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$([r, s], [t, u])$-INTERVAL-VALUED INTUITIONISTIC FUZZY GENERALIZED PRECONTINUOUS MAPPINGS

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Abstract. In this paper, we introduce the concepts of $([r, s], [t, u])$-interval-valued intuitionistic fuzzy generalized preclosed sets and $([r, s], [t, u])$-interval-valued intuitionistic fuzzy generalized preopen sets in the interval-valued intuitionistic smooth topological space and $([r, s], [t, u])$-interval-valued intuitionistic fuzzy generalized precontinuous mappings and then investigate some of their properties.

1. Introduction

After Zadeh [16] introduced the concept of fuzzy sets, there have been various generalizations of the concept of fuzzy sets. Chang [5] introduced the concept of fuzzy topology on a set X by axiomatizing a collection T of fuzzy subsets of X and Coker [7] introduced the concept of intuitionistic fuzzy topology on a set by axiomatizing a collection T of intuitionistic fuzzy subsets of X. Chattopadhyay, Hazra and Samanta [6,9] introduced the concept of gradation of openness of fuzzy subsets. Zadeh [17] introduced the concept of interval-valued fuzzy sets and Atanassov [1]

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introduced the concept of intuitionistic fuzzy sets. Atanassov and Gargov [2] introduced the concept of interval-valued intuitionistic fuzzy sets which is a generalization of both interval-valued fuzzy sets and intuitionistic fuzzy sets. Mondal and Samanta [10,15] introduced the concept of intuitionistic gradation of openness and defined an intuitionistic smooth topological space. In [13], we introduced the concept of interval-valued intuitionistic gradation of openness and defined an interval-valued intuitionistic smooth topological space. Fukutake, Saraf, Caldas and Mishra [8] introduced the concepts of generalized preclosed fuzzy sets and fuzzy generalized precontinuous mappings in fuzzy topological spaces.

In this paper, we introduce the concepts of $([r,s],[t,u])$-interval-valued intuitionistic fuzzy generalized preclosed sets and $([r,s],[t,u])$-interval-valued intuitionistic fuzzy generalized preopen sets in the interval-valued intuitionistic smooth topological space and $([r,s],[t,u])$-interval-valued intuitionistic fuzzy generalized precontinuous mappings and then investigate some of their properties.

2. Preliminaries

Throughout this paper, let X be a nonempty set, $I = [0,1]$, $I_0 = (0,1]$ and $I_1 = [0,1)$. The family of all fuzzy sets of X will be denoted by I^X. By 0_X and 1_X we denote the characteristic functions of ϕ and X, respectively. For any $A \in I^X$, A^c denotes the complement of A, i.e., $A^c = 1_X - A$.

Definition 2.1.[3,6,14]. A *gradation of openness* (for short, GO) on X, which is also called a *smooth topology* on X, is a mapping $\tau : I^X \to I$ satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$,

(O2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ for each $A, B \in I^X$,

(O3) $\tau(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau(A_i)$, for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

The pair (X, τ) is called a *smooth topological space* (for short, STS).

Definition 2.2.[10]. An *intuitionistic gradation of openness* (for short, IGO) on X, which is also called an *intuitionistic smooth topology* on X, is an ordered pair (τ, τ^*) of mappings from I^X to I satisfying the following conditions:

(IGO1) $\tau(A) + \tau^*(A) \leq 1$ for each $A \in I^X$,

(IGO2) $\tau(0_X) = \tau(1_X) = 1$ and $\tau^*(0_X) = \tau^*(1_X) = 0$, (IGO3) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ and $\tau^*(A \cap B) \leq \tau^*(A) \vee \tau^*(B)$ for each $A, B \in I^X$,

(IGO4) $\tau(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau(A_i)$ and $\tau^*(\cup_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \tau^*(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

The triple (X, τ, τ^*) is called an *intuitionistic smooth topological space* (for short, ISTS). τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Definition 2.3.[10]. Let (X, τ, τ^*) and (Y, η, η^*) be two ISTSs and let $f : X \to Y$ be a mapping. f is called a *gradation preserving mapping* (for short, a GP-mapping) if for each $A \in I^Y$, $\eta(A) \leq \tau(f^{-1}(A))$ and $\eta^*(A) \geq \tau^*(f^{-1}(A))$.

Let $D(I)$ be the set of all closed subintervals of the unit interval I. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$ and $M = [M^L, M^U]$, where M^L and M^U are respectively the lower and the upper end points. Especially, we denote $\mathbf{r} = [r, r]$ for each $r \in I$. The complement of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$. Note that $M = N$ iff $M^L = N^L$ and $M^U = N^U$ and that $M \leq N$ iff $M^L \leq N^L$ and $M^U \leq N^U$.

Definition 2.4.[17]. A mapping $A = [A^L, A^U] : X \to D(I)$ is called an *interval-valued fuzzy set* (for short, IVFS) on X, where $A(x) = [A^L(x), A^U(x)]$ for each $x \in X$. $A^L(x)$ and $A^U(x)$ are called the *lower* and *upper end points* of $A(x)$, respectively.

Definition 2.5.[11]. Let A and B be IVFSs on X.

(i) $A = B$ iff $A^L(x) = B^L(x)$ and $A^U(x) = B^U(x)$ for all $x \in X$.

(ii) $A \subseteq B$ iff $A^L(x) \leq B^L(x)$ and $A^U(x) \leq B^U(x)$ for all $x \in X$.

(iii) The *complement* A^c of A is defined by $A^c(x) = [1 - A^U(x), 1 - A^L(x)]$ for all $x \in X$.

(iv) For a family of IVFSs $\{A_i : i \in \Gamma\}$, the union $\cup_{i\in\Gamma} A_i$ and the intersection $\cap_{i\in\Gamma} A_i$ are respectively defined by

$$\cup_{i\in\Gamma} A_i(x) = [\vee_{i\in\Gamma} A_i^L(x), \vee_{i\in\Gamma} A_i^U(x)],$$
$$\cap_{i\in\Gamma} A_i(x) = [\wedge_{i\in\Gamma} A_i^L(x), \wedge_{i\in\Gamma} A_i^U(x)]$$

for all $x \in X$.

Definition 2.6.[2]. A mapping $A = (\mu_A, \nu_A) : X \to D(I) \times D(I)$ is called an *interval-valued intuitionistic fuzzy set* (for short, IVIFS) on X, where $\mu_A : X \to D(I)$ and $\nu_A : X \to D(I)$ are interval-valued fuzzy sets on X with the condition $\sup_{x\in X}\mu_A^U(x)+\sup_{x\in X}\nu_A^U(x)\le 1$. The intervals $\mu_A(x)=[\mu_A^L(x),\mu_A^U(x)]$ and $\nu_A(x)=[\nu_A^L(x),\nu_A^U(x)]$ denote the degree of belongingness and the degree of nonbelongingness of the element x to the set A, respectively.

Definition 2.7.[12]. Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be IVIFSs on X.

(i) $A\subseteq B$ iff $\mu_A^L(x)\le\mu_B^L(x)$, $\mu_A^U(x)\le\mu_B^U(x)$ and $\nu_A^L(x)\ge\nu_B^L(x)$, $\nu_A^U(x)\ge\nu_B^U(x)$ for all $x\in X$.

(ii) $A=B$ iff $A\subseteq B$ and $B\subseteq A$.

(iii) The *complement* A^c of A is defined by $\mu_{A^c}(x)=\nu_A(x)$ and $\nu_{A^c}(x)=\mu_A(x)$ for all $x\in X$.

(iv) For a family of IVIFSs $\{A_i : i\in\Gamma\}$, the union $\cup_{i\in\Gamma}A_i$ and the intersection $\cap_{i\in\Gamma}A_i$ are respectively defined by

$$\mu_{\cup_{i\in\Gamma}A_i}(x)=\cup_{i\in\Gamma}\mu_{A_i}(x),\nu_{\cup_{i\in\Gamma}A_i}(x)=\cap_{i\in\Gamma}\nu_{A_i}(x),$$
$$\mu_{\cap_{i\in\Gamma}A_i}(x)=\cap_{i\in\Gamma}\mu_{A_i}(x),\nu_{\cap_{i\in\Gamma}A_i}(x)=\cup_{i\in\Gamma}\nu_{A_i}(x)$$

for all $x\in X$.

Definition 2.8.[4]. Let $(X,\mathcal{T})$ be a fuzzy topological space.

(i) A fuzzy set A in X is called a *preopen fuzzy set* if $A\subseteq int(cl(A))$ and a *preclosed fuzzy set* if $cl(int(A))\subseteq A$.

(ii) The *preclosure* of a fuzzy set A in X is the intersection of all preclosed fuzzy sets containing A and is denoted by $pcl(A)$.

(iii) A fuzzy set A in X is called a *generalized preclosed fuzzy set* if $pcl(A)\subseteq U$ whenever $A\subseteq U$ and U is an open fuzzy set in X. The complement of a generalized preclosed fuzzy set is called a *generalized preopen fuzzy set.*

Definition 2.9.[4]. Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be two fuzzy topological spaces. A mapping $f:X\to Y$ is called *fuzzy generalized precontinuous* if $f^{-1}(A)$ is a generalized preclosed fuzzy set in X for every closed fuzzy set A of Y.

3. $([r,s],[t,u])$-interval-valued intuitionistic fuzzy generalized preclosed and preopen sets

Definition 3.1.[13]. An *interval-valued intuitionistic gradation of openness* (for short, IVIGO) on X, which is also called an *interval-valued intuitionistic smooth topology* on X, is an ordered pair (τ,τ^*) of mappings $\tau = [\tau^L, \tau^U] : I^X \to D(I)$ and $\tau^* = [\tau^{*L}, \tau^{*U}] : I^X \to D(I)$ satisfying the following conditions:

(IVIGO1) $\tau^L(A) \leq \tau^U(A)$, $\tau^{*L}(A) \leq \tau^{*U}(A)$ and $\tau^U(A) + \tau^{*U}(A) \leq 1$ for each $A \in I^X$,

(IVIGO2) $\tau(0_X) = \tau(1_X) = \mathbf{1}$ and $\tau^*(0_X) = \tau^*(1_X) = \mathbf{0}$,

(IVIGO3) $\tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B)$, $\tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B)$ and $\tau^{*L}(A \cap B) \leq \tau^{*L}(A) \vee \tau^{*L}(B)$, $\tau^{*U}(A \cap B) \leq \tau^{*U}(A) \vee \tau^{*U}(B)$ for each $A, B \in I^X$,

(IVIGO4) $\tau^L(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau^L(A_i)$, $\tau^U(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau^U(A_i)$ and $\tau^{*L}(\cup_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \tau^{*L}(A_i)$, $\tau^{*U}(\cup_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \tau^{*U}(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

The triple (X, τ, τ^*) is called an *interval-valued intuitionistic smooth topological space* (for short, IVISTS). τ and τ^* may be interpreted as interval-valued gradation of openness and interval-valued gradation of nonopenness, respectively.

Definition 3.2.[13]. An *interval-valued intuitionistic gradation of closedness* (for short, IVIGC) on X, which is also called an *interval-valued intuitionistic smooth cotopology* on X, is an ordered pair $(\mathcal{F}, \mathcal{F}^*)$ of mappings $\mathcal{F} = [\mathcal{F}^L, \mathcal{F}^U] : I^X \to D(I)$ and $\mathcal{F}^* = [\mathcal{F}^{*L}, \mathcal{F}^{*U}] : I^X \to D(I)$ satisfying the following conditions:

(IVIGC1) $\mathcal{F}^L(A) \leq \mathcal{F}^U(A)$, $\mathcal{F}^{*L}(A) \leq \mathcal{F}^{*U}(A)$ and $\mathcal{F}^U(A) + \mathcal{F}^{*U}(A) \leq 1$ for each $A \in I^X$,

(IVIGC2) $\mathcal{F}(0_X) = \mathcal{F}(1_X) = \mathbf{1}$ and $\mathcal{F}^*(0_X) = \mathcal{F}^*(1_X) = \mathbf{0}$,

(IVIGC3) $\mathcal{F}^L(A \cup B) \geq \mathcal{F}^L(A) \wedge \mathcal{F}^L(B)$, $\mathcal{F}^U(A \cup B) \geq \mathcal{F}^U(A) \wedge \mathcal{F}^U(B)$ and $\mathcal{F}^{*L}(A \cup B) \leq \mathcal{F}^{*L}(A) \vee \mathcal{F}^{*L}(B)$, $\mathcal{F}^{*U}(A \cup B) \leq \mathcal{F}^{*U}(A) \vee \mathcal{F}^{*U}(B)$ for each $A, B \in I^X$,

(IVIGC4) $\mathcal{F}^L(\cap_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \mathcal{F}^L(A_i)$, $\mathcal{F}^U(\cap_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \mathcal{F}^U(A_i)$ and $\mathcal{F}^{*L}(\cap_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \mathcal{F}^{*L}(A_i)$, $\mathcal{F}^{*U}(\cap_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \mathcal{F}^{*U}(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

For an IVIGO (τ, τ^*) and an IVIGC $(\mathcal{F}, \mathcal{F}^*)$ on X, we define

$$\tau_{\mathcal{F}}(A) = \mathcal{F}(A^c),\ \tau^*_{\mathcal{F}^*}(A) = \mathcal{F}^*(A^c),$$
$$\mathcal{F}_\tau(A) = \tau(A^c),\ \mathcal{F}^*_{\tau^*}(A) = \tau^*(A^c)$$

for each $A \in I^X$.

Definition 3.3. Let (X, τ, τ^*) be an IVISTS, $A \in I^X$ and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$.

(i) A is called an $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy open set* (for short, $([r,s],[t,u])$-IVIFOS) if $\tau(A) \geq [r,s]$ and $\tau^*(A) \leq [t,u]$.

(ii) A is called an $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy closed set* (for short, $([r,s],[t,u])$-IVIFCS) if $\mathcal{F}_\tau(A) \geq [r,s]$ and $\mathcal{F}^*_{\tau^*}(A) \leq [t,u]$.

Definition 3.4. Let (X,τ,τ^*) be an IVISTS, $A \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$. The $([r,s],\ [t,u])$-*interval-valued intuitionistic fuzzy closure* and $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy interior* of A are defined by

$cl_{[r,s],[t,u]}(A) = \cap\{K \in I^X : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r,s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t,u]\}$,

$int_{[r,s],[t,u]}(A) = \cup\{G \in I^X : G \subseteq A,\ \tau(G) \geq [r,s],\ \tau^*(G) \leq [t,u]\}$.

Definition 3.5. Let (X,τ,τ^*) be an IVISTS, $A \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$.

(i) A is called an $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy preopen set* (for short, $([r,s],[t,u])$-IVIFPOS) if $A \subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(A))$.

(ii) A is called an $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy preclosed set* (for short, $([r,s],[t,u])$-IVIFPCS) if A^c is an $([r,s],[t,u])$-IVIFPOS, or equivalently, $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(A)) \subseteq A$.

Note that if A is an $([r,s],[t,u])$-IVIFOS then A is an $([r,s],[t,u])$-IVIFPOS and that if A is an $([r,s],[t,u])$-IVIFCS then A is an $([r,s],[t,u])$-IVIFPCS.

Remark 3.6. Let (X,τ,τ^*) be an IVISTS, $A \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$. Then

(i) Any intersection of $([r,s],[t,u])$-IVIFPCSs is an $([r,s],[t,u])$-IVIFPCS.

(ii) Any union of $([r,s],[t,u])$-IVIFPOSs is an $([r,s],[t,u])$-IVIFPOS.

Definition 3.7. Let (X,τ,τ^*) be an IVISTS, $A \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$. The $([r,s],\ [t,u])$-*interval-valued intuitionistic fuzzy preclosure* and $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy preinterior* of A are defined by

$pcl_{[r,s],[t,u]}(A) = \cap\{K \in I^X : A \subseteq K,\ K \text{ is an } ([r,s],[t,u])\text{-IVIFPCS}\}$,

$pint_{[r,s],[t,u]}(A) = \cup\{G \in I^X : G \subseteq A,\ G \text{ is an } ([r,s],[t,u])\text{-IVIFPOS}\}$.

Note that $int_{[r,s],[t,u]}(A) \subseteq pint_{[r,s],[t,u]}(A) \subseteq A \subseteq pcl_{[r,s],[t,u]}(A)$
$\subseteq cl_{[r,s],[t,u]}(A)$.

Definition 3.8. Let (X,τ,τ^*) be an IVISTS, $A \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$.

(i) A is called an $([r,s],[t,u])$*-interval-valued intuitionistic fuzzy generalized preclosed set* (for short, $([r,s],[t,u])$-IVIFGPCS) if $pcl_{([r,s],[t,u])}(A) \subseteq U$ whenever $A \subseteq U$ and U is an $([r,s],[t,u])$-IVIFOS.

(ii) A is called an $([r,s],[t,u])$*-interval-valued intuitionistic fuzzy generalized preopen set* (for short, $([r,s],[t,u])$-IVIFGPOS) if A^c is an $([r,s],[t,u])$-IVIFGPCS, or equivalently, $U \subseteq pint_{[r,s],[t,u]}(A))$ whenever $U \subseteq A$ and U is an $([r,s],[t,u])$-IVIFCS.

Note that if A is an $([r,s],[t,u])$-IVIFPCS then A is an $([r,s],[t,u])$-IVIFGPCS and that if A is an $([r,s],[t,u])$-IVIFPOS then A is an $([r,s],[t,u])$-IVIFGPOS.

Example 3.9. The intersection of two $([r,s],[t,u])$-IVIFGPCSs need not be an $([r,s],[t,u])$-IVIFGPCS and the union of two $([r,s],[t,u])$-IVIFGPOSs need not be an $([r,s],[t,u])$-IVIFGPOS.

Let $X = \{a,b,c\}$. Define $G_1, G_2, G_3 \in I^X$ as follows:

$$G_1 = \{(a,1),(b,0),(c,0)\},\ G_2 = \{(a,1),(b,1),(c,0)\},$$
$$G_3 = \{(a,1),(b,0),(c,1)\}.$$

Define $\tau, \tau^* : I^X \to D(I)$ as follows:

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_X, 1_X\}, \\ [0.7,0.8] & \text{if } A = G_1, \\ [0.5,0.6] & \text{if } A = G_2, \\ [0.3,0.4] & \text{if } A = G_3, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\tau^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_X, 1_X\}, \\ [0.1,0.2] & \text{if } A = G_1, \\ [0.3,0.4] & \text{if } A = G_2, \\ [0.5,0.6] & \text{if } A = G_3, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Then (X,τ,τ^*) is an IVISTS. Let $[r,s] = [0.6,0.7]$ and $[t,u] = [0.2,0.3]$. Then $int_{[r,s],[t,u]}(G_2) = G_1$ and $cl_{[r,s],[t,u]}(G_1) = 1_X$. Hence $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}\,(G_2)) = 1_X \nsubseteq G_2$. Thus G_2 is not an $([r,s],[t,u])$-IVIFPCS. Similarly, G_3 is not also an $([r,s],[t,u])$-IVIFPCS. Let $G_2 \subseteq U$ and let U be an $([r,s],[t,u])$-IVIFOS. Then $U = 1_X$ and so $pcl_{[r,s],[t,u]}(G_2) \subseteq 1_X = U$. Hence G_2 is an $([r,s],[t,u])$-IVIFGPCS. Similarly, G_3 is also an $([r,s],[t,u])$-IVIFGPCS. Since $G_2 \cap G_3 = G_1$ and $int_{[r,s],[t,u]}(G_1) = G_1$ and $cl_{[r,s],[t,u]}(G_1) = 1_X$, $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(G_1)) = 1_X \nsubseteq G_1$. Hence G_1 is not an $([r,s],[t,u])$-IVIFPCS. Also G_1 is an $([r,s],[t,u])$-IVIFOS. Let $G = \{(a,1),(b,s),(c,t)\} \in I^X$, where $s,t \in (0,1)$. Then $int_{[r,s],[t,u]}(G) = G_1$ and so $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(G)) = 1_X \nsubseteq G$. Hence G is not an $([r,s],[t,u])$-IVIFPCS. Therefore the set for which K is an $([r,s],[t,u])$-IVIFPCS with $G_1 \subseteq K$ is only $K = 1_X$. Hence $pcl_{[r,s],[t,u]}(G_1) = \cap\{K \in I^X : G_1 \subseteq K,\ K \text{ is an } ([r,s],[t,u])\text{-IVIFPCS}\} = 1_X \nsubseteq G_1$. Therefore $G_2 \cap G_3 = G_1$ is not an $([r,s],[t,u])$-IVIFGPCS.

By taking the complement in the above example, the union of two $([r,s],[t,u])$-IVIFGPOSs need not be an $([r,s],[t,u])$-IVIFGPOS.

DEFINITION 3.10. Let (X,τ,τ^*) be an IVISTS, $A \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$. The $([r,s],[t,u])$*-interval-valued intuitionistic fuzzy generalized preclosure* and $([r,s],[t,u])$*-interval-valued intuitionistic fuzzy generalized preinterior* of A are defined by

$gpcl_{[r,s],[t,u]}(A) = \cap\{K \in I^X : A \subseteq K,\ K \text{ is an } ([r,s],[t,u])\text{-IVIFGPCS}\}$,

$gpint_{[r,s],[t,u]}(A) = \cup\{G \in I^X : G \subseteq A,\ G \text{ is an } ([r,s],[t,u])\text{-IVIFGPOS}\}$.

Note that $int_{[r,s],[t,u]}(A) \subseteq pint_{[r,s],[t,u]}(A) \subseteq gpint_{[r,s],[t,u]}(A) \subseteq A \subseteq gpcl_{[r,s],[t,u]}(A) \subseteq pcl_{[r,s],[t,u]}(A) \subseteq cl_{[r,s],[t,u]}(A)$.

THEOREM 3.11. *Let (X,τ,τ^*) be an IVISTS, $A,B \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$. Then*

(i) $gpcl_{[r,s],[t,u]}(0_X) = 0_X$.

(ii) $A \subseteq gpcl_{[r,s],[t,u]}(A)$.

(iii) $A = gpcl_{[r,s],[t,u]}(A)$ *if A is an $([r,s],[t,u])$-IVIFGPCS.*

(iv) $gpcl_{[r,s],[t,u]}(A) \subseteq gpcl_{[r,s],[t,u]}(B)$ *if* $A \subseteq B$.

(v) $gpcl_{[r,s],[t,u]}(A \cup B) \supseteq gpcl_{[r,s],[t,u]}(A) \cup gpcl_{[r,s],[t,u]}(B)$,
$gpcl_{[r,s],[t,u]}(A \cap B) \subseteq gpcl_{[r,s],[t,u]}(A) \cap gpcl_{[r,s],[t,u]}(B)$.

(vi) $gpcl_{[r,s],[t,u]}(gpcl_{[r,s],[t,u]}(A)) = gpcl_{[r,s],[t,u]}(A)$.

(vii) $gpcl_{[r,s],[t,u]}(A^c) = (gpint_{[r,s],[t,u]}(A))^c$.

Proof. (i), (ii), (iii) and (iv) follow directly from Definition 3.10.

(v) It follows directly from (iv).

(vi) By (ii) and (iv), $gpcl_{[r,s],[t,u]}(A) \subseteq gpcl_{[r,s],[t,u]}(gpcl_{[r,s],[t,u]}(A))$. Suppose that $gpcl_{[r,s],[t,u]}(gpcl_{[r,s],[t,u]}(A)) \nsubseteq gpcl_{[r,s],[t,u]}(A)$. Then there exists $x \in X$ such that $(gpcl_{[r,s],[t,u]}(A))(x) < (gpcl_{[r,s],[t,u]}(gpcl_{[r,s],[t,u]}(A)))(x)$. Choose $a \in (0,1)$ with $(gpcl_{[r,s],[t,u]}(A))(x) < a < (gpcl_{[r,s],[t,u]}(gpcl_{[r,s],[t,u]}(A)))(x)$. Since $(gpcl_{[r,s],[t,u]}(A))(x) < a$, by Definition 3.10, there exists an $([r,s],[t,u])$-IVIFGPCS K such that $A \subseteq K$ and $K(x) < a$.

Since K is an $([r,s],[t,u])$-IVIFGPCS with $A \subseteq K$, $gpcl_{[r,s],[t,u]}(A) \subseteq K$ and also $gpcl_{[r,s],[t,u]}(gpcl_{[r,s],[t,u]}(A)) \subseteq K$. Hence $(gpcl_{[r,s],[t,u]}(gpcl_{[r,s],[t,u]}(A)))(x) \leq K(x) < a$. This is a contradiction. Hence $gpcl_{[r,s],[t,u]}(gpcl_{[r,s],[t,u]}(A)) \subseteq gpcl_{[r,s],[t,u]}(A)$. Therefore $gpcl_{[r,s],[t,u]}(gpcl_{[r,s],[t,u]}(A)) = gpcl_{[r,s],[t,u]}(A)$.

(vii) By Definition 3.10, we have

$$
\begin{aligned}
& gpcl_{[r,s],[t,u]}(A^c) \\
& = \cap\{K \in I^X : A^c \subseteq K,\ K \text{ is an } ([r,s],[t,u])\text{-IVIFGPCS}\} \\
& = \cap\{G^c \in I^X : A^c \subseteq G^c,\ G^c \text{ is an } ([r,s],[t,u])\text{-IVIFGPCS}\} \\
& = (\cup\{G \in I^X : G \subseteq A,\ G \text{ is an } ([r,s],[t,u])\text{-IVIFGPOS}\})^c \\
& = (gpint_{[r,s],[t,u]}(A))^c.
\end{aligned}
$$

□

THEOREM 3.12. *Let (X,τ,τ^*) be an IVISTS, $A, B \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$. Then*

(i) $gpint_{[r,s],[t,u]}(1_X) = 1_X$.

(ii) $gpint_{[r,s],[t,u]}(A) \subseteq A$.

(iii) $A = gpint_{[r,s],[t,u]}(A)$ *if A is an $([r,s],[t,u])$-IVIFGPOS.*

(iv) $gpint_{[r,s],[t,u]}(A) \subseteq gpint_{[r,s],[t,u]}(B)$ *if* $A \subseteq B$.

(v) $gpint_{[r,s],[t,u]}(A \cup B) \supseteq gpint_{[r,s],[t,u]}(A) \cup gpint_{[r,s],[t,u]}(B)$,
$gpint_{[r,s],[t,u]}(A \cap B) \subseteq gpint_{[r,s],[t,u]}(A) \cap gpint_{[r,s],[t,u]}(B)$.

(vi) $gpint_{[r,s],[t,u]}(gpint_{[r,s],[t,u]}(A)) = gpint_{[r,s],[t,u]}(A)$.

(vii) $gpint_{[r,s],[t,u]}(A^c) = (gpcl_{[r,s],[t,u]}(A))^c$.

Proof. The proof is similar to Theorem 3.11.

□

4. $([r,s],[t,u])$-interval-valued intuitionistic fuzzy generalized precontinuous mappings

DEFINITION 4.1. Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping.

(i) f is called an $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy generalized precontinuous mapping* (for short, $([r,s],[t,u])$-IVIFG precontinuous mapping) if $f^{-1}(A)$ is an $([r,s],[t,u])$-IVIFGPCS of X for each $([r,s],[t,u])$-IVIFCS A of Y.

(ii) f is called an $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy generalized preopen mapping* (for short, $([r,s],[t,u])$-IVIFG preopen mapping) if $f(A)$ is an $([r,s],[t,u])$-IVIFGPOS of Y for each $([r,s],\ [t,u])$-IVIFOS A of X.

(iii) f is called an $([r,s],\ [t,u])$-*interval-valued intuitionistic fuzzy generalized preclosed mapping* (for short, $([r,s],[t,u])$-IVIFG preclosed mapping) if $f(A)$ is an $([r,s],[t,u])$-IVIFGPCS of Y for each $([r,s],$ $[t,u])$IVIFCS A of X.

(iv) f is called an $([r,s],\ [t,u])$-*interval-valued intuitionistic fuzzy generalized preirresolute mapping* (for short, $([r,s],[t,u])$-IVIFG preirresolute mapping) if $f^{-1}(A)$ is an $([r,s],[t,u])$-IVIFGPCS of X for each $([r,s],\ [t,u])$-IVIFGPCS A of Y.

Note that $f : X \to Y$ is an $([r,s],[t,u])$-IVIFG precontinuous mapping if and only if $f^{-1}(A)$ is an $([r,s],[t,u])$-IVIFGPOS of X for each $([r,s],\ [t,u])$-IVIFOS A of Y.

THEOREM 4.2. *Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. If f is an $([r,s],[t,u])$-IVIFG precontinuous mapping, then $f(gpcl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$.*

Proof. Let $A \in I^X$. Then $cl_{[r,s],[t,u]}(f(A))$ is an $([r,s],\ [t,u])$-IVIFCS of Y. Since f is $([r,s],[t,u])$-IVIFG precontinuous, $f^{-1}(cl_{[r,s],[t,u]}(f(A)))$ is an $([r,s],\ [t,u])$-IVIFGPCS of X. Since $A \subseteq f^{-1}(cl_{[r,s],[t,u]}(f(A)))$, by Definition 3.10 $gpcl_{[r,s],[t,u]}(A) \subseteq f^{-1}(cl_{[r,s],[t,u]}(f(A)))$. Hence $f(gpcl_{[r,s],[t,u]}$ $(A)) \subseteq f(f^{-1}(cl_{[r,s],[t,u]}(f(A)))) \subseteq cl_{[r,s],[t,u]}(f(A))$.

□

THEOREM 4.3. *Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. If $f(pcl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$, then f is an $([r,s],[t,u])$-IVIFG precontinuous mapping.*

Proof. Let A be an $([r,s],\ [t,u])$-IVIFCS of Y. Then $f^{-1}(A) \in I^X$. Let $f^{-1}(A) \subseteq U$ and let U be an $([r,s],\ [t,u])$-IVIFOS. By hypothesis, $f(pcl_{[r,s],[t,u]}\ (f^{-1}(A))) \subseteq cl_{[r,s],[t,u]}(f(f^{-1}(A))) \subseteq cl_{[r,s],[t,u]}(A) = A$. Hence $pcl_{[r,s],[t,u]}\ (f^{-1}(A)) \subseteq f^{-1}(f(pcl_{[r,s],[t,u]}(f^{-1}(A)))) \subseteq f^{-1}(A) \subseteq U$. Thus $f^{-1}(A)$ is an $([r,s],\ [t,u])$-IVIFGPCS of X. Therefore f is an $([r,s],[t,u])$-IVIFG precontinuous mapping.

□

Definition 4.4. An IVISTS (X,τ,τ^*) is called an *interval-valued intuitionistic fuzzy pre $T_{1/2}$ space* (for short, IVIFPT$_{1/2}$ space) if for each $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ with $s+u\leq 1$, every $([r,s],[t,u])$-IVIFGPCS in X is an $([r,s],[t,u])$-IVIFPCS in X.

Theorem 4.5. *Let (X,τ,τ^*) be an IVIFPT$_{1/2}$ space and (Y,η,η^*) an IVISTS and $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ with $s+u\leq 1$ and let $f: X\to Y$ be a mapping. Then the following statements are equivalent.*

(i) *f is an $([r,s],[t,u])$-IVIFG precontinuous mapping.*

(ii) *$f(gpcl_{[r,s],[t,u]}(A))\subseteq cl_{[r,s],[t,u]}(f(A))$ for each $A\in I^X$.*

(iii) *$gpcl_{[r,s],[t,u]}(f^{-1}(A))\subseteq f^{-1}(cl_{[r,s],[t,u]}(A))$ for each $A\in I^Y$.*

(iv) *$f^{-1}(int_{[r,s],[t,u]}(A))\subseteq gpint_{[r,s],[t,u]}(f^{-1}(A))$ for each $A\in I^Y$.*

Proof. (i)⇒(ii). It follows from Theorem 4.2.

(ii)⇒(iii). Let $A\in I^Y$. Then $f^{-1}(A)\in I^X$. By (ii), we have

$$\begin{aligned} f(gpcl_{[r,s],[t,u]}(f^{-1}(A))) &\subseteq cl_{[r,s],[t,u]}(f(f^{-1}(A))) \\ &\subseteq cl_{[r,s],[t,u]}(A). \end{aligned}$$

Hence we have

$$\begin{aligned} gpcl_{[r,s],[t,u]}(f^{-1}(A)) &\subseteq f^{-1}(f(gpcl_{[r,s],[t,u]}(f^{-1}(A)))) \\ &\subseteq f^{-1}(cl_{[r,s],[t,u]}(A)). \end{aligned}$$

(iii)⇒(iv). Let $A\in I^Y$. By (iii), we have

$$\begin{aligned} gpcl_{[r,s],[t,u]}((f^{-1}(A))^c) &= gpcl_{[r,s],[t,u]}(f^{-1}(A^c)) \\ &\subseteq f^{-1}(cl_{[r,s],[t,u]}(A^c)). \end{aligned}$$

Thus $(gpint_{[r,s],[t,u]}(f^{-1}(A)))^c\subseteq (f^{-1}(int_{[r,s],[t,u]}(A)))^c$. Hence $f^{-1}(int_{[r,s],[t,u]}(A))\subseteq gpint_{[r,s],[t,u]}(f^{-1}(A))$.

(iv)⇒(i). Let A be an $([r,s],[t,u])$-IVIFCS of Y. Then $f^{-1}(A)\in I^X$ and A^c is an $([r,s],[t,u])$-IVIFOS of Y and so $int_{[r,s],[t,u]}(A^c)=A^c$. Let $f^{-1}(A)\subseteq U$ and let U be an $([r,s],[t,u])$-IVIFOS of X. By (iv), we have

$$\begin{aligned} (f^{-1}(A))^c = f^{-1}(A^c) &= f^{-1}(int_{[r,s],[t,u]}(A^c)) \\ &\subseteq gpint_{[r,s],[t,u]}(f^{-1}(A^c)) \\ &= (gpcl_{[r,s],[t,u]}(f^{-1}(A)))^c. \end{aligned}$$

Hence $gpcl_{[r,s],[t,u]}(f^{-1}(A))\subseteq f^{-1}(A)$ and so $gpcl_{[r,s],[t,u]}(f^{-1}(A))=f^{-1}(A)$. Since (X,τ,τ^*) is an IVIFPT$_{1/2}$ space, $gpcl_{[r,s],[t,u]}(f^{-1}(A))=pcl_{[r,s],[t,u]}(f^{-1}(A))$. Hence $pcl_{[r,s],[t,u]}(f^{-1}(A))=f^{-1}(A)\subseteq U$. Thus $f^{-1}(A)$ is an $([r,s],[t,u])$-IVIFGPCS of X. Therefore f is an $([r,s],[t,u])$-IVIFG precontinuous mapping.

□

THEOREM 4.6. *Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ with $s+u\le 1$ and let $f:X\to Y$ be a mapping. If $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A)))\subseteq f^{-1}(cl_{[r,s],[t,u]}(A))$ for each $A\in I^Y$, then f is an $([r,s],[t,u])$-IVIFG precontinuous mapping.*

Proof. Let A be an $([r,s],\ [t,u])$-IVIFCS of Y. Then $cl_{[r,s],[t,u]}(A)=A$. By hypothesis, $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A)))\subseteq f^{-1}(cl_{[r,s],[t,u]}(A))=f^{-1}(A)$. Thus $f^{-1}(A)$ is an $([r,s],\ [t,u])$-IVIFPCS of X. So $f^{-1}(A)$ is an $([r,s],\ [t,u])$-IVIFGPCS of X. Hence f is an $([r,s],[t,u])$-IVIFG precontinuous mapping.

□

We can obtain the following corollary from Theorem 4.6.

COROLLARY 4.7. *Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ with $s+u\le 1$ and let $f:X\to Y$ be a mapping. If $f^{-1}(int_{[r,s],[t,u]}(A))\subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(A)))$ for each $A\in I^Y$, then f is an $([r,s],[t,u])$-IVIFG precontinuous mapping.*

THEOREM 4.8. *Let (X,τ,τ^*) be an IVIFPT$_{1/2}$ space and (Y,η,η^*) an IVISTS and $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ with $s+u\le 1$ and let $f:X\to Y$ be a mapping. Then the following statements are equivalent.*

(i) *f is an $([r,s],[t,u])$-IVIFG precontinuous mapping.*

(ii) *$cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A)))\subseteq f^{-1}(cl_{[r,s],[t,u]}(A))$ for each $A\in I^Y$.*

(iii) *$f^{-1}(int_{[r,s],[t,u]}(A))\subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(A)))$ for each $A\in I^Y$.*

Proof. (i)⇒(ii). Let $A\in I^Y$. Then $cl_{[r,s],[t,u]}(A)$ is an $([r,s],\ [t,u])$-IVIFCS of Y. Since f is an $([r,s],[t,u])$-IVIFG precontinuous mapping, $f^{-1}(cl_{[r,s],[t,u]}\ (A))$ is an $([r,s],\ [t,u])$-IVIFGPCS of X. Since X is an IVIFPT$_{1/2}$ space, $f^{-1}(cl_{[r,s],[t,u]}(A))$ is an $([r,s],\ [t,u])$-IVIFPCS of X. Thus $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}\ (f^{-1}(cl_{[r,s],[t,u]}(A))))\subseteq f^{-1}(cl_{[r,s],[t,u]}(A))$. Hence

$$\begin{aligned}cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A)))&\subseteq cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(cl_{[r,s],[t,u]}(A))))\\&\subseteq f^{-1}(cl_{[r,s],[t,u]}(A)).\end{aligned}$$

(ii)⇒(iii). Let $A\in I^Y$. Then $A^c\in I^Y$. By (ii), $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A^c)))\subseteq f^{-1}(cl_{[r,s],[t,u]}(A^c))$. Thus $(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(A)))^c\subseteq$

$(f^{-1}\ (int_{[r,s],[t,u]}(A)))^c$. Hence $f^{-1}(int_{[r,s],[t,u]}(A)) \subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}$ $(f^{-1}(A)))$.

(iii)⇒(i). It follows from Corollary 4.7.

□

THEOREM 4.9. *Let (X, τ, τ^*) be an IVISTS and (Y, η, η^*) an IVIFPT$_{1/2}$ space and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. Then the following statements are equivalent.*

(i) *f is an $([r,s],[t,u])$-IVIFG preopen mapping.*

(ii) *$f(int_{[r,s],[t,u]}(A)) \subseteq gpint_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$.*

(iii) *$int_{[r,s],[t,u]}(f^{-1}(A)) \subseteq f^{-1}(gpint_{[r,s],[t,u]}(A))$ for each $A \in I^Y$.*

Proof. (i)⇒(ii). Let $A \in I^X$. Then $int_{[r,s],[t,u]}(A)$ is an $([r,s],[t,u])$-IVIFOS of X. Since f is an $([r,s],[t,u])$-IVIFG preopen mapping, $f(int_{[r,s],[t,u]}(A))$ is an $([r,s], [t,u])$-IVIFGPOS of Y and $f(int_{[r,s],[t,u]}(A))$ $\subseteq f(A)$. By Definition 3.10, $f(int_{[r,s],[t,u]}(A)) \subseteq gpint_{[r,s],[t,u]}(f(A))$.

(ii)⇒(iii). Let $A \in I^Y$. Then $f^{-1}(A) \in I^X$. By (ii), we have

$$\begin{aligned} f(int_{[r,s],[t,u]}(f^{-1}(A))) &\subseteq gpint_{[r,s],[t,u]}(f(f^{-1}(A))) \\ &\subseteq gpint_{[r,s],[t,u]}(A). \end{aligned}$$

Hence

$$\begin{aligned} int_{[r,s],[t,u]}(f^{-1}(A)) &\subseteq f^{-1}(f(int_{[r,s],[t,u]}(f^{-1}(A)))) \\ &\subseteq f^{-1}(gpint_{[r,s],[t,u]}(A)). \end{aligned}$$

(iii)⇒(i). Let A be an $([r,s],[t,u])$-IVIFOS of X. Then $int_{[r,s],[t,u]}(A)$ $= A$ and $f(A) \in I^Y$. Let $(f(A))^c \subseteq U$ and let U be an $([r,s],[t,u])$-IVIFOS of Y. By (iii), we have

$$\begin{aligned} A = int_{[r,s],[t,u]}(A) &\subseteq int_{[r,s],[t,u]}(f^{-1}(f(A))) \\ &\subseteq f^{-1}(gpint_{[r,s],[t,u]}(f(A))). \end{aligned}$$

Hence $f(A) \subseteq f(f^{-1}(gpint_{[r,s],[t,u]}(f(A)))) \subseteq gpint_{[r,s],[t,u]}(f(A))$ and so $(f(A))^c \supseteq (gpint_{[r,s],[t,u]}(f(A)))^c = gpcl_{[r,s],[t,u]}((f(A))^c)$. Thus $(f(A))^c = gpcl_{[r,s],[t,u]}((f(A))^c)$. Since Y is an IVIFPT$_{1/2}$ space, $gpcl_{[r,s],[t,u]}((f(A))^c)$ $= pcl_{[r,s],[t,u]}((f(A))^c)$. Hence $pcl_{[r,s],[t,u]}((f(A))^c) = (f(A))^c \subseteq U$. So $(f(A))^c$ is an $([r,s],[t,u])$-IVIFGPCS of Y. Thus $f(A)$ is an $([r,s],[t,u])$-IVIFGPOS of Y. Therefore f is an $([r,s],[t,u])$-IVIFG preopen mapping.

□

THEOREM 4.10. *Let (X, τ, τ^*) be an IVISTS and (Y, η, η^*) an IVIFPT$_{1/2}$ space and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \to Y$ be a mapping. Then f is an $([r, s], [t, u])$-IVIFG preclosed mapping if and only if $gpcl_{[r,s],[t,u]}(f(A)) \subseteq f(cl_{[r,s],[t,u]}(A))$ for each $A \in I^X$.*

Proof. Suppose that f is an $([r, s], [t, u])$-IVIFG preclosed mapping. Let $A \in I^X$. Then $cl_{[r,s],[t,u]}(A)$ is an $([r, s], [t, u])$-IVIFCS of X. Since f is an $([r, s], [t, u])$-IVIFG preclosed mapping, $f(cl_{[r,s],[t,u]}(A))$ is an $([r, s], [t, u])$-IVIFGPCS of Y and $f(A) \subseteq f(cl_{[r,s],[t,u]}(A))$. By Definition 3.10, $gpcl_{[r,s],[t,u]}$ $(f(A)) \subseteq f(cl_{[r,s],[t,u]}(A))$.

Conversely, suppose that $gpcl_{[r,s],[t,u]}(f(A)) \subseteq f(cl_{[r,s],[t,u]}(A))$ for each $A \in I^X$. Let A be an $([r, s], [t, u])$-IVIFCS of X. Then $cl_{[r,s],[t,u]}(A) = A$. Let $f(A) \subseteq U$ and let U be an $([r, s], [t, u])$-IVIFOS of Y. By hypothesis, $gpcl_{[r,s],[t,u]}(f(A)) \subseteq f(cl_{[r,s],[t,u]}(A)) = f(A)$. Thus $gpcl_{[r,s],[t,u]}(f(A)) = f(A)$. Since Y is an IVIFPT$_{1/2}$ space, $gpcl_{[r,s],[t,u]}(f(A)) = pcl_{[r,s],[t,u]}(f(A))$. Hence $pcl_{[r,s],[t,u]}(f(A)) = f(A) \subseteq U$. Thus $f(A)$ is an $([r, s], [t, u])$-IVIFGPCS of Y. Therefore f is an $([r, s], [t, u])$-IVIFG preclosed mapping.

□

THEOREM 4.11. *Let (X, τ, τ^*) be an IVISTS and (Y, η, η^*) an IVIFPT$_{1/2}$ space and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \to Y$ be a bijective mapping. Then f is an $([r, s], [t, u])$-IVIFG preclosed mapping if and only if $f^{-1}(gpcl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f^{-1}(A))$ for each $A \in I^Y$.*

Proof. Suppose that f is an $([r, s], [t, u])$-IVIFG preclosed mapping. Let $A \in I^Y$. Then $cl_{[r,s],[t,u]}(f^{-1}(A))$ is an $([r, s], [t, u])$-IVIFCS of X. Since f is an $([r, s], [t, u])$-IVIFG preclosed mapping, $f(cl_{[r,s],[t,u]}(f^{-1}(A)))$ is an $([r, s], [t, u])$-IVIFGPCS of Y. Since f is surjective, $A = f(f^{-1}(A)) \subseteq f(cl_{[r,s],[t,u]}$ $(f^{-1}(A)))$. By Definition 3.10, $gpcl_{[r,s],[t,u]}(A) \subseteq f(cl_{[r,s],[t,u]}(f^{-1}$ $(A)))$. Since f is injective, we have

$$\begin{aligned} f^{-1}(gpcl_{[r,s],[t,u]}(A)) &\subseteq f^{-1}(f(cl_{[r,s],[t,u]}(f^{-1}(A)))) \\ &= cl_{[r,s],[t,u]}(f^{-1}(A)). \end{aligned}$$

Conversely, suppose that $f^{-1}(gpcl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f^{-1}(A))$ for each $A \in I^Y$. Let A be an $([r, s], [t, u])$-IVIFCS of X. Then $cl_{[r,s],[t,u]}(A)$ $= A$. Let $f(A) \subseteq U$ and let U be an $([r, s], [t, u])$-IVIFOS of Y. By hypothesis and the injectivity of f, we have

$$\begin{aligned} f^{-1}(gpcl_{[r,s],[t,u]}(f(A))) &\subseteq cl_{[r,s],[t,u]}(f^{-1}(f(A))) \\ &= cl_{[r,s],[t,u]}(A) = A. \end{aligned}$$

Since f is surjective, $gpcl_{[r,s],[t,u]}(f(A)) = f(f^{-1}(gpcl_{[r,s],[t,u]}(f(A)))) \subseteq f(A)$. Thus $gpcl_{[r,s],[t,u]}(f(A)) = f(A)$. Since Y is an IVIFPT$_{1/2}$ space, $gpcl_{[r,s],[t,u]}$ $(f(A)) = pcl_{[r,s],[t,u]}(f(A))$. Thus $pcl_{[r,s],[t,u]}(f(A)) = f(A) \subseteq U$. Hence $f(A)$ is an $([r,s],[t,u])$-IVIFGPCS of Y. Therefore f is an $([r,s],[t,u])$-IVIFG preclosed mapping.

□

THEOREM 4.12. *Let (X,τ,τ^*) be an IVIFPT$_{1/2}$ space and (Y,η,η^*) an IVISTS and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. Then the following statements are equivalent.*

(i) *f is an $([r,s],[t,u])$-IVIFG preirresolute mapping.*

(ii) *$f(gpcl_{[r,s],[t,u]}(A)) \subseteq gpcl_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$.*

(iii) *$gpcl_{[r,s],[t,u]}(f^{-1}(A)) \subseteq f^{-1}(gpcl_{[r,s],[t,u]}(A))$ for each $A \in I^Y$.*

(iv) *$f^{-1}(gpint_{[r,s],[t,u]}(A)) \subseteq gpint_{[r,s],[t,u]}(f^{-1}(A))$ for each $A \in I^Y$.*

Proof. (i)⇒(ii). Let $A \in I^X$. Then $f(A) \in I^Y$. Since f is an $([r,s],[t,u])$-IVIFG preirresolute mapping, we have

$$\begin{aligned} &f^{-1}(gpcl_{[r,s],[t,u]}(f(A))) \\ &= f^{-1}(\cap\{K \in I^Y : f(A) \subseteq K,\ K \text{ is an } ([r,s],[t,u])\text{-IVIFGPCS}\}) \\ &\supseteq f^{-1}(\cap\{K \in I^Y : A \subseteq f^{-1}(K),\ K \text{ is an } ([r,s],[t,u])\text{-IVIFGPCS}\}) \\ &= \cap\{f^{-1}(K) \in I^X : A \subseteq f^{-1}(K),\ K \text{ is an } ([r,s],[t,u])\text{-IVIFGPCS}\} \\ &\supseteq \cap\{f^{-1}(K) \in I^X : A \subseteq f^{-1}(K),\ f^{-1}(K) \text{ is an } ([r,s],[t,u])\text{-IVIFGPCS}\} \\ &\supseteq \cap\{W \in I^X : A \subseteq W,\ W \text{ is an } ([r,s],[t,u])\text{-IVIFGPCS}\} \\ &= gpcl_{[r,s],[t,u]}(A). \end{aligned}$$

Hence

$$\begin{aligned} f(gpcl_{[r,s],[t,u]}(A)) &\subseteq f(f^{-1}(gpcl_{[r,s],[t,u]}(f(A)))) \\ &\subseteq gpcl_{[r,s],[t,u]}(f(A)). \end{aligned}$$

The proofs of (ii)⇒(iii), (iii)⇒(iv) and (iv)⇒(i) are similar to Theorem 4.5.

□

Theorem 4.13. *Let* (X,τ,τ^*) *and* (Y,η,η^*) *be two IVISTSs and* $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ *with* $s+u\leq 1$ *and let* $f:X\to Y$ *be a mapping. If* $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A)))\subseteq f^{-1}(gpcl_{[r,s],[t,u]}(A))$ *for each* $A\in I^Y$, *then* f *is an* $([r,s],[t,u])$*-IVIFG preirresolute mapping.*

Proof. It is similar to Theorem 4.6. □

We can obtain the following corollary from Theorem 4.13.

Corollary 4.14. *Let* (X,τ,τ^*) *and* (Y,η,η^*) *be two IVISTSs and* $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ *with* $s+u\leq 1$ *and let* $f:X\to Y$ *be a mapping. If* $f^{-1}(gpint_{[r,s],[t,u]}(A))\subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(A)))$ *for each* $A\in I^Y$, *then* f *is an* $([r,s],[t,u])$*-IVIFG preirresolute mapping.*

Theorem 4.15. *Let* (X,τ,τ^*) *be an IVIFPT*$_{1/2}$ *space and* (Y,η,η^*) *an IVISTS and* $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ *with* $s+u\leq 1$ *and let* $f:X\to Y$ *be a mapping. Then the following statements are equivalent.*

(i) f *is an* $([r,s],[t,u])$*-IVIFG preirresolute mapping.*

(ii) $gpcl_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A))))\subseteq f^{-1}(A)$ *for each* $([r,s],[t,u])$*-IVIFGPCS* A *of* Y.

(iii) $f^{-1}(A)\subseteq gpint_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(A))))$ *for each* $([r,s],[t,u])$*-IVIFGPOS* A *of* Y.

Proof. (i)⇒(ii). Let A be an $([r,s],[t,u])$-IVIFGPCS of Y. Since f is an $([r,s],[t,u])$-IVIFG preirresolute mapping, $f^{-1}(A)$ is an $([r,s],[t,u])$-IVIFGPCS of X and so $gpcl_{[r,s],[t,u]}(f^{-1}(A))=f^{-1}(A)$. Since X is an IVIFPT$_{1/2}$ space, $f^{-1}(A)$ is an $([r,s],[t,u])$-IVIFPCS of X and so $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A)))\subseteq f^{-1}(A)$. Hence

$$\begin{aligned}&gpcl_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A))))\\&\subseteq gpcl_{[r,s],[t,u]}(f^{-1}(A))=f^{-1}(A).\end{aligned}$$

(ii)⇒(iii). Let A be an $([r,s],[t,u])$-IVIFGPOS of Y. Then A^c is an $([r,s],[t,u])$-IVIFGPCS of Y. By (ii), $gpcl_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A^c))))\subseteq f^{-1}(A^c)$. Thus $(gpint_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(A)))))^c\subseteq(f^{-1}(A))^c$. Hence $f^{-1}(A)\subseteq gpint_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(A))))$.

(iii)⇒(i). Let A be an $([r,s],[t,u])$-IVIFGPCS of Y. Then A^c is an $([r,s],[t,u])$-IVIFGPOS of Y. By (iii), $f^{-1}(A^c)\subseteq gpint_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(A^c))))$. Thus $(f^{-1}(A))^c\subseteq(gpcl_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A)))))^c$. Hence $gpcl_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A))))\subseteq f^{-1}(A)$. Since $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A)))\subseteq gpcl_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}

$(A)))),\ cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(A)))) \subseteq f^{-1}(A)$. Hence $f^{-1}(A)$ is an $([r,s],\ [t,u])$-IVIFPCS of X and so $f^{-1}(A)$ is an $([r,s],\ [t,u])$-IVIFGPCS of X. Therefore f is an $([r,s],[t,u])$-IVIFG preirresolute mapping.

$\square$

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