

## ON STATISTICALLY SEQUENTIALLY QUOTIENT MAPS

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ABSTRACT. In this paper, we introduce the concept of statistically sequentially quotient map which is a generalization of sequence covering map and discuss the relation with covering maps by some examples. Using this concept, we give an affirmative answer for a question by Fucai Lin and Shou Lin.

### 1. Introduction

Finding the internal characterizations of certain images of metric spaces is one of the central questions in general topology [8–10, 12, 16, 22]. In 1971, Siwec [20] introduced the concept of sequence covering maps which is closely related to the question about compact-covering and  $s$ -images of metric spaces. Lin and Yan in [15] proved that each sequence-covering and compact map on metric spaces is an 1-sequence covering map. Later Lin Fucai and Lin Shou in [13] proved that each sequence-covering and boundary-compact map on metric spaces is an 1-sequence covering map and posed Question 1 below. In [14], they answered this

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question. In this paper, we consider a similar question for statistically sequentially quotient and boundary-compact maps and prove the main result Theorem 1.

QUESTION 1.1. [13] *Let  $f : X \rightarrow Y$  be a sequence-covering and boundary-compact map. Is  $f$  an 1-sequence-covering map, if  $X$  is a space with a point-countable base or a developable space?*

THEOREM 1. *Let  $f : X \rightarrow Y$  be a statistically sequentially quotient and boundary-compact map. Suppose also that at least one of the following conditions holds:*

- (1)  *$X$  has a point-countable base;*
- (2)  *$X$  is a developable space.*

*Then  $f$  is an 1-sequence-covering map.*

Throughout this paper, all spaces are  $T_2$ , all maps are continuous and onto, and  $\mathbb{N}$  is the set of positive integers.  $x_n \rightarrow x$  denote a sequence  $\{x_n\}$  converging to  $x$ . Let  $X$  be a space and  $P \subset X$ . A sequence  $\{x_n\}$  converging to  $x$  in  $X$  is eventually in  $P$  if  $\{x_n \mid n > k\} \cup \{x\} \subset P$  for some  $k \in \mathbb{N}$ ; it is frequently in  $P$  if  $\{x_{n_k}\}$  is eventually in  $P$  for some subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$ . Let  $\mathcal{P}$  be a family of subsets of  $X$ . Then  $\cup\mathcal{P}$  and  $\cap\mathcal{P}$  denote the union  $\cup\{P \mid P \in \mathcal{P}\}$  and the intersection  $\cap\{P \mid P \in \mathcal{P}\}$ , respectively.

DEFINITION 1.2. Let  $X$  be a space and  $P \subset X$ .

- (a) Let  $x \in P$ .  $P$  is called a *sequential neighborhood* [6] of  $x$  in  $X$  if whenever  $\{x_n\}$  is a sequence converging to the point  $x$ , then  $\{x_n\}$  is eventually in  $P$ .
- (b)  $P$  is called a *sequentially open* [6] subset in  $X$  if  $P$  is a sequential neighborhood of  $x$  in  $X$  for each  $x \in P$ .

DEFINITION 1.3. Let  $(X, \tau)$  be a topological space. We define a *sequential closure-topology*  $\sigma_\tau$  [6] on  $X$  as follows:  $O \in \sigma_\tau$  if and only if  $O$  is a sequentially open subset in  $(X, \tau)$ . The topological space  $(X, \sigma_\tau)$  is denoted by  $\sigma X$ .

DEFINITION 1.4. Let  $\mathcal{P} = \cup\{\mathcal{P}_x \mid x \in X\}$  be a cover of a space  $X$  such that for each  $x \in X$ , the following conditions (a) and (b) are satisfied:

- (a)  $\mathcal{P}_x$  is a *network* at  $x$  in  $X$ , i.e.,  $x \in \cap\mathcal{P}_x$  and for each neighborhood  $U$  of  $x$  in  $X$ ,  $P \subset U$  for some  $P \in \mathcal{P}_x$ ;
- (b) If  $U, V \in \mathcal{P}_x$ , then  $W \subset U \cap V$  for some  $W \in \mathcal{P}_x$ .

- (i)  $\mathcal{P}$  is called an *sn-network* [7] of  $X$  if each element of  $\mathcal{P}_x$  is a sequential neighborhood of  $x$  for each  $x \in X$ , where  $\mathcal{P}_x$  is called an *sn-network* at  $x$  in  $X$ . In this paper, when we say an *sn  $f$ -countable space*  $Y$ , it is always assumed that  $Y$  has an *sn-network*  $\mathcal{P} = \bigcup\{\mathcal{P}_y \mid y \in Y\}$  such that  $\mathcal{P}_y$  is countable and closed under finite intersections for each point  $y \in Y$ .
- (ii)  $\mathcal{P}$  is called a *weak base* [2] of  $X$  if whenever  $G \subset X$ ,  $G$  is open in  $X$  if and only if for each  $x \in G$ , there exists  $P \in \mathcal{P}_x$  such that  $P \subset G$ .

DEFINITION 1.5. [17] Let  $A$  be a subset of a space  $X$ . We call an open family  $\mathcal{N}$  of subsets of  $X$  is an *external base* of  $A$  in  $X$  if for any  $x \in A$  and open subset  $U$  with  $x \in U$  there is a  $V \in \mathcal{N}$  such that  $x \in V \subset U$ .

Similarly, we can define an *externally weak base* for a subset  $A$  of a space  $X$ .

DEFINITION 1.6. Let  $f : X \rightarrow Y$  be a map.

- (a)  $f$  is a *boundary compact map* [13] if  $\partial f^{-1}(y)$  is compact in  $X$  for every  $y \in Y$ .
- (b)  $f$  is *sequence covering* [12] if for every convergent sequence  $S$  in  $Y$ , there is a convergent sequence  $L$  in  $X$  such that  $f(L) = S$ . Equivalently, if whenever  $\{y_n\}$  is a convergent sequence in  $Y$ , there is a convergent sequence  $\{x_n\}$  in  $X$  with each  $x_n \in f^{-1}(y_n)$  [20].
- (c)  $f$  is *sequentially quotient* [12] if for every convergent sequence  $S$  in  $Y$ , there is a convergent sequence  $L$  in  $X$  such that  $f(L)$  is an infinite subsequence of  $S$ . Equivalently, if whenever  $\{y_n\}$  is a convergent sequence in  $Y$ , there is a convergent sequence  $\{x_k\}$  in  $X$  with each  $x_k \in f^{-1}(y_{n_k})$  [20].
- (d)  $f$  is *1-sequence covering* [11] if for each  $y \in Y$ , there is  $x \in f^{-1}(y)$  such that whenever  $\{y_n\}$  is a sequence converging to  $y$  in  $Y$ , there is a sequence  $\{x_n\}$  converging to  $x$  in  $X$  with each  $x_n \in f^{-1}(y_n)$ .

DEFINITION 1.7. [5, 19] If  $K \subset \mathbb{N}$ , then  $K_n$  will denote the set  $\{k \in K, k \leq n\}$  and  $|K_n|$  stands for the cardinality of  $K_n$ . The *natural density* of  $K$  is defined by  $d(K) = \lim_{n \rightarrow \infty} \frac{|K_n|}{n}$ , if limit exists. And  $K$  is called *statistically dense* [4] if  $d(K) = 1$ .

DEFINITION 1.8. A subsequence  $S$  of the sequence  $L$  is called *statistically dense* [4] in  $L$  if the set of all indices of elements from  $S$  is statistically dense.

## 2. Statistically Sequentially Quotient Maps

In this section, we introduce statistically sequentially quotient maps and give their properties. A map  $f : X \rightarrow Y$  is said to be a *statistically sequentially quotient* map if for given  $y_n \rightarrow y$  in  $Y$ , there exist  $x_k \rightarrow x$ ,  $x \in f^{-1}(y)$  and  $x_k \in f^{-1}(y_{n_k})$  such that  $d(K) = 0$  where  $K = \{n \in \mathbb{N} \mid x_k \notin f^{-1}(y_n) \text{ for all } k \in \mathbb{N}\}$ . That is,  $\{f(x_n)\}$  is statistically dense in  $\{y_n\}$ , since  $d(K) + d(N \setminus K) = 1$  [18].

**PROPOSITION 2.1.** *Let  $f : X \rightarrow Y$  be a map and  $g = f|_{\sigma X} : \sigma X \rightarrow \sigma Y$ . Then  $f$  is a statistically sequentially quotient if and only if  $g$  is statistically sequentially quotient.*

**PROPOSITION 2.2.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be any two maps. Then the following hold.*

- (a) *If  $f$  and  $g$  are statistically sequentially quotient, then  $f \circ g$  is statistically sequentially quotient.*
- (b) *If  $f \circ g$  is statistically sequentially quotient, then  $g$  is statistically sequentially quotient.*

**PROPOSITION 2.3.** *The following hold:*

- (a) *Finite product of statistically sequentially quotient map is statistically sequentially quotient.*
- (b) *Statistically sequentially quotient maps are hereditarily statistically sequentially quotient maps.*

*Proof.*

- (a) Let  $\prod_{i=1}^{\mathcal{N}} f_i : \prod_{i=1}^{\mathcal{N}} X_i \rightarrow \prod_{i=1}^{\mathcal{N}} Y_i$  be a map where each  $f_i : X_i \rightarrow Y_i$  is statistically sequentially quotient map for  $i = 1, 2, 3, \dots, \mathcal{N}$ . Let  $\{(y_{i,n})\}_{n \in \mathbb{N}}$  be a sequence converges to  $(y_i)$  in  $\prod_{i=1}^{\mathcal{N}} Y_i$ . Then each  $\{y_{i,n}\}$  is a sequence converges to  $y_i$  in  $Y_i$ . Since each  $f_i$  is a statistically sequentially quotient map, there exists a sequence  $\{x_{i,k}\}$  converges to  $x_i$  such that  $f_i(x_{i,k}) = y_{i,n_k}$ . Take  $(x_i) \in \prod_{i=1}^{\mathcal{N}} X_i$ . Then  $\{(x_{i,k})\}$  converges to  $(x_i)$ . And for each  $i = 1, 2, 3, \dots, \mathcal{N}$ ,  $\mathcal{N}_i = \{n_k \in \mathbb{N} / x_{i,k} \in f_i^{-1}(y_{i,n_k})\}$  is statistically dense in  $\mathbb{N}$ . By Remark 1.1 (3) in [23],  $\mathcal{N}' = \cap \mathcal{N}_i$  is statistically dense in  $\mathbb{N}$ . That implies a sequence  $\{(x_{i,k})\}_{n \in \mathbb{N}}$  is converges to  $(x_i)$  and  $f((x_{i,k}))$  is statistically dense in  $(y_{i,n})$ . Therefore,  $\prod_{i=1}^{\mathcal{N}} f_i$  is a statistically sequentially quotient map.

- (b) Let  $f : X \rightarrow Y$  be a statistically sequentially quotient map and  $H$  be a subspace of  $Y$ . Take  $g = f|_{f^{-1}(H)}$  such that  $g : f^{-1}(H) \rightarrow H$  be a map.

Given a sequence  $\{y_n\}$  converging to  $y$  in  $H$ , there exists a sequence  $x_k \in f^{-1}(y_{n_k}) \in f^{-1}(H)$  such that  $(x_k)$  converges to  $x \in f^{-1}(y) \in f^{-1}(H)$ , since  $f$  is statistically sequentially quotient map and  $\{y_n\}$  converges to  $y$  in  $Y$ . Therefore,  $g$  is a statistically sequentially quotient map.

□

We observe the following implications.

sequence covering map  $\implies$  statistically sequentially quotient map  $\implies$  sequentially quotient map.

But none of the reverse implications need not be true as shown by the following examples.

EXAMPLE 2.4. Let  $X = (\bigoplus_{\alpha \in I} S_\alpha) \bigoplus I$  where  $I$  is the closed unit interval with usual topology and  $S_\alpha = \{x_{\alpha,n}/n \in \mathbb{N}\} \cup \{x_\alpha\}$  with the topology defined as follows:

- (i) Each point  $x_{\alpha,n}$  is open
- (ii) Open set containing  $x_\alpha$  is of the form  $\{x_{\alpha,n}/n \geq n_0\} \cup \{x_\alpha\}$  for some  $n_0 \in \mathbb{N}$ ,

and  $Y$  be the space obtained from  $X$  by identifying the limit point of  $S_\alpha$  with  $\alpha$ . Take  $Y$  as a quotient topology that is open sets of  $Y$  are as follows:

- (i) Each point  $x_{\alpha,n}$  is open
- (ii) open set  $U$  containing  $\alpha$  is of the form  $\{x_{\alpha,n}/n \geq n_0\} \cup \{x_\alpha\} \cup U'$  where  $U'$  is open neighborhood of  $\alpha$  in  $I$  and  $n_0 \in \mathbb{N}$ .

Let  $f : X \rightarrow Y$  be the map defined by

$$f(x) = \begin{cases} x & \text{if } x = x_{\alpha,n} \in S_\alpha \\ \alpha & \text{if } x = x_\alpha \in S_\alpha \\ \alpha & \text{if } x = \alpha \in I \end{cases}$$

- (a)  $f$  is sequentially quotient

Let  $S$  be a non-trivial convergent sequence in  $Y$  with its limit  $y$ . Clearly  $y \in I$ . Take  $S \cap S_y = S'$  and  $S \cap I = S''$ . Either  $S'$  or  $S''$  must be infinite, since  $S$  is a non-trivial convergent sequence. Then the infinite sequence  $S'$  or  $S''$  is a convergent sequence in  $X$

with its image is a subsequence of  $S$ . Therefore,  $f$  is sequentially quotient map.

(b)  $f$  is not statistically sequentially quotient

Let  $\{\alpha_n\}$  be a sequence in  $I$  converging to  $\alpha \in I$ . Define the sequence  $S = \{y_n\}$  in  $Y$  in the following way:

$$y_n = \begin{cases} x_{\alpha,n} & \text{if } n \text{ is even} \\ \alpha_n & \text{if } n \text{ is odd} \end{cases}.$$

Then  $\{y_n\}$  converges to  $\alpha \in Y$ . Consider  $S \cap S_\alpha$  and  $S \cap I$  in  $X$ . We have  $d(\{n/y_n \in S \cap S_\alpha\}) = \frac{1}{2}$ ,  $d(\{n/y_n \in S \cap I\}) = \frac{1}{2}$  and  $S \cap S_\alpha$  converges to  $x_\alpha$ ,  $S \cap I$  converges to  $\alpha$ . Since  $X$  is Hausdorff, we conclude that there is no sequence  $S'$  in  $X$  such that  $f(S')$  is statistically dense subsequence of  $S$ . Therefore,  $f$  is not statistically sequentially quotient.

EXAMPLE 2.5. Let  $\wedge = \{K/d(K) = 1, K \text{ is a subsequence of } \mathbb{N} \text{ obtained by deleting infinitely many elements}\}$  and  $S_\alpha = \{x_{\alpha,n}/n \in \mathbb{N}\} \cup \{x_\alpha\}$  be a topological space as defined in Example 2.4, where  $\alpha \in \wedge$ . Let  $X = \bigoplus_{\alpha \in \wedge} S_\alpha$  and  $Y = \{y_n/n \in \mathbb{N}\} \cup \{y\}$ . Then  $Y$  be a topological space as defined in  $S_\alpha$  and  $f : X \rightarrow Y$  defined by  $f(x_{\alpha,n}) = \alpha(n)$  and  $f(x_\alpha) = y$  where  $\alpha(n)$  is an  $n^{\text{th}}$  element in the sequence  $\alpha$ .

(1)  $f$  is a statistically sequentially quotient

Let  $S$  be a non-trivial convergent sequences in  $Y$  with its limit  $y$ . If  $S$  is a statistically dense subsequence of  $Y$ , then there is an element  $\alpha \in \wedge$  such that  $f(S_\alpha)$  is a statistically dense subsequence of  $S$ . If  $S$  is a non-statistically dense subsequence of  $Y$ , then there is an element  $\alpha \in \wedge$  such that  $f(S_\alpha) \cap S = S$  that is  $S' = f^{-1}(f(S_\alpha) \cap S) \cap S_\alpha$  is a convergent sequence in  $X$  whose image is  $S$ . Therefore,  $f$  is a statistically sequentially quotient map.

(2)  $f$  is not a sequence covering map

Since corresponding to the sequence  $\{y_n\}$ , there are no sequence in  $X$  whose image is  $\{y_n\}$ . Therefore,  $f$  is not a sequence covering map.

convergent sequence with its limit  $x_\alpha$ . Then for each  $\alpha \in \wedge$ ,  $S_\alpha = \{x_{\alpha,i}, x_\alpha \mid i \in \alpha\}$ . Let  $X$  be a disjoint union of  $S_\alpha$  and  $Y$  be a convergent sequence  $\{y_n\}$  with its limit  $y$ . Then  $f : X \rightarrow Y$  defined by  $f(x_{\alpha,i}) = y_i$  and  $f(x_\alpha) = y$  is a statistically sequentially quotient map but not a sequence covering map,

### 3. Proof of Theorem 1

The proof of Theorem 1 will be divided into Theorems 3.1, 3.2, 3.4 and Lemma 3.3. Let  $\Omega$  be the set of all topological spaces such that each compact subset  $K \subset X$  is metrizable and has a countable neighborhood base in  $X$ . In fact, Michael and Nagami in [17] has proved that  $X \in \Omega$  if and only if  $X$  is the image of some metric space under an open and compact-covering map. It is easy to see that if a space  $X$  is developable or has a point-countable base, then  $X \in \Omega$ . (see [1] and [21], respectively)

**THEOREM 3.1.** *Let  $f : X \rightarrow Y$  be a statistically sequentially quotient and boundary compact map, where  $Y$  is sn  $f$ -countable. For each non-isolated point  $y \in Y$ , there exists a point  $x_y \in \partial f^{-1}(y)$  such that whenever  $U$  is an open subset with  $x_y \in U$ , there exists a  $P \in \mathcal{P}_y$  satisfying  $P \subset f(U)$ .*

*Proof.* Suppose not, that is there exists a non-isolated point  $y \in Y$  such that for every point  $x \in \partial f^{-1}(y)$ , there is an open neighborhood  $U_x$  of  $x$  such that  $P \not\subseteq f(U_x)$  for every  $P \in \mathcal{P}_y$ . Then  $\partial f^{-1}(y) \subset \cup \{U_x \mid x \in \partial f^{-1}(y)\}$ . Since  $\partial f^{-1}(y)$  is compact, there exists a finite subfamily  $\mathcal{U} \subset \{U_x \mid x \in \partial f^{-1}(y)\}$  such that  $\partial f^{-1}(y) \subset \cup \mathcal{U}$ . We denote  $\mathcal{U}$  by  $\{U_i \mid 1 \leq i \leq n_0\}$ . Assume that  $\mathcal{P}_y = \{P_n \mid n \in \mathbb{N}\}$  and  $\mathcal{W}_y = \{F_n = \bigcap_{i=1}^n P_i \mid n \in \mathbb{N}\}$ . It is obvious that  $\mathcal{W}_y \subset \mathcal{P}_y$  and  $F_{n+1} \subset F_n$  for every  $n \in \mathbb{N}$ . For each  $1 \leq m \leq n_0, n \in \mathbb{N}$ , it follows that there exists  $x_{n,m} \in F_n \setminus f(U_m)$ . Denote  $y_k = x_{n,m}$  where  $k = (n-1)n_0 + m$ . Since  $\mathcal{P}_y$  is a network at a point  $y$  and  $F_{n+1} \subset F_n$  for every  $n \in \mathbb{N}$ ,  $\{y_n\}$  is a sequence converging to  $y$  in  $Y$ . Since  $f$  is a statistically sequentially quotient map,  $\{y_{n_k}\}$  is an image of some sequence  $\{x_k\}$  converging to  $x \in \partial f^{-1}(y)$  in  $X$ . From  $x \in \partial f^{-1}(y) \subset \cup \mathcal{U}$  it follows that there exists  $1 \leq m_0 \leq n_0$  such that  $x \in U_{m_0}$ . Therefore,  $\{x\} \cup \{x_k \mid k \geq k_0\} \subseteq U_{m_0}$  for some  $k_0 \in \mathbb{N}$ . Hence  $\{y\} \cup \{y_{n_k} \mid n_k \geq k_0\} \subset f(U_{m_0})$ . However, we can choose an  $n > k_0$  such that  $n_k = (n-1)n_0 + m_0 \geq k_0$ ,  $y_{n_k} = x_{n,m_0}$  which implies that  $x_{n,m_0} \in f(U_{m_0})$ . Suppose there is no  $n > k_0$  such that  $n_k = (n-1)n_0 + m_0 \geq k_0$ ,  $y_{n_k} \in f(U_{m_0})$ . That is, for all  $n > k_0$ ,  $n_k = (n-1)n_0 + m_0 \geq k_0$  such that  $y_{n_k} \notin f(U_{m_0})$ . This implies  $\{n_k \in \mathbb{N} \mid n \geq n_k, n_k = (n'-1)n_0 + m_0, n' > k_0\} \subset K_n$  where  $k_0 = q_1 n_0 + r_1$ ,  $n = q_1 n_0 + r_1$  and  $K = \{n/y_n \notin f(U_{m_0})\}$ . Now  $|K_n| > q_1 - q$  implies  $d(K) = \lim_{n \rightarrow \infty} \frac{|K_n|}{n} > \lim_{q_1 \rightarrow \infty} \frac{q_1 - q}{q_1 n_0 + r_1} = \frac{1}{n_0}$  which is

a contradiction to  $d(K) = 0$ . Therefore, there exists  $n > k_0$  such that  $n_k = (n - 1)n_0 + m_0 \geq k_0$ ,  $y_{n_k} = x_{n,m_0}$  and  $x_{n,m_0} \in f(U_{m_0})$  which contradicts  $x_{n,m_0} \in F_n \setminus f(U_{m_0})$ .  $\square$

**THEOREM 3.2.** *Let  $f : X \rightarrow Y$  be a statistically sequentially quotient and boundary-compact map, where  $X$  is first countable. Then  $Y$  is  $snf$ -countable if and only if  $f$  is an 1-sequence-covering map.*

*Proof.* Necessity. Let  $y$  be a non-isolated point in  $Y$ . Since  $Y$  is  $snf$ -countable, by Theorem 3.1, there exists a point  $x_y \in \partial f^{-1}(y)$  such that whenever  $U$  is an open neighborhood of  $x_y$ , there is a  $P \in \mathcal{P}_y$  satisfying  $P \subset f(U)$ . Let  $\{B_n \mid n \in \mathbb{N}\}$  be a countable neighborhood base at  $x_y$  such that  $B_{n+1} \subset B_n$  for each  $n \in \mathbb{N}$ . Now for each  $B_n$ , there exists a  $P_n \in \mathcal{P}_y$  such that  $P_n \subset f(B_n)$  for each  $n \in \mathbb{N}$  which implies that every  $f(B_n)$  is a sequential neighborhood of  $y$  in  $Y$ , since every  $P \in \mathcal{P}_y$  is a sequential neighborhood of  $y$ .

Suppose that  $\{y_n\}$  is a sequence in  $Y$  which converges to  $y$ . Then for each  $n \in \mathbb{N}$ , there is an  $i_n \in \mathbb{N}$  such that  $y_i \in f(B_n)$  for every  $i \geq i_n$ . Let it  $1 < i_n < i_{n+1}$  for every  $n \in \mathbb{N}$ . Take

$$x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1 \\ f^{-1}(y_j) \cap B_n, & \text{if } i_n \leq j < i_{n+1} \end{cases}$$

Denote  $S = \{x_j \mid j \in \mathbb{N}\}$ . It is easy to see that  $S$  converges to  $x_y$  in  $X$  and  $f(S) = \{y_n\}$ . Therefore,  $f$  is an 1-sequence-covering map. Converse part is easy to see.  $\square$

**LEMMA 3.3.** *Let  $f : X \rightarrow Y$  be a sequentially quotient and boundary-compact map. If  $X \in \Omega$ , then  $Y$  is  $snf$ -countable*

*Proof.* Let  $y$  be a non-isolated point for  $Y$ . Since  $X \in \Omega$  and  $\partial f^{-1}(y)$  is non-empty and compact for  $X$ , there exist a countable external base  $\mathcal{U}$  for  $\partial f^{-1}(y)$  in  $X$ . Let it be  $\mathcal{V} = \{\cup \mathcal{F} \mid \text{there is a finite subfamily } \mathcal{F} \subset \mathcal{U} \text{ with } \partial f^{-1}(y) \subset \cup \mathcal{F}\}$ . Obviously,  $\mathcal{V}$  is countable. We now prove that  $f(\mathcal{V})$  is a countable  $sn$ -network at point  $y$ .

(1)  $f(\mathcal{V})$  is a network at  $y$ .

Let  $y \in U$ . Obviously,  $\partial f^{-1}(y) \subset f^{-1}(U)$ . For each  $x \in \partial f^{-1}(y)$ , there exists an  $U_x \in \mathcal{U}$  such that  $x \in U_x \subset f^{-1}(U)$ . Therefore,  $\partial f^{-1}(y) \subset \cup \{U_x \mid x \in \partial f^{-1}(y)\}$ . Since  $\partial f^{-1}(y)$  is compact, it follows that there exists a finite subfamily  $\mathcal{F} \subset \{U_x \mid x \in \partial f^{-1}(y)\}$  such that  $\partial f^{-1}(y) \subset \cup \mathcal{F} \subset f^{-1}(U)$ . It is easy to see that  $\cup \mathcal{F} \in \mathcal{V}$  and  $y \in f(\cup \mathcal{F}) \subset U$ .



(2) For any  $P_1, P_2 \in f(\mathcal{V})$ , there exists a  $P_3 \in f(\mathcal{V})$  such that  $P_3 \subset P_1 \cap P_2$ .

It is obvious that there exist  $V_1, V_2$  in  $\mathcal{V}$  such that  $f(V_1) = P_1, f(V_2) = P_2$ , respectively. Since  $\partial f^{-1}(y) \subset V_1 \cap V_2$ , it follows from the similar proof of (1) that there exist a  $V_3 \in \mathcal{V}$  such that  $\partial f^{-1}(y) \subset V_3 \subset V_1 \cap V_2$ . Let  $P_3 = f(V_3)$ . Thus,  $P_3 \subset f(V_1 \cap V_2) \subset f(V_1) \cap f(V_2) = P_1 \cap P_2$ .

(3) For each  $P \in f(\mathcal{V})$ ,  $P$  is a sequential neighborhood of  $y$ .

Let  $\{y_n\}$  be any sequence in  $Y$  converges to  $y$  in  $Y$ . Since  $f$  is a sequentially quotient map, there exist a sequence  $\{x_k\}$  converging to  $x \in \partial f^{-1}(y) \subset X$  where  $x_k \in f^{-1}(y_{n_k})$ . Since  $P \in f(\mathcal{V})$ , there exists a  $V \in \mathcal{V}$  such that  $P = f(V)$ . Therefore,  $\{x_n\}$  is eventually in  $V$ , and this implies that  $\{y_n\}$  is eventually in  $P$ . Suppose not, there exists a subsequence  $\{y'_n\}$  such that  $y'_n \notin P$  and it converges to  $y$ . Then there exists a sequence  $\{x'_n\}$  converges to  $x' \in \partial f^{-1}(y)$  and it's image is a subsequence of  $\{y'_n\}$ . Since  $x' \in \partial f^{-1}(y) \subset V$ ,  $y'_n$  is frequently in  $P$ , which is a contradiction. Therefore,  $f(\mathcal{V})$  is a countable  $sn$ -network at a point  $y$ .  $\square$

**THEOREM 3.4.** *Let  $f : X \rightarrow Y$  be a statistically sequentially quotient and boundary-compact map. If  $X \in \Omega$ , then  $f$  is an 1-sequence-covering map.*

*Proof.* From Lemma 3.3, it follows that  $Y$  is  $snf$ -countable. Therefore,  $f$  is an 1-sequence covering map, since  $\partial f^{-1}(y)$  is compact subset of  $X \in \Omega$ , by Theorem 3.2.  $\square$

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