-centralizable point.

Before proving Theorem 2.1, we need the following several lemmas.

**Lemma 2.2.** Let \( \mathcal{A} \) be a unital Banach algebra with the form \( \mathcal{A} = \sum_{i \in \Lambda} \mathcal{A}_i \). Then \( \phi(\mathcal{A}_i) \subseteq \mathcal{A}_i \). Moreover, suppose \( G = \sum_{i \in \Lambda} G_i \), where \( G_i \in \mathcal{A}_i \). If \( G_i \) is a full-centralizable point in \( \mathcal{A}_i \) for every \( i \in \Lambda \), then \( G \) is a full-centralizable point in \( \mathcal{A} \).

**Proof.** Let \( I_i \) be the unit in \( \mathcal{A}_i \). Suppose that \( A_i \) is an invertible element in \( \mathcal{A}_i \), and \( t \) is an arbitrary nonzero element in \( \mathbb{C} \). It is easy to check that

\[
(I - I_i + t^{-1}GA_i^{-1})((I - I_i)G + tA_i) = G.
\]

So we have

\[
(I - I_i + t^{-1}GA_i^{-1})\phi((I - I_i)G + tA_i) = \phi(G).
\]

Considering the coefficient of \( t \), since \( t \) is arbitrarily chosen, we have \( (I - I_i)\phi(A_i) = 0 \). It follows that \( \phi(A_i) = I_i\phi(A_i) \in \mathcal{A}_i \) for all invertible elements. Since \( \mathcal{A}_i \) is a Banach algebra, every element can be written into the sum of two
invertible elements. So the above equation holds for all elements in $A_i$. That is to say $\phi(A_i) \subseteq A_i$.

Let $\phi_i = \phi |_{A_i}$. For every $A$ in $A$, we write $A = \sum_{i \in A} A_i$. Assume $AB = G$.

Since $A_iB_i = G_i$ and $\phi(A_i) \subseteq A_i$, we have

$$\sum_{i \in A} \phi(G_i) = \sum_{i \in A} \phi(A_i) \sum_{i \in A} B_i = \sum_{i \in A} \phi(A_i)B_i.$$ 

It implies that $\phi_i(G_i) = \phi_i(A_i)B_i$. Similarly, we can obtain $\phi_i(G_i) = A_i\phi_i(B_i)$.

By assumption, $G_i$ is a full-centralizable point, so $\phi_i$ is a centralizer. Hence

$$\phi(A) = \sum_{i \in A} \phi_i(A_i) = \sum_{i \in A} \phi_i(I_i)A_i = \sum_{i \in A} \phi_i(I_i) \sum_{i \in A} A_i = \phi(I)A.$$ 

Similarly, we can prove $\phi(A) = A\phi(I)$. Hence $G$ is a full-centralizable point. \qed

**Lemma 2.3.** Let $A$ be a $C^*$-algebra. If $G^*$ is a full-centralizable point in $A$, then $G$ is a full-centralizable point in $A$.

**Proof.** Define a linear mapping $\tilde{\phi} : A \to A$ by: $\tilde{\phi}(A) = (\phi(A^*))^*$ for every $A$ in $A$. For each $A$ and $B$ in $A$ with $AB = G^*$, we have $B^*A^* = G$. It follows that $\phi(G) = \phi(B^*)A^* = B^*\phi(A^*)$. By the definition of $\tilde{\phi}$, we obtain $\tilde{\phi}(G^*) = \tilde{\phi}(A)B = A\tilde{\phi}(B)$. Since $G^*$ is a full-centralizable point in $A$, we have that $\tilde{\phi}$ is a centralizer. Thus $\tilde{\phi}$ is also a centralizer. Hence $G$ is a full-centralizable point in $A$. \qed

For a unital algebra $A$ and a unital $A$-bimodule $M$, an element $A \in A$ is called a left separating point (resp. right separating point) of $M$ if $AM = 0$ implies $M = 0$ ($MA = 0$ implies $M = 0$) for every $M \in M$.

**Lemma 2.4.** Let $A$ be a unital Banach algebra and $G$ be a left and right separating point in $A$. Then $G$ is a full-centralizable point.

**Proof.** For every invertible element $X$ in $A$, we have

$$\phi(I)G = \phi(G) = \phi(XX^{-1}G) = \phi(X)X^{-1}G.$$ 

Since $G$ is a right separating point, we obtain $\phi(I) = \phi(X)X^{-1}$. It follows that $\phi(X) = \phi(I)X$ for each invertible element $X$ and so for all elements in $A$. Similarly, we have that $\phi(X) = X\phi(I)$. Hence $G$ is a full-centralizable point. \qed

**Lemma 2.5.** Let $A$ be a von Neumann algebra. Then $G = 0$ is a full-centralizable point.

**Proof.** For any projection $P$ in $A$, since $P(I - P) = (I - P)P = 0$, we have

$$\phi(P)(I - P) = P\phi(I - P) = \phi(I - P)P = (I - P)\phi(P) = 0.$$ 

It follows that $\phi(P) = \phi(I)P = P\phi(I)$. By [6, Proposition 2.4] and [4, Corollary 1.2], we know that $\phi$ is continuous. Since $A = \text{span}\{P \in A : P = P^* = P^2\}$,
it follows that \( \phi(A) = \phi(I)A = A\phi(I) \) for every \( A \in \mathcal{A} \). Hence \( G \) is a full-centralizable point. \( \square \)

**Lemma 2.6.** Let \( \mathcal{A} \) be a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \) and \( P \) be the range projection of \( G \). If \( C(P) = C(I - P) = I \), then \( G \) is a full-centralizable point.

**Proof.** Set \( P_1 = P \), \( P_2 = I - P \), and denote \( P_i \mathcal{A} P_j \) by \( \mathcal{A}_{ij} \), \( i, j = 1, 2 \). For every \( A \in \mathcal{A} \), denote \( P_i \mathcal{A} P_j \) by \( \mathcal{A}_{ij} \).

Firstly, we claim that the condition \( A \mathcal{A}_{ij} = 0 \) implies \( AP_i = 0 \), and similarly, \( A_j A = 0 \) implies \( P_j A = 0 \). Indeed, since \( C(P_j) = I \), by [9, Proposition 5.5.2], the range of \( AP_j \) is dense in \( \mathcal{H} \). So \( AP_i AP_j = 0 \) implies \( AP_i = 0 \). On the other hand, if \( A_j A = 0 \), then \( A^* P_j = 0 \) and \( P_j A = 0 \).

Besides, since \( P_1 = P \) is the range projection of \( G \), we have \( P_1 G = G \). Moreover, if \( AG = 0 \), then \( AP_i = 0 \).

In the following, we assume that \( A_{ij} \) is an arbitrary element in \( \mathcal{A}_{ij} \), \( i, j = 1, 2 \), and \( t \) is an arbitrary nonzero element in \( \mathbb{C} \). Without loss of generality, we may assume that \( A_{11} \) is invertible in \( \mathcal{A}_{11} \).

**Claim 1** \( \phi(A_{12}) \subseteq \mathcal{A}_{12} \).

Since \( (P_1 + tA_{12})G = G \), we have \( \phi(G) = \phi(P_1 + tA_{12})G \). It implies that \( \phi(A_{12})G = 0 \). Hence \( \phi(A_{12})P_1 = 0 \).

By \( (P_1 + tA_{12})G = G \), we also have \( \phi(G) = \phi(P_1 + tA_{12}) \phi(G) \). It follows that \( A_{12} \phi(G) = \phi(P_1)G = 0 \). So \( A_{12} \phi(P_1)P_1 = 0 \). Hence \( P_2 \phi(P_1)P_1 = 0 \).

Since \( (A_{11} + tA_{11} A_{12})(A_{11}^{-1}G - A_{12}A_{22} + t^{-1}A_{22}) = G \), we have
\[
(2.1) \quad \phi(A_{11} + tA_{11} A_{12})(A_{11}^{-1}G - A_{12}A_{22} + t^{-1}A_{22}) = \phi(G).
\]

Since \( t \) is arbitrarily chosen in (2.1), we obtain
\[
\phi(A_{11})(A_{11}^{-1}G - A_{12}A_{22}) + \phi(A_{11} A_{12})A_{22} = \phi(G).
\]

Since \( A_{12} \) is also arbitrarily chosen, we can obtain
\[
\phi(A_{11})A_{12}A_{22} = \phi(A_{11} A_{12})A_{22}.
\]

Taking \( A_{22} = P_2 \), since \( \phi(A_{12})P_1 = 0 \), we have
\[
(2.2) \quad \phi(A_{11} A_{12}) = \phi(A_{11})A_{12}.
\]

Taking \( A_{11} = P_1 \), since \( P_2 \phi(P_1)P_1 = 0 \), we have
\[
(2.3) \quad P_2 \phi(A_{12}) = P_2 \phi(P_1)A_{12} = 0.
\]

So
\[
\phi(A_{12}) = \phi(A_{12})P_1 + P_1 \phi(A_{12})P_2 + P_2 \phi(A_{12})P_2
\]
\[
= P_1 \phi(A_{12})P_2 \subseteq \mathcal{A}_{12}.
\]

**Claim 2** \( \phi(A_{11}) \subseteq \mathcal{A}_{11} \).

Considering the coefficient of \( t^{-1} \) in (2.1), we have \( \phi(A_{11})A_{22} = 0 \). Thus \( \phi(A_{11})P_2 = 0 \). By (2.2), we obtain \( P_2 \phi(A_{11})A_{12} = P_2 \phi(A_{11} A_{12}) = 0 \). It follows that \( P_2 \phi(A_{11})P_1 = 0 \). Therefore, \( \phi(A_{11}) = P_1 \phi(A_{11})P_1 \subseteq \mathcal{A}_{11} \).
Therefore,

\[ P \]

Thus

\[ \phi \]

Hence

\( (2.5) \)

\[ Q \]

Case 1

\[ \ker \]

that

\( \phi \)

3

\[ I \]

A

\( \Box \)

point.

Similarly, we can prove that

\[ P \]

Therefore,

\[ \phi(A_{22}) = A_{12}\phi(A_{22}). \]

Claim 4

\[ \phi(A_{21}) \subseteq A_{21}. \]

Since

\[ (A_{11} + tA_{11}A_{12})(A_{11}^{-1}G - A_{12}A_{21} + t^{-1}A_{21}) = G, \]

we have

\( (A_{11} + tA_{11}A_{12})\phi(A_{11}^{-1}G - A_{12}A_{21} + t^{-1}A_{21}) = \phi(G). \)

According to this equation, we can similarly obtain that

\( P_{1}\phi(A_{21}) = 0 \) and

\( (2.5) \)

\[ A_{12}\phi(A_{21}) = \phi(A_{12}A_{21}). \]

Hence

\( A_{12}\phi(A_{21})P_{2} = \phi(A_{12}A_{21})P_{2} = 0. \)

Therefore, \( \phi(A_{21}) = P_{2}\phi(A_{21})P_{1} \subseteq A_{21}. \)

Claim 5

\[ \phi(A_{ij}) = \phi(P_{i})A_{ij} = A_{ij}\phi(P_{j}) \]

for each \( i, j \in \{1, 2\}. \)

By taking \( A_{11} = P_{1} \) in (2.2), we have

\( \phi(A_{12}) = \phi(P_{1})A_{12}. \)

By taking

\[ A_{22} = P_{2} \text{ in (2.4), we have } \phi(A_{12}) = A_{12}\phi(P_{2}). \]

By (2.2), we have

\( \phi(A_{11})A_{12} = \phi(A_{11}A_{12}) = \phi(P_{1})A_{11}A_{12}. \)

It follows that

\( \phi(A_{11}) = \phi(P_{1})A_{11}. \)

On the other hand, \( \phi(A_{11})A_{12} = \phi(A_{11}A_{12}) = A_{11}A_{12}\phi(P_{2}) = A_{11}\phi(A_{12}) = A_{11}\phi(P_{1})A_{12}. \)

It follows that

\( \phi(A_{11}) = A_{11}\phi(P_{1}). \)

By (2.4) and (2.5), through a similar discussion as above, we can obtain that

\( \phi(A_{22}) = A_{22}\phi(P_{2}) = \phi(P_{2})A_{22} \)

and

\( \phi(A_{21}) = A_{21}\phi(P_{1}) = \phi(P_{2})A_{21}. \)

Now we have proved that \( \phi(A_{ij}) \subseteq A_{ij} \) and

\( \phi(A_{ij}) = \phi(P_{i})A_{ij} = A_{ij}\phi(P_{j}). \)

It follows that

\( \phi(A) = \phi(A_{11} + A_{12} + A_{21} + A_{22}) = \phi(P_{1})(A_{11} + A_{12} + A_{21} + A_{22}) + \phi(P_{2})(A_{11} + A_{12} + A_{21} + A_{22}) = \phi(I)A. \)

Similarly, we can prove that \( \phi(A) = A\phi(I). \)

Hence \( G \) is a full-centralizable point.

Proof of Theorem 2.1. Suppose the range projection of \( G \) is \( P. \) Set

\[ Q_{1} = I - C(I - P), \]

\[ Q_{2} = I - C(P), \]

and

\[ Q_{3} = I - Q_{1} - Q_{2}. \]

Since \( Q_{1} \subseteq P \) and

\[ Q_{2} \subseteq I - P, \]

\( \{Q_{i}\}_{i=1,2,3} \) are mutually orthogonal central projections. Therefore

\[ A = \sum_{i=1}^{3} A_{i} = \sum_{i=1}^{3} (Q_{i}A). \]

Obviously, \( A_{i} \) is also a von Neumann algebra acting on \( Q_{i}H. \) For each element \( A \) in \( A, \) we write

\[ A = \sum_{i=1}^{3} A_{i} = \sum_{i=1}^{3} Q_{i}A. \]

We divide our proof into two cases.

Case 1

\[ \ker(G) = \{0\}. \]
Since $Q_1 \leq P$, we have $\text{ran} G_1 = \text{ran} Q_1 G = Q_1 \mathcal{H}$. Since $G$ is injective on $\mathcal{H}$, $G_1 = Q_1 G$ is also injective on $Q_1 \mathcal{H}$. Hence $G_1$ is a separating point (both right and left) in $A$. By Lemma 2.4, $G_1$ is a full-centralizable point in $A$.

Since $Q_2 \leq I - P$, we have $G_2 = Q_2 G = 0$. By Lemma 2.5, $G_2$ is a full-centralizable point in $A$.

We write $G = [A M]$. By Lemma 2.2, $G$ is a full-centralizable point in $A$.

**Case 2** $\ker(G) \neq \{0\}$.

In this case, $G_2$ and $G_3$ are still full-centralizable points. Since $\text{ran} G_1 = Q_1 H$, we have $\ker(G_1) = \{0\}$. By Case 1, $G_1$ is a full-centralizable point in $A$.

By Lemma 2.3, $G_3$ is also a full-centralizable point in $A$.

By Lemma 2.2, $G$ is a full-centralizable point.

**3. Centralizers on triangular algebras**

In this section, we characterize the full-centralizable points on triangular algebras. The following theorem is our main result.

**Theorem 3.1.** Let $\mathcal{J} = [A M]$ be a triangular algebra, where $A$ and $B$ are two unital Banach algebras. Then every $G$ in $\mathcal{J}$ is a full-centralizable point.

**Proof.** Let $\phi : \mathcal{J} \to \mathcal{J}$ be a centralizable mapping at $G$.

Since $\phi$ is linear, for every $[X Y Z]$ in $\mathcal{J}$, we write

$$\phi \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \begin{bmatrix} f_{11}(X) + g_{11}(Y) + h_{11}(Z) & f_{12}(X) + g_{12}(Y) + h_{12}(Z) \\ 0 & f_{22}(X) + g_{22}(Y) + h_{22}(Z) \end{bmatrix},$$

where $f_{11} : A \to A$, $f_{12} : A \to M$, $f_{22} : A \to B$, $g_{11} : M \to A$, $g_{12} : M \to M$, $g_{22} : M \to B$, $h_{11} : B \to A$, $h_{12} : B \to M$, $h_{22} : B \to B$, are all linear mappings.

In the following, we denote the units of $A$ and $B$ by $I_1$ and $I_2$, respectively. We write $G = [A M]$ and

\begin{equation}
\phi \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} = \begin{bmatrix} f_{11}(A) + g_{11}(M) + h_{11}(B) & f_{12}(A) + g_{12}(M) + h_{12}(B) \\ 0 & f_{22}(A) + g_{22}(M) + h_{22}(B) \end{bmatrix}.
\end{equation}

We divide our proof into several steps.

**Claim 1** $f_{12} = f_{22} = 0$.

Let $S = [X M]$ and $T = \begin{bmatrix} X^{-1} A & \phi \end{bmatrix}$, where $X$ is an invertible element in $A$. Since $ST = G$, we have
φ(G) = φ(S)T
= \begin{bmatrix}
    f_{11}(X) + g_{11}(M) + h_{11}(B) & f_{12}(X) + g_{12}(M) + h_{12}(B) \\
    0 & f_{22}(X) + g_{22}(M) + h_{22}(B)
\end{bmatrix}
\begin{bmatrix}
    X^{-1}A & 0 \\
    0 & I_2
\end{bmatrix}
(3.2)
= \begin{bmatrix}
    f_{11}(X) + g_{11}(M) + h_{11}(B) & f_{12}(X) + g_{12}(M) + h_{12}(B) \\
    f_{22}(X) + g_{22}(M) + h_{22}(B) & 0
\end{bmatrix}.

By comparing (3.1) with (3.2), we obtain
\begin{align*}
    & f_{12}(X) = f_{12}(A) \quad \text{and} \quad f_{22}(X) = f_{22}(A) \\
    & \text{for each invertible element } X \text{ in } \mathcal{A}.
\end{align*}
Noting that \( A \) is a fixed element, for any nonzero element \( \lambda \) in \( \mathcal{C} \), we have \( f_{12}(\lambda X) = f_{12}(A) = \lambda f_{12}(X) = \lambda f_{12}(A) \). It follows that \( f_{12}(X) = 0 \) for each invertible element \( X \). Thus \( f_{12}(X) = 0 \) for all \( X \) in \( \mathcal{A} \). Similarly, we can obtain \( f_{22}(X) = 0 \).

**Claim 2** \( h_{12} = h_{11} = 0 \).

Let \( S = \begin{bmatrix}
    I_1 & 0 \\
    0 & BZ^{-1}
\end{bmatrix} \) and \( T = \begin{bmatrix}
    A & Y \\
    0 & h_1
\end{bmatrix} \), where \( Z \) is an invertible element in \( \mathcal{B} \).

Since \( ST = G \), we have
\[
\phi(G) = S\phi(T)
= \begin{bmatrix}
    I_1 & 0 \\
    0 & BZ^{-1}
\end{bmatrix}
\begin{bmatrix}
    f_{11}(A) + g_{11}(M) + h_{11}(Z) & f_{12}(A) + g_{12}(M) + h_{12}(Z) \\
    f_{22}(A) + g_{22}(M) + h_{22}(Z) & 0
\end{bmatrix}
(3.3)
= \begin{bmatrix}
    f_{11}(A) + g_{11}(M) + h_{11}(Z) & f_{12}(A) + g_{12}(M) + h_{12}(Z) \\
    0 & *
\end{bmatrix}.
\]

By comparing (3.1) with (3.3), we obtain \( h_{12}(Z) = h_{12}(B) \) and \( h_{11}(Z) = h_{11}(B) \) for each invertible element \( Z \) in \( \mathcal{B} \). Similarly as the previous discussion, we can obtain \( h_{12}(Z) = h_{11}(Z) = 0 \) for all \( Z \) in \( \mathcal{B} \).

**Claim 3** \( g_{22} = g_{11} = 0 \).

For every \( Y \) in \( \mathcal{M} \), we set \( S = \begin{bmatrix}
    I_1 & M - Y \\
    0 & B
\end{bmatrix} \), \( T = \begin{bmatrix}
    A & Y \\
    0 & h_1
\end{bmatrix} \). Obviously, \( ST = G \).

Thus we have
\[
\phi(G) = \phi(S)T
= \begin{bmatrix}
    * & * \\
    0 & f_{22}(I_1) + g_{22}(M - Y) + h_{22}(B)
\end{bmatrix}
\begin{bmatrix}
    A & Y \\
    0 & I_2
\end{bmatrix}
(3.4)
= \begin{bmatrix}
    * & * \\
    0 & f_{22}(I_1) + g_{22}(M - Y) + h_{22}(B)
\end{bmatrix}.
\]

By comparing (3.1) with (3.4), we obtain
\[
    f_{22}(I_1) + g_{22}(M - Y) + h_{22}(B) = f_{22}(A) + g_{22}(M) + h_{22}(B).
\]
Hence \( g_{22}(Y) = f_{22}(I_1 - A) \). It means \( g_{22}(Y) = 0 \) immediately.

On the other hand,
\[
\phi(G) = S\phi(T)
= \begin{bmatrix}
    I_1 & M - Y \\
    0 & B
\end{bmatrix}
\begin{bmatrix}
    f_{11}(A) + g_{11}(Y) + h_{11}(I_2) & * \\
    0 & *
\end{bmatrix}
(3.5)
= \begin{bmatrix}
    f_{11}(A) + g_{11}(Y) + h_{11}(I_2) & * \\
    0 & *
\end{bmatrix}.
By comparing (3.1) with (3.5), we obtain $g_{11}(Y) = g_{11}(M) + h_{11}(B - I_2)$. Hence $g_{11}(Y) = 0$.

According to the above three claims, we obtain that

$$
\phi \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \begin{bmatrix} f_{11}(X) & g_{12}(Y) \\ 0 & h_{22}(Z) \end{bmatrix}
$$

for every $\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ in $J$.

**Claim 4** $f_{11}(X) = f_{11}(I_1)X$ for all $X$ in $A$, and $g_{12}(Y) = f_{11}(I_1)Y$ for all $Y$ in $M$.

Let $S = \begin{bmatrix} X & M-XY \\ 0 & B \end{bmatrix}$ and $T = \begin{bmatrix} X^{-1}A & Y \\ 0 & I_2 \end{bmatrix}$, where $X$ is an invertible element in $A$, and $Y$ is an arbitrary element in $M$. Since $ST = G$, we have

$$
\phi(G) = \phi(S)T
= \begin{bmatrix} f_{11}(X) & g_{12}(M - XY) \\ 0 & h_{22}(B) \end{bmatrix} \begin{bmatrix} X^{-1}A & Y \\ 0 & I_2 \end{bmatrix}
= \begin{bmatrix} * & f_{11}(X)Y + g_{12}(M - XY) \\ 0 & * \end{bmatrix}
= \begin{bmatrix} f_{11}(A) & g_{12}(M) \\ 0 & h_{22}(B) \end{bmatrix}.
$$

(3.6)

So we have $f_{11}(X)Y = g_{12}(XY)$. It follows that

$$
g_{12}(Y) = f_{11}(I_1)Y
$$

by taking $X = I_1$. Replacing $Y$ in (3.7) with $XY$, we can obtain $g_{12}(XY) = f_{11}(I_1)XY = f_{11}(X)Y$ for each invertible element $X$ in $A$ and $Y$ in $M$. Since $M$ is faithful, we have

$$
f_{11}(X) = f_{11}(I_1)X
$$

for all invertible elements $X$ and so for all elements in $A$.

**Claim 5** $h_{22}(Z) = Zh_{22}(I_2)$ for all $Z$ in $B$, and $g_{12}(Y) = Yh_{22}(I_2)$ for all $Y$ in $M$.

Let $S = \begin{bmatrix} I_1 & Y \\ 0 & BZ^{-1} \end{bmatrix}$ and $T = \begin{bmatrix} A & M - YZ \\ 0 & Z \end{bmatrix}$, where $Z$ is an invertible element in $B$, and $Y$ is an arbitrary element in $M$. Since $ST = G$, we have

$$
\phi(G) = S\phi(T)
= \begin{bmatrix} I_1 & Y \\ 0 & BZ^{-1} \end{bmatrix} \begin{bmatrix} f_{11}(A) & g_{12}(M - YZ) \\ 0 & h_{22}(Z) \end{bmatrix}
= \begin{bmatrix} * & g_{12}(M - YZ) + Yh_{22}(Z) \\ 0 & * \end{bmatrix}
= \begin{bmatrix} f_{11}(A) & g_{12}(M) \\ 0 & h_{22}(B) \end{bmatrix}.
$$

(3.9)
So we have \( g_{12}(YZ) = Y h_{22}(Z) \). Through a similar discussion as the proof of Claim 4, we obtain \( h_{22}(Z) = Z h_{22}(I_2) \) for all \( Z \) in \( B \) and \( g_{12}(Y) = Y h_{22}(I_2) \) for all \( Y \) in \( M \).

Thus we have that

\[
\phi \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \begin{bmatrix} f_{11}(I_1)X & f_{11}(I_1)Y \\ 0 & Z h_{22}(I_2) \end{bmatrix} = \begin{bmatrix} f_{11}(I_1)X & Y h_{22}(I_2) \\ 0 & Z h_{22}(I_2) \end{bmatrix}
\]

for every \([X \ Y]_2 \in J\). So it is sufficient to show that \( f_{11}(I_1)X = X f_{11}(I_1) \) for all \( X \) in \( \mathcal{A} \), and \( h_{22}(I_2)Z = Z h_{22}(I_2) \) for all \( Z \) in \( B \). Since \( f_{11}(I_1)Y = Y h_{22}(I_2) \) for all \( Y \) in \( M \), we have \( f_{11}(I_1)XY = X Y h_{22}(I_2) = X f_{11}(I_1)Y \). It implies that \( f_{11}(I_1)X = X f_{11}(I_1) \). Similarly, \( h_{22}(I_2)Z = Z h_{22}(I_2) \). Now we can obtain that \( \phi(J) = \phi(I)J = J \phi(I) \) for all \( J \) in \( J \), where \( I = \begin{bmatrix} 1 & 0 \\ 0 & I_2 \end{bmatrix} \) is the unit of \( J \). Hence, \( G \) is a full-centralizable point.

As applications of Theorem 3.1, we have the following corollaries.

**Corollary 3.2.** Let \( \mathcal{A} \) be a nest algebra on a Hilbert space \( \mathcal{H} \). Then every element in \( \mathcal{A} \) is a full-centralizable point.

**Proof.** If \( \mathcal{A} = B(\mathcal{H}) \), then the result follows from Theorem 2.1. Otherwise, \( \mathcal{A} \) is isomorphic to a triangular algebra. By Theorem 3.1, the result follows.

**Corollary 3.3.** Let \( \mathcal{A} \) be a CDCSL (completely distributive commutative sub-space lattice) algebra on a Hilbert space \( \mathcal{H} \). Then every element in \( \mathcal{A} \) is a full-centralizable point.

**Proof.** It is known that \( \mathcal{A} \cong \sum_{i \in \Lambda} \bigoplus \mathcal{A}_i \), where each \( \mathcal{A}_i \) is either \( B(\mathcal{H}_i) \) for some Hilbert space \( \mathcal{H}_i \) or a triangular algebra \( \text{Tri}(B, \mathcal{M}, \mathcal{C}) \) such that the conditions of Theorem 3.1 hold (see in [8] and [15]). By Lemma 2.2, the result follows.

**Remark.** For the definition of a CDCSL algebra, we refer to [5].

## 4. Derivations on von Neumann algebras

In this section, we characterize the derivable mappings at a given point in a von Neumann algebra.

**Lemma 4.1.** Let \( \mathcal{A} \) be a von Neumann algebra. Suppose \( \Delta : \mathcal{A} \to \mathcal{A} \) is a linear mapping such that \( \Delta(A)B + A \Delta(B) = 0 \) for each \( A \) and \( B \) in \( \mathcal{A} \) with \( AB = 0 \). Then \( \Delta = D + \phi \), where \( D : \mathcal{A} \to \mathcal{A} \) is a derivation, and \( \phi : \mathcal{A} \to \mathcal{A} \) is a centralizer. In particular, \( \Delta \) is bounded.

**Proof.** Case 1. \( \mathcal{A} \) is an abelian von Neumann algebra. In this case, \( \mathcal{A} \cong C(\mathcal{X}) \) for some compact Hausdorff space \( \mathcal{X} \). If \( AB = 0 \), then the supports of \( A \) and \( B \) are disjoint. So the equation \( \Delta(A)B + A \Delta(B) = 0 \) implies that \( \Delta(A)B = A \Delta(B) = 0 \). By Lemma 2.5, \( \Delta \) is a centralizer.
Case 2. \( A \cong M_n(B)(n \geq 2) \), where \( B \) is also a von Neumann algebra. By [1, Theorem 2.3], \( \Delta \) is a generalized derivation with \( \Delta(I) \) in the center. That is to say, \( \Delta \) is a sum of a derivation and a centralizer.

For general cases, we know \( A \cong \sum_{i=1}^{n} \oplus A_i \), where each \( A_i \) coincides with either Case 1 or Case 2. We write \( A = \sum_{i=1}^{n} A_i \) with \( A_i \in A_i \) and denote the restriction of \( \Delta \) in \( A_i \) by \( \Delta_i \). It is not difficult to check that \( \Delta(A_i) \in A_i \). Moreover, setting \( A_i B_i = 0 \), we have \( \Delta(A_i)B_i + A_i \Delta(B_i) = \Delta_i(A_i)B_i + A_i \Delta_i(B_i) = 0 \). By Case 1 and Case 2, each \( \Delta_i \) is a sum of a derivation and a centralizer. Hence, \( \Delta = \sum_{i=1}^{n} \Delta_i \) is a sum of a derivation and a centralizer. \( \square \)

Remark. In [10], the authors prove that for a prime semisimple Banach algebra \( A \) with nontrivial idempotents and a linear mapping \( \Delta \) from \( A \) to itself, the condition \( \Delta(A)B + AB \Delta(B) = 0 \) for each \( A \) and \( B \) in \( A \) implies that \( \Delta \) is bounded. By Lemma 4.1, we have that for a von Neumann algebra \( A \), the result holds still even if \( A \) is not prime.

Now we prove our main result in this section.

Theorem 4.2. Let \( A \) be a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \), and \( G \) be a given point in \( A \). If \( \Delta : A \to A \) is a linear mapping derivable at \( G \), then \( \Delta = D + \phi \), where \( D \) is a derivation, and \( \phi \) is a centralizer. Moreover, \( G \) is a full-derivable point if and only if \( \mathcal{C}(G) = I \).

Proof. Suppose the range projection of \( G \) is \( P \). We note that \( \mathcal{C}(G) = \mathcal{C}(P) \).

Set \( Q_1 = I - \mathcal{C}(I - P) \), \( Q_2 = I - \mathcal{C}(P) \), and \( Q_3 = I - Q_1 - Q_2 \). Then we have \( A = \sum_{i=1}^{3} \oplus A_i = \sum_{i=1}^{3} \oplus (Q_iA) \). For every \( A \) in \( A \), we write \( A = \sum_{i=1}^{3} A_i = \sum_{i=1}^{3} Q_iA \).

For any central projection \( Q \), setting \( Q^\perp = I - Q \), we have

\[
(Q^\perp + t^{-1}QGA^{-1})(Q^\perp G + tQA) = G,
\]

where \( A \) is an arbitrary invertible element in \( A \), and \( t \) is an arbitrary nonzero element in \( \mathbb{C} \). So we obtain

\[
\Delta(G) = (Q^\perp + t^{-1}QGA^{-1})\Delta(Q^\perp G + tQA) + \Delta(Q^\perp G + tQA - Q^\perp G - tQA).
\]

Considering the coefficient of \( t \), we obtain \( Q^\perp \Delta(QA) + \Delta(Q^\perp)(QA) = 0 \). Since the ranges of \( Q \) and \( Q^\perp \) are disjoint, it follows that \( Q^\perp \Delta(QA) = 0 \) and so \( \Delta(QA) \in QA \). Since \( Q_i \) are central projections, we have \( \Delta(A_i) \subseteq A_i \).

Denote the restriction of \( \Delta \) to \( A_i \) by \( \Delta_i \). Setting \( A_i B_i = G_i \), it is not difficult to check that \( \Delta_i(G_i) = \Delta(A_i)B_i + A_i \Delta(B_i) \).

Since \( Q_1 \leq P \), we have \( \text{ran}G_1 = \text{ran}Q_1G = Q_1H \). So \( G_1 \) is a right separating point in \( A_1 \). By [13, Corollary 2.5], \( \Delta_1 \) is a Jordan derivation and so is a derivation on \( A_1 \).

Since \( Q_2 \leq I - P \), we have \( G_2 = Q_2G = 0 \). By Lemma 4.1, \( \Delta_2 \) is a sum of a derivation and a centralizer on \( A_2 \).

Note that \( \text{ran}G_3 = \text{ran}Q_3G = Q_3P = P_3 \). As we proved before, \( \mathcal{C}_{A_3}(P_3) = \mathcal{C}_{A_3}(Q_3 - P_3) = Q_3 \). So by [16, Theorem 3.1], \( \Delta_3 \) is a derivation on \( A_3 \).
Hence, $\Delta = \sum_{i=1}^{3} \Delta_i$ is a sum of a derivation and a centralizer.

If $C(G) = I$, then $Q_2 = 0$, $A = A_1 \oplus A_3$ and $G = G_1 + G_3$ is a full-derivable point. If $C(G) \neq I$, then $Q_2 \neq 0$. Define a linear mapping $\delta : A \to A$ by $\delta(A) = A_2$ for all $A \in A$. One can check that $\delta$ is not a derivation but derivable at $G$. Thus $G$ is not a full-derivable point.

As an application, we obtain the following corollary.

**Corollary 4.3.** Let $A$ be a von Neumann algebra. Then $A$ is a factor if and only if every nonzero element $G$ in $A$ is a full-derivable point.

**Proof.** If $A$ is a factor, for each nonzero element $G$ in $A$, we know that $C(G) = I$. By Theorem 4.2, $G$ is a full-derivable point.

If $A$ is not a factor, then there exists a nontrivalent central projection $P$. Define a linear mapping $\delta : A \to A$ by $\delta(A) = (I - P)A$ for all $A \in A$. One can check that $\delta$ is not a derivation but derivable at $P$. Thus $P$ is not a full-derivable point.

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