WEAK HOPF ALGEBRAS CORRESPONDING TO NON-STANDARD QUANTUM GROUPS

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Abstract. We construct a weak Hopf algebra \( \mathfrak{m} X_q(A_1) \) corresponding to non-standard quantum group \( X_q(A_1) \). The PBW basis of \( \mathfrak{m} X_q(A_1) \) is described and all the highest weight modules of \( \mathfrak{m} X_q(A_1) \) are classified. Finally we give the Clebsch-Gordan decomposition of the tensor product of two highest weight modules of \( \mathfrak{m} X_q(A_1) \).

Introduction

In this paper, we always assume that the base closed field is \( \mathbb{F} \) with characteristic 0. All algebras, modules are over the field \( \mathbb{F} \). The parameter \( q \in \mathbb{F} \) is non-zero and not a root of unity.

Quantum groups play an important role in mathematics and physics. A new quantum group was constructed in [2] solving exotic solution of quantum Yang-Baxter equation. This new quantum group is called the non-standard quantum group. Jing et al. [4] derived a new quantum group \( X_q(2) \) by employing the FRT method. All finite dimensional irreducible representations of \( X_q(2) \) were classified. It is noted that dimensions of the irreducible representations are only one or two. In 1993, Aghamohammadi et al. (see [1]) used the method of FRT to obtain the non-standard quantum group \( X_q(A_{n-1}) \) corresponding to type \( A_{n-1} \). Note that \( X_q(A_1) \) is just quantum algebra \( X_q(2) \). It is shown that this kind of quantum group has a Hopf algebra structure (see [3, 5]). On the other hand, Li defined a kind of weak Hopf algebra on a bialgebra with a weak antipode in [6] and many interesting results are obtained. Yang constructed weak Hopf algebras corresponding to Cartan matrices in [9] and gave their PBW bases. It is noted that finite dimensional integrable representations of \( \mathfrak{ms} l_q(2) \) were described and the decomposition of the tensor product of two finite dimensional integrable modules were considered in [10].
In this paper, we intend to study the weak Hopf algebra structure corresponding to the non-standard quantum group $X_q(A_1)$. By definition, $X_q(A_1)$ is the associative algebra over the field $\mathbb{F}$ with 1 generated by six generators $K_1^\pm, K_2^\pm, E, F$ with the following relations

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad i = 1, 2, \quad K_1 K_2 = K_2 K_1,
\]

\[
K_1 E = q_1^{-1} EK_1, \quad K_1 F = q_1 FK_1,
\]

\[
K_2 E = q_2 EK_2, \quad K_2 F = q_2^{-1} FK_2,
\]

\[
E^2 = F^2 = 0,
\]

\[
EF - FE = \frac{K_2 K_1^{-1} - K_1 K_2^{-1}}{q - q^{-1}},
\]

where $q_1 = q$ and $q_2 = -q^{-1}$.

First we add a central generator $J$ and weaken the group-likes to get an algebra $\mathfrak{m} X_q(A_1)$. It is verified that $\mathfrak{m} X_q(A_1)$ is a weak Hopf algebra but not a Hopf algebra. Then the PBW basis of $\mathfrak{m} X_q(A_1)$ is given in the similar way as [9]. We also give the sufficient and necessary conditions of the isomorphism between $\mathfrak{m} X_q(A_1)$ and $\mathfrak{m} X_p(A_1)$ as weak Hopf algebras. By applying the idea in [10] and some well-known facts, we can construct all highest weight representations of $\mathfrak{m} X_q(A_1)$ and the Clebsch-Gordan decomposition of $\mathfrak{m} X_q(A_1)$-modules. It is indicated that the indecomposable modules of $\mathfrak{m} X_q(A_1)$ are not necessarily irreducible. These results for $\mathfrak{m} X_q(A_1)$ are not the same as those in [10]. In fact they just extend the results in [4].

The paper is arranged as follows. In Section 1, we introduce some notions and define the algebra $\mathfrak{m} X_q(A_1)$, then we prove that $\mathfrak{m} X_q(A_1)$ is a weak Hopf algebra. In Section 2, We investigate the PBW basis of $\mathfrak{m} X_q(A_1)$. In Section 3, we describe the conditions of the weak Hopf isomorphisms between $\mathfrak{m} X_q(A_1)$ and $\mathfrak{m} X_p(A_1)$. In Section 4, we classify all the highest weight modules of $\mathfrak{m} X_q(A_1)$. Then in Section 5, we give the Clebsch-Gordan decomposition of tensor product of two highest weight modules of $\mathfrak{m} X_q(A_1)$.

1. Preliminaries

In this section, we construct the weak Hopf algebra $\mathfrak{m} X_q(A_1)$ by weaken $K_i$ of $X_q(A_1)$ and the defining relation $K_i K_i^{-1} = K_i^{-1} K_i = 1$ ($i = 1, 2$). Firstly, we replace $\{K_i, K_i^{-1} | i = 1, 2\}$ by $\{\overline{K}_i, \overline{K}_i^{-1} | i = 1, 2\}$ and introduce the new generator $J$ such that

\[
K_i \overline{K}_i = \overline{K}_i K_i = J \quad (i = 1, 2).
\]

Secondly, we give the following the definition.

**Definition 1.1** (see [9]). If $E$ satisfies

\[
K_1 E = q_1^{-1} EK_1, \quad K_2 E = q_2 EK_2 \quad \text{and} \quad \overline{K}_1 E = q_1 \overline{E} \overline{K}_1, \quad \overline{K}_2 E = q_2^{-1} \overline{E} \overline{K}_2,
\]
we say that $E$ is of type I. If $E$ satisfies
\[ K_1 E \bar{K}_1 = q_1^{-1} E, \quad K_2 E \bar{K}_2 = q_2 E, \]
we say that $E$ is of type II.

Similarly, we can define $F$ is of type I (type II). That is, if $F$ satisfies
\[ K_1 F = q_1 FK_1, \quad K_2 F = q_2^{-1} FK_2, \]
we say that $F$ is of type I. If $F$ satisfies
\[ K_1 \bar{F}_1 = q_1^{-1} \bar{F}_1, \quad K_2 \bar{F}_2 = q_2 \bar{F}_2, \]
we say that $F$ is of type II.

**Notation.** (See [9]) The notation $d = (k|\overline{k})$, $k, \overline{k} = 0$ or $1$ indicated that if $k = 1$ (resp. $0$), the corresponding generator $E$ is of type I (resp. type II), and if $\overline{k} = 1$ (resp. $0$), the corresponding generator $F$ is of type II (resp. type I). The information before $|$ is related to $E$. The information after $|$ is related to $F$. $E$ and $F$ are said to be of type $d$ if $E$ and $F$ are of type I or type II according to $d$.

Now, we can give the definition of the algebra $wX_q(A_1)$.

**Definition 1.2.** The algebra $wX_q(A_1)$ is defined as an associative algebra over the field $F$ with $1$ generated by $J, \overline{K}_i, K_i, K_1, K_2, E, F$ with the relations
\[ K_1 K_2 = K_2 K_1, \quad \overline{K}_1 \overline{K}_2 = \overline{K}_2 \overline{K}_1, \quad K_i \overline{K}_j = \overline{K}_j K_i, \quad i, j = 1, 2, \]
\[ K_i \overline{K}_i = J = K_i \overline{K}_i, \quad K_i J = K_i, \quad \overline{K}_i J = \overline{K}_i, \quad i = 1, 2, \]
\[ E \text{ and } F \text{ are of type } d, \]
\[ E^2 = F^2 = 0, \]
\[ EF - FE = \frac{K_2 \overline{K}_1 - K_1 \overline{K}_2}{q - q^{-1}}. \]

In this case, we say $wX_q(A_1)$ is of type $d$.

**Lemma 1.3.** In $wX_q(A_1)$ of type $d$, the following statements hold.

1. $J, 1 - J$ are idempotent elements.
2. $J$ is in the center of $wX_q(A_1)$.
3. If $E$ (resp. $F$) is of type II, then it enjoys type I.
4. $K_i^n E^m = q_1^{-mn} E^m K_i^n, \quad K_i^n F^m = q_1^{mn} F^m K_i^n,$
\[ K_2^n E^m = q_2^{-mn} E^m K_2^n, \quad K_2^n F^m = q_2^{mn} F^m K_2^n, \]
\[ \overline{K}_1^n E^m = q_1^{-mn} E^m \overline{K}_1^n, \quad \overline{K}_1^n F^m = q_1^{mn} F^m \overline{K}_1^n, \]
\[ \overline{K}_2^n E^m = q_2^{-mn} E^m \overline{K}_2^n, \quad \overline{K}_2^n F^m = q_2^{mn} F^m \overline{K}_2^n. \]
Proof. (1) Easy.
(2) By definition, we have
\[ K_i J = JK_i, \quad \overline{K_i} J = J \overline{K_i}. \]
If \( E \) is type I, then
\[ JE = \overline{K_1} K_1 E = q_1^{-1} \overline{K_1} E K_1 = q_1 q_1^{-1} E \overline{K_1} K_1 = E J. \]
If \( E \) is type II, then
\[ JE = K_1 \overline{K_1} E = q_1 K_1 \overline{K_1} K_1 E = q_1 K_1 E \overline{K_1} K_1 = E K_1 \overline{K_1} = E J. \]
It is similar to get \( J F = F J \). Therefore, \( J \) is in the center of \( W X_q(A_1) \).
(3) If \( E \) is type II, the relation \( K_1 E \overline{K_1} = q_1^{-1} E \) implies that \( K_1 E \overline{K_1} K_1 = q_1^{-1} E K_1 \). The left hand side is
\[ K_1 E J = K_1 JE = K_1 E. \]
Hence, we get \( K_1 E = q_1^{-1} E K_1 \). Similarly, \( K_2 E = q_2 E K_2 \).
For the generator \( F \), the statement is similar to prove.
(4) Straightforward. \( \Box \)

The concept of weak Hopf algebra was defined by [6], and was studied by [7, 9]. By definition a weak Hopf algebra \( W \) is a bialgebra with a weak antipode \( T \) such that
\[ T^* \text{Id}^* T = T \text{ and } \text{Id}^* T^* \text{Id} = \text{Id}, \]
where \( ^* \) is the multiplication of convolution algebra \( \text{Hom}_F(W, W) \).

In the following, we can equip a coalgebra structure with \( W X_q(A_1) \) such that \( W X_q(A_1) \) is a weak Hopf algebra. Indeed, we define the coalgebra structure in \( W X_q(A_1) \) as follows.

The comultiplication \( \Delta : W X_q(A_1) \rightarrow W X_q(A_1) \otimes W X_q(A_1) \) is
\[ \Delta(J) = J \otimes J, \quad \Delta(K_i) = K_i \otimes K_i, \quad \Delta(\overline{K_i}) = \overline{K_i} \otimes \overline{K_i}, \quad i = 1, 2; \]
\[ \Delta(E) = \begin{cases} (K_1 \overline{K_2}) \otimes E + E \otimes 1, & \text{if } E \text{ is of type I}, \\ (K_1 \overline{K_2}) \otimes E + E \otimes J, & \text{if } E \text{ is of type II}; \end{cases} \]
\[ \Delta(F) = \begin{cases} 1 \otimes F + F \otimes (K_2 \overline{K_1}), & \text{if } F \text{ is of type I}, \\ J \otimes F + F \otimes (K_2 \overline{K_1}), & \text{if } F \text{ is of type II}. \end{cases} \]

The counit \( \varepsilon : W X_q(A_1) \rightarrow F \) is
\[ \varepsilon(K_i) = \varepsilon(\overline{K_i}) = \varepsilon(J) = 1, \quad i = 1, 2; \]
\[ \varepsilon(E) = \varepsilon(F) = 0. \]

It is obvious that \( W X_q(A_1) \) is a coalgebra by the definition of \( \Delta \) and \( \varepsilon \). In fact:

**Theorem 1.4.** Keeping all notations as above. Then \( W X_q(A_1) \) is a weak Hopf algebra with \( J \neq 1 \), the comultiplication \( \Delta \), counit \( \varepsilon \) and weak antipode \( T \), but it is not a Hopf algebra.
Proof. Indeed, it is straightforward to see that \( \mathfrak{w}X_q(A_1) \) is a bialgebra (as the proof in [9, Theorem 3.1]). To see that \( \mathfrak{w}X_q(A_1) \) is a weak Hopf algebra, we need to find a weak antipode \( T \) such that \( T \ast \text{Id} \ast T = T \) and \( \text{Id} \ast T \ast \text{Id} = \text{Id} \).

For the purpose, we define \( T : \mathfrak{w}X_q(A_1) \rightarrow \mathfrak{w}X_q(A_1) \) by
\[
T(J) = J, \ T(K_i) = \overline{K}_i, \ T(\overline{K}_i) = K_i, \ i = 1, 2,
\]
\[
T(E) = -\overline{K}_1K_2E, \ T(F) = -FK_1\overline{K}_2.
\]

The left is to prove \( T \) is an weak antipode of \( \mathfrak{w}X_q(A_1) \). The proof is more or less the same as that in [9, Theorem 3.1].

We now prove that \( \mathfrak{w}X_q(A_1) \) is not a Hopf algebra. Otherwise, we assume that \( \mathfrak{w}X_q(A_1) \) is a Hopf algebra and \( S : \mathfrak{w}X_q(A_1) \rightarrow \mathfrak{w}X_q(A_1) \) is an antipode. Then \( (S \ast \text{id})(J) = w \varepsilon(J) = (\text{id} \ast S)(J) \) implies that \( S(J)J = 1 = JS(J) \).

It follows that \( J \) is invertible. However, \( J(1 - J) = 0 \) and \( J \neq 1 \). It is contradiction. Therefore, \( \mathfrak{w}X_q(A_1) \) is a weak Hopf algebra not a Hopf algebra.

\[\Box\]

2. The PBW basis of \( \mathfrak{w}X_q(A_1) \)

Let \( \omega_q = \mathfrak{w}X_q(A_1)J, \ \overline{\omega}_q = \mathfrak{w}X_q(A_1)(J - 1) \), we have:

**Proposition 2.1.** Assume that \( \mathfrak{w}X_q(A_1) \) is of type \( d \). Then \( \mathfrak{w}X_q(A_1) = \omega_q \oplus \overline{\omega}_q \) as algebras. Furthermore, \( \omega_q \) and \( X_q(A_1) \) are isomorphic as Hopf algebras.

**Proof.** It is easy to see that
\[
\mathfrak{w}X_q(A_1) = \omega_q \oplus \overline{\omega}_q
\]
as algebras for \( J \) is a center idempotent element. Consider the algebra \( \omega_q \), it can be viewed as an algebra generated by \( EJ, FJ, K_i, \overline{K}_i, \overline{K}_j, K_j \), satisfying the following relations:
\[
K_1K_2 = K_2K_1, \ \overline{K}_1\overline{K}_2 = \overline{K}_2\overline{K}_1, \ K_i\overline{K}_j = \overline{K}_jK_i, \ i, j = 1, 2,
\]
\[
K_1K_1 = J = K_2\overline{K}_2, \ K_iJ = JK_i = K_i, \ K_i\overline{J} = J\overline{K}_i = K_i, \ i = 1, 2,
\]
\[
K_1EJ = q_1^{-1}EJK_1, \ K_1FJ = q_1FJK_1,
\]
\[
K_2EJ = q_2EJK_2, \ K_2FJ = q_2^{-1}FJK_2,
\]
\[
\overline{K}_1EJ = q_1EJK_1, \ \overline{K}_1FJ = q_1^{-1}FJK_1,
\]
\[
\overline{K}_2EJ = q_2EJK_2, \ \overline{K}_2FJ = q_2FJK_2,
\]
\[
(EJ)^2 = (FJ)^2 = 0,
\]
\[
(EJ)(FJ) - (FJ)(EJ) = \frac{K_1\overline{K}_1 - K_1\overline{K}_2}{q - q^{-1}},
\]

where \( J \) is the identity of \( \omega_q \). By the comultiplication of \( \mathfrak{w}X_q(A_1) \), it is deduced in \( \mathfrak{w}X_q(A_1) \) that
\[
\Delta(K_i) = K_i \otimes K_i, \ \Delta(\overline{K}_i) = \overline{K}_i \otimes \overline{K}_i, \ i = 1, 2,
\]
\[
\Delta(EJ) = (K_1\overline{K}_2) \otimes EJ + EJ \otimes J,
\]
\[
\Delta(FJ) = J \otimes FJ + FJ \otimes (K_2\overline{K}_1),
\]
\[ \varepsilon(K_i) = \varepsilon(K_i^\epsilon) = \varepsilon(J) = 1, \ i = 1, 2, \varepsilon(EJ) = \varepsilon(FJ) = 0, \]
\[ T(J) = J, T(K_i) = K_i, \ T(K_i^\epsilon) = K_i, \ i = 1, 2, \]
\[ T(EJ) = -(K_iK_2(EJ)), \ T(FJ) = -(FJ)K_1K_2. \]

Let \( \rho : X_q(A_1) \to \omega_q \) be the map defined by
\[ \rho(K_i) = K_i, \ \rho(K_i^{-1}) = K_i, \ i = 1, 2, \ \rho(E') = EJ, \ \rho(F') = FJ, \]
where \( K_i, K_i^{-1} (i = 1, 2) \), \( E' \), and \( F' \) are the generators of \( X_q(A_1) \). It is straightforward to see that \( \rho \) is a well-defined surjective algebra homomorphism.

Let \( \phi : \mathfrak{m}X_q(A_1) \to X_q(A_1) \) be a map given by
\[ \phi(1) = 1, \ \phi(J) = 1, \ \phi(E) = E, \ \phi(F) = F, \ \phi(K_i) = K_i, \ \phi(K_i^\epsilon) = K_i^{-1}. \]

We can check that \( \phi \) is a well-defined algebra homomorphism. If we consider the restricted homomorphism \( \phi|_{\omega_q} \), then we have \( \phi|_{\omega_q} \circ \rho = \text{id}_{X_q(A_1)} \). Hence \( \rho \) is injective. Therefore, \( \omega_q \cong X_q(A_1) \).

It is noted that
\[ \mathfrak{m}X_q(A_1)/\langle J - 1 \rangle \cong X_q(A_1) \]
as Hopf algebras, where \( \langle J - 1 \rangle \) is the two-sided ideal generated by \( J - 1 \) (see the proof of Proposition 2.1).

Let us describe the structure of \( \mathfrak{m}q \).

- If \( E \) (resp. \( F \)) is of type II, then \( E(1 - J) = 0 \) (resp. \( F(1 - J) = 0 \)). Indeed, if \( E \) is of type II, then \( q_1^{-1}E = K_1E = K_1EJ = K_1EJ \) and \( E(1 - J) = 0 \). Similarly for \( F \).
- If \( E \) (resp. \( F \)) is of type I, then \( E(1 - J) \neq 0 \) (resp. \( F(1 - J) \neq 0 \)).

To see this, if \( E \) and \( F \) are of type \( d = (1|1) \), we apply the actions of \( E(1 - J) \) and \( F(1 - J) \) on the \( \mathfrak{m}X_q(A_1) \)-module \( M(1, 1) \) in Section 4, we have \( E(1 - J)X^{0}Y^{0} = X^{1}Y^{0} \neq 0 \) and \( F(1 - J)X^{0}Y^{0} = X^{0}Y^{1} \neq 0 \). Hence \( E(1 - J) \neq 0 \) and \( F(1 - J) \neq 0 \).

If \( E \) (resp. \( F \)) is of type I, we assume \( X = E(1 - J) \) (resp. \( Y = F(1 - J) \)).

There are the following four cases.

1. If \( d = (1 | 1) \), then \( \mathfrak{m}q = FX + FY + FYX + F(1 - J) \). It is easy to see that \( XY = YX \);
2. If \( d = (0 | 0) \), then \( \mathfrak{m}q = \mathbb{F}(1 - J) \);
3. If \( d = (1 | 0) \), then \( \mathfrak{m}q = \mathbb{F}X + F(1 - J) \);
4. If \( d = (0 | 1) \), then \( \mathfrak{m}q = \mathbb{F}Y + F(1 - J) \).

Let \( X_q^+(A_1) \) (resp. \( X_q^{-}(A_1) \), and \( X_q^0(A_1) \)) be the subalgebra generated by \( E \) (resp. \( F \), and \( K_i^{\pm 1}, K_i^\pm ) \). Considering the \( X_q^+(A_1) \)-module \( V \) with basis \( \{v_0, v_1\} \), defined by \( Ev_0 = 0, Ev_1 = v_0, 1v_i = v_i \ (i = 0, 1) \), accordingly we have \( \{1, E\} \) is a basis of \( X_q^+(A_1) \). Similarly, \( \{1, F\} \) is a basis of \( X_q^-(A_1) \). On the other hand, \( X_q^0(A_1) \cong \mathbb{F}[K_i^{\pm 1}, K_i^\pm ] \) as \( \mathbb{F} \)-algebras, where \( \mathbb{F}[K_i^{\pm 1}, K_i^\pm ] \) is
the algebra of Laurent polynomials. Hence, \( \{ K_1^m K_2^n \mid m, n \in \mathbb{Z} \} \) is a basis of \( X_q^0(A_1) \). Moreover, one has
\[
X_q(A_1) \cong X_q^-(A_1) \otimes X_q^0(A_1) \otimes X_q^+(A_1).
\]

To see these, one can refer to the statements of [3, Lemma 4.14–Theorem 4.21].

We set
\[
P_{s_i} = \begin{cases} 
K_i^{s_i}, & \text{if } s_i > 0, \\
J_i, & \text{if } s_i = 0, \\
K_i^{-s_i}, & \text{if } s_i < 0.
\end{cases}
\]

We denote \( P^* = P_1^{s_1} P_2^{s_2} \) if \( s = (s_1, s_2) \). It is easy to see \( P^* \) is the basis of \( \omega_q^0 \).

By Proposition 2.1, we have:

**Proposition 2.2.** Assume that \( \mathfrak{m} X_q(A_1) \) is of type \( d \). Then the set
\[
\{ F^b P^* E^a J \mid s = (s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}, \text{ and } a, b \in \mathbb{Z}_2 \} \cup \{ 0 \neq F^b E^a (1-J) \mid a, b \in \mathbb{Z}_2 \}
\]
forms a basis of \( \mathfrak{m} X_q(A_1) \).

3. The isomorphisms among weak quantum algebras

We assume that \( X_p(A_1) \) is generated by \( E', F', K'_1, K'_2^{-1} \), \( i = 1, 2 \). The defining relations and comultiplications of \( X_p(A_1) \) are the same as those of \( X_q(A_1) \) replaced \( q \) by \( p \).

In this section, we give the sufficient and necessary conditions as weak Hopf algebra isomorphisms between \( \mathfrak{m} X_q(A_1) \) and \( \mathfrak{m} X_p(A_1) \).

In first, we recall some concepts about group-like elements and primitive elements of a coalgebra.

Let \( C \) be a coalgebra, \( x \in C \). If \( \Delta(x) = x \otimes x \), and \( \epsilon(x) = 1 \), then \( x \) is called a group-like element in \( C \). Let \( G(C) \) denote the set of group-like elements. Let \( g, h \in G(C) \). If
\[
\Delta(x) = g \otimes x + x \otimes h,
\]
then \( x \) is called a \((g : h)\)-primitive element. Let \( P_{g,h}(C) \) denote the space consisting of \((g : h)\)-primitive elements.

**Lemma 3.1.** The space of \((K_1^{l_1} K_2^{l_2} : 1)\)-primitive elements of \( X_q(A_1) \) is
\[
P_{K_1^{l_1} K_2^{l_2},1}(X_q(A_1)) = \begin{cases} 
\mathbb{F} E + \mathbb{F} K_1 K_2^{-1} + \mathbb{F}(1 - K_1 K_2^{-1}), & \text{if } l_1 = 1, l_2 = -1, \\
\mathbb{F}(1 - K_1^{l_1} K_2^{l_2}), & \text{others.}
\end{cases}
\]

**Proof.** Assume that \( x \in X_q(A_1) \) is a \((K_1^{l_1} K_2^{l_2} : 1)\)-primitive element, then
\[
\Delta(x) = K_1^{l_1} K_2^{l_2} \otimes x + x \otimes 1.
\]

We suppose that
\[
x = \sum_{i,j \in \mathbb{Z}_2, m_1, m_2} a_{i,j,m_1,m_2} E^i F^j K_1^{m_1} K_2^{m_2},
\]
we have
\[
\Delta(x) = \Delta \left( \sum_{i,j,m_1,m_2} a_{i,j,m_1,m_2} E^i F^j K_1^{m_1} K_2^{m_2} \right)
\]
\[
= \sum_{m_1,m_2} a_{0,0,m_1,m_2} K_1^{m_1} K_2^{m_2} \otimes K_1^{m_1} K_2^{m_2}
\]
\[
+ \sum_{m_1,m_2} a_{1,0,m_1,m_2} \left( K_1^{m_1+1} K_2^{m_2-1} \otimes E K_1^{m_1} K_2^{m_2} + E K_1^{m_1} K_2^{m_2} \otimes K_1^{m_1} K_2^{m_2} \right)
\]
\[
+ \sum_{m_1,m_2} a_{0,1,m_1,m_2} \left( K_1^{m_1} K_2^{m_2} \otimes F K_1^{m_1} K_2^{m_2} + F K_1^{m_1} K_2^{m_2} \otimes K_1^{m_1} K_2^{m_2} \right)
\]
\[
+ \sum_{m_1,m_2} a_{1,1,m_1,m_2} \left( K_1^{m_1+1} K_2^{m_2-1} \otimes E F K_1^{m_1} K_2^{m_2}
\right.
\]
\[
\left. + K_1^{m_1} K_2^{m_2} \otimes E F K_1^{m_1} K_2^{m_2} \right)
\]
\[
+ E K_1^{m_1} K_2^{m_2} \otimes F K_1^{m_1} K_2^{m_2} + E F K_1^{m_1} K_2^{m_2} \otimes K_1^{m_1} K_2^{m_2-1} \right)
\]
(3.1)

On the other hand
\[
K_1^{l_1} K_2^{l_2} \otimes x + x \otimes 1 = K_1^{l_1} K_2^{l_2} \otimes \sum_{m_1,m_2} a_{0,0,m_1,m_2} K_1^{m_1} K_2^{m_2}
\]
\[
+ K_1^{l_1} K_2^{l_2} \otimes \sum_{m_1,m_2} a_{1,0,m_1,m_2} E K_1^{m_1} K_2^{m_2}
\]
\[
+ K_1^{l_1} K_2^{l_2} \otimes \sum_{m_1,m_2} a_{0,1,m_1,m_2} F K_1^{m_1} K_2^{m_2}
\]
\[
+ K_1^{l_1} K_2^{l_2} \otimes \sum_{m_1,m_2} a_{1,1,m_1,m_2} E F K_1^{m_1} K_2^{m_2}
\]
\[
+ \sum_{m_1,m_2} a_{0,0,m_1,m_2} K_1^{m_1} K_2^{m_2} \otimes 1
\]
\[
+ \sum_{m_1,m_2} a_{1,0,m_1,m_2} E K_1^{m_1} K_2^{m_2} \otimes 1
\]
\[
+ \sum_{m_1,m_2} a_{0,1,m_1,m_2} F K_1^{m_1} K_2^{m_2} \otimes 1
\]
\[
+ \sum_{m_1,m_2} a_{1,1,m_1,m_2} E F K_1^{m_1} K_2^{m_2} \otimes 1.
\]
(3.2)

Comparing the equations (3.1) and (3.2), we have if \( l_1 = 1 \) and \( l_2 = -1 \), then \( x \) can be written as
\[
aE + bFK_1 K_2^{-1} + c(1 - K_1 K_2^{-1}) \), \( a, b, c \in \mathbb{F}.
\]

If \( l_1 \neq 1 \) or \( l_2 \neq -1 \), then \( x \) can be written as
\[
x = d(1 - K_1^{l_1} K_2^{l_2}), \ d \in \mathbb{F}.
\]

Therefore, we finish the proof. \( \square \)

We now give the first main result.

**Proposition 3.2.** \( X_p(A_1) \cong X_q(A_1) \) as Hopf algebras if and only if \( p = \pm q^{-1} \).
Proof. \( \Rightarrow \) Let \( \phi : X_p(A_1) \rightarrow X_q(A_1) \) be a Hopf algebra isomorphism. Then \( \phi \) must map group-like elements to group-like elements. Therefore we can assume that
\[
\phi(K_1') = K_1^{m_1}K_2^{m_2}, \quad \phi(K_2') = K_1^{n_1}K_2^{n_2}.
\]
Then we have
\[
\Delta(\phi(E')) = (\phi \otimes \phi)(\Delta(E')) = \phi(K_1'K_2'^{-1}) \otimes \phi(E') + \phi(E') \otimes 1
\]
\[
= K_1^{m_1-n_1}K_2^{m_2-n_2} \otimes \phi(E') + \phi(E') \otimes 1.
\]
So \( \phi(E') \) is a \((K_1^{m_1-n_1}K_2^{m_2-n_2} : 1)\)-primitive element. By Lemma 3.1, if \( m_1 - n_1 \neq 1 \), or \( m_2 - n_2 \neq -1 \), we can assume \( \phi(E') = d(1 - K_1^{m_1-n_1}K_2^{m_2-n_2}) \neq 0 \). This contradicts to the fact that \( \phi(K_1')\phi(E') = p^{-1}\phi(E')\phi(K_1') \).

Now, we focus on \( m_1 - n_1 = 1, m_2 - n_2 = -1 \). By Lemma 3.1, we can assume that
\[
\phi(E') = aE + bFK_1K_2^{-1} + c(1 - K_1K_2^{-1}).
\]
Applying the algebra isomorphism \( \phi \) to the relation \( K_1'E' = p^{-1}E'K_1' \), we get
\[
\phi(K_1')\phi(E') = K_1^{m_1}K_2^{m_2}(aE + bFK_1K_2^{-1} + c(1 - K_1K_2^{-1}))
\]
\[
= aK_1^{m_1}K_2^{m_2}E + bK_1^{m_1}K_2^{m_2}FK_1K_2^{-1}
\]
\[
+ cK_1^{m_1}K_2^{m_2}(1 - K_1K_2^{-1})
\]
\[
= (-1)^{-m_2}aq^{-m_1-m_2}EK_1^{m_1}K_2^{m_2}
\]
\[
+ (-1)^{m_2}bq^{m_1+m_2}FK_1^{m_1+1}K_2^{m_2-1}
\]
\[
+ cK_1^{m_1}K_2^{m_2}(1 - K_1K_2^{-1}),
\]
\[
p^{-1}\phi(E')\phi(K_1') = K_1^{m_1}K_2^{m_2}(aE + bFK_1K_2^{-1} + c(1 - K_1K_2^{-1}))
\]
\[
= p^{-1}(aE + bFK_1K_2^{-1} + c(1 - K_1K_2^{-1}))K_1^{m_1}K_2^{m_2}
\]
\[
= p^{-1}aEK_1^{m_1}K_2^{m_2} + p^{-1}bFK_1^{m_1+1}K_2^{m_2-1}
\]
\[
+ p^{-1}c(1 - K_1K_2^{-1})K_1^{m_1}K_2^{m_2}
\]
\[
\Longrightarrow \quad (-1)^{-m_2}aq^{-m_1-m_2} = p^{-1}a, \quad (-1)^{m_2}bq^{m_1+m_2} = p^{-1}b, \quad c = p^{-1}c.
\]
Hence \( c = 0 \) since \( p \) and \( q \) are not a root of unity.

(1) If \( a \neq 0 \), then
\[
(-1)^{m_2}q^{m_1+m_2} = p, \quad b = 0, \quad \phi(E') = aE.
\]
Let us determine \( \phi(E') \) as follows. Since \( F'^{K_1}K_2^{m_1-n_1} \) is a \((K_1^{m_1-n_1}K_2^{m_2-n_2} : 1)\)-primitive element, we can assume that
\[
\phi(F'^{K_1}K_2^{-1}) = a'E + b'FK_1K_2^{-1} + c'(1 - K_1K_2^{-1}) = \phi(F')K_1K_2^{-1}.
\]
This implies that
\[
\phi(E') = b'FK_1^{m_1-n_1}K_2^{-1} + c'FK_1K_2^{-1}.
\]
by the defining relations. Moreover, applying $\phi$ to the relation
\[ E' F' - F' E' = \frac{K_1^{l-1} K_2^{m+1} - K_1^{l} K_2^{m+1}}{p - p^{-1}}, \]
we get that
\[ b' = \frac{q - q^{-1}}{a(p - p^{-1})}, \text{ and that } \phi(F') = \frac{q - q^{-1}}{a(p - p^{-1})} F. \]
Therefore, we may assume that
\[ m_1 + m_2 = n_1 + n_2 = l, m_2 = m. \]
Then $(-1)^m q^l = p$, the corresponding isomorphism has the form
\[ \phi(K_1^a) = K_1^{1-m} K_2^{m}, \phi(K_2^a) = K_1^{1-m} K_2^{m+1}, \]
\[ \phi(E') = aE, \phi(F') = \frac{q - q^{-1}}{a(p - p^{-1})} F, (a \neq 0). \]
This isomorphism forces that there are $a, b \in \mathbb{Z}$ such that
\[ \phi(K_1^{a})\phi(K_2^{b}) = K_1 \text{ or } \phi(K_1^{a})\phi(K_2^{b}) = K_2. \]
It concludes that $a(l - m) + b(l - m - 1) = 1, am + b(m + 1) = 0$ or $a(l - m) + b(l - m - 1) = 0, am + b(m + 1) = 1$. For the first case, we have $l = 1, a = 1 + m, b = -m$, or $l = -1, a = -1 - m, b = m$. For the last case, we have $l = 1, a = m, b = 1 - m$, or $l = -1, a = -2 - m, b = m + 1$. Therefore $p = (-1)^m q^{l+1}$.
If $p = (-1)^m q$, then we get the weak Hopf algebra isomorphism
\[ \phi(K_1^a) = K_1^{1-m} K_2^{m}, \phi(K_2^a) = K_1^{1-m} K_2^{m+1}, \]
\[ \phi(E') = aE, \phi(F') = (-1)^m a^{-1} F, (a \neq 0). \]
The inverse $\phi'$ of $\phi$ is
\[ \phi'(K_1) = (K_1^{a})^{1+m} (K_2^{a})^{-m}, \phi'(K_2) = (K_1^{a})^{m} (K_2^{a})^{1-m}, \]
\[ \phi'(E) = a^{-1} E', \phi'(F') = (-1)^m a F'. \]
If $p = (-1)^m q^{-1}$, then we get the weak Hopf algebra isomorphism
\[ \phi(K_1^a) = K_1^{1-m} K_2^{m}, \phi(K_2^a) = K_1^{2-m} K_2^{m+1}, \]
\[ \phi(E') = aE, \phi(F') = (-1)^m a^{-1} F, (a \neq 0). \]
The inverse $\phi'$ of $\phi$ is
\[ \phi'(K_1) = (K_1^{a})^{-1-m} (K_2^{a})^{m}, \phi'(K_2) = (K_1^{a})^{-2-m} (K_2^{a})^{m+1}, \]
\[ \phi'(E) = a^{-1} E', \phi'(F') = (-1)^m a F'. \]
(2) If $b \neq 0$, then
\[ (-1)^{m_2} q^{m_1 + m_2} = p^{-1}, a = 0, \phi(E') = bFK_1 K_2^{-1}. \]
We assume that
\[ \phi(F'K'_1K'_2^{-1}) = a'E + b'FK_1K_2^{-1} + c'(1 - K_1K_2^{-1}). \]
By the defining relations and more or less than the above discussion, we have
\[ \phi(F') = a'EK_1^{-1}K_2. \]
In fact,
\[ a' = \frac{q - q^{-1}}{b(p - p^{-1})} \]
b by applying the isomorphism \( \phi \) to the relation
\[ E'aE' = \frac{K'_2K'_1^{-1} - K'_1K'_2^{-1}}{p - p^{-1}}. \]
Therefore, we have that in this case
\[ \phi(F') = \frac{q - q^{-1}}{b(p - p^{-1})} K_1^{-1}K_2 E. \]
Let \( m_1 + m_2 = l, m_2 = m \), then \( p = (-1)^m q^{-1} \), the corresponding isomorphism
\[ \phi(K'_1) = K_1^{-l-m}K_2^m, \quad \phi(K'_2) = K_1^{l-m-1}K_2^{m+1}, \]
\[ \phi(E') = bFK_1K_2^{-1}, \quad \phi(F') = \frac{q - q^{-1}}{b(p - p^{-1})} EK_1^{-1}K_2, (b \neq 0). \]
The similar arguments as the case (1) show that \( p = (-1)^m q^{\pm 1} \).
If \( p = (-1)^m q \), we get the weak Hopf algebra isomorphism
\[ \phi(K'_1) = K_1^{-1-m}K_2^m, \quad \phi(K'_2) = K_1^{-2-m}K_2^{m+1}, \]
\[ \phi(E') = bFK_1K_2^{-1}, \quad \phi(F') = (-1)^m b^{-1} EK_1^{-1}K_2, (b \neq 0). \]
The inverse \( \phi' \) of \( \phi \) is
\[ \phi'(K_1) = (K'_1)^{-1-m}(K'_2)^m, \quad \phi'(K_2) = (K'_1)^{-2-m}(K'_2)^{m+1}, \]
\[ \phi'(E) = (-1)^m bF'K'_1(K'_2)^{-1}, \quad \phi'(F) = b^{-1} E'(K'_1)^{-1}K'_2. \]
If \( p = (-1)^m q^{-1} \), then we get the weak Hopf algebra isomorphism
\[ \phi(K'_1) = K_1^{-1-m}K_2^m, \quad \phi(K'_2) = K_1^{-m-1}K_2^{m+1}, \]
\[ \phi(E') = bFK_1K_2^{-1}, \quad \phi(F') = (-1)^{m+1} b^{-1} EK_1^{-1}K_2, (b \neq 0). \]
The inverse \( \phi' \) of \( \phi \) is
\[ \phi'(K_1) = (K'_1)^{1+m}(K'_2)^{-m}, \quad \phi'(K_2) = (K'_1)^m(K'_2)^{1-m}, \]
\[ \phi'(E) = (-1)^{m+1} bF'K'_1(K'_2)^{-1}, \quad \phi'(F) = b^{-1} E'(K'_1)^{-1}K'_2. \]
\( (\Leftarrow) \) If \( p = \pm q^{\pm 1} \), we can assume that \( p = (-1)^m q^n (n = \pm 1) \) and define the map \( \psi : X_p(A_1) \rightarrow X_q(A_1) \) as
\[ \psi(K'_1) = K_1^{-n-m}K_2^m, \quad \psi(K'_2) = K_1^{-n-m-1}K_2^{m+1}, \]
\[ \psi(E') = aE, \quad \psi(F') = (-1)^{m+\delta_{n,1}} a^{-1} F. \]
where
\[
\delta_{-1,n} = \begin{cases} 
1, & \text{if } n = -1, \\
0, & \text{if } n \neq -1.
\end{cases}
\]

It is easy to see that \(\psi\) is a Hopf algebra isomorphism. \(\square\)

Recall that
\[
\mathfrak{m}X_q(A_1) \cong \omega_q \oplus \overline{\omega}_q.
\]

Let us consider the weak Hopf algebra isomorphism between \(\mathfrak{m}X_q(A_1)\) and \(\mathfrak{m}X_p(A_1)\).

**Theorem 3.3.** For the weak Hopf algebra \(\mathfrak{m}X_q(A_1)\) of type \((1|1)\), we have
\[
\mathfrak{m}X_p(A_1) \cong \mathfrak{m}X_q(A_1)
\]

as weak Hopf algebras if and only if
\[
p = \pm q^{\pm 1}.
\]

**Proof.** Let \(\gamma : \mathfrak{m}X_p(A_1) \rightarrow \mathfrak{m}X_q(A_1)\) be an isomorphism of weak Hopf algebra. It is easy to see that \(\gamma(J') = J\) since \(\gamma\) sends group-likes to group-likes.

By Proposition 2.1 it is well-known that
\[
\mathfrak{m}X_p(A_1) = w_p \oplus \overline{w}_p, \quad \mathfrak{m}X_q(A_1) = w_q \oplus \overline{w}_q,
\]

and \(w_p \cong X_p(A_1)\), \(w_q \cong X_q(A_1)\). Note that \(\overline{w}_p\) is spanned by \(\{E^iF^j(1 - J) \mid i, j = 0, 1\}\), and \(\overline{w}_q\) is spanned by \(\{E^iF^j(1 - J) \mid i, j = 0, 1\}\).

Assume that \(\text{inj}_p : w_p \rightarrow \mathfrak{m}X_p(A_1)\) is defined by
\[
J' \mapsto J', \quad E'J' \mapsto E'J', \quad F'J' \mapsto F'J', \quad K'_i \mapsto K'_i, \quad \overline{K}'_i \mapsto \overline{K}'_i, \quad i = 1, 2.
\]

It is easy to see that \(\text{inj}_p\) is a bialgebra homomorphism (see [8]). Moreover, we have \(w_q = \gamma \circ \text{inj}_p(w_p)\). Since \(\mathfrak{m}X_p(A_1) \cong \mathfrak{m}X_q(A_1)\), it follows that \(X_p(A_1) \cong X_q(A_1)\). By Proposition 3.2, \(p = \pm q^{\pm 1}\).

(\(\Rightarrow\)) Assume that \(p = \pm q^{\pm 1}\). Without loss of generality, we assume that \(p = (-1)^mq^n(n = \pm 1)\) and define the map \(\gamma : \mathfrak{m}X_p(A_1) \rightarrow \mathfrak{m}X_q(A_1)\) as follows
\[
\gamma(1) = 1, \quad \gamma(J') = J,
\]
\[
\gamma(P'_1) = P_1^{n-m}P_2^m, \quad \gamma(P'_2) = P_1^{n-m-1}P_2^{m+1},
\]
\[
\gamma(E') = E, \quad \gamma(F') = (-1)^{m+\delta_{1,n}}F,
\]

where \(P_i\) and \(P'_i\) are defined by (2.1) respectively. It is straightforward to see that \(\gamma\) indeed can be extended to a weak Hopf algebra isomorphism.

The proof is finished. \(\square\)

**Remark 3.4.** In general, if \(E,F\) are of type \((1|0), (0|1), \) or \((0|0), \) more or less the same arguments show that Theorem 3.3 also hold.
4. The representations of \( \mathfrak{w}X_q(A_1) \)

In this section, we consider the representation theory of \( \mathfrak{w}X_q(A_1) \) of type \( d \).

Let \( V \) be a \( \mathfrak{w}X_q(A_1) \)-module and \( 0 \neq v \in V \). If \( K_1v = \lambda_1v, K_2v = \lambda_2v \), then \( \lambda = (\lambda_1, \lambda_2) \) is called a weight of \( V \) and \( v \) is called a weight vector. The subspace

\[
\{0\} \neq V_\lambda = \{v \in V \mid K_1v = \lambda_1v, K_2v = \lambda_2v\}
\]

is called a weight space of \( \lambda = (\lambda_1, \lambda_2) \). If

\[
Ev = 0 = K_1v = \lambda_1v, \quad K_2v = \lambda_2v,
\]

then \( v \) is called a highest weight vector of \( \lambda = (\lambda_1, \lambda_2) \). If \( V = \mathfrak{w}X_q(A_1)v \) and \( v \) is a highest weight vector, then \( V \) is called a highest weight module of \( \mathfrak{w}X_q(A_1) \) generated by the highest weight vector \( v \).

**Lemma 4.1.** Let \( \mathfrak{w}X_q(A_1) \) be the weak Hopf algebra of type \( d \), \( V \) be a \( \mathfrak{w}X_q(A_1) \)-module and \( 0 \neq v \in V \). If \( K_i v = \lambda_i v, i = 1, 2, \lambda_i \in \mathbb{F} \), then there are elements \( \overline{\lambda}_i \in \mathbb{F} \) such that \( \overline{K}_i v = \overline{\lambda}_i v \). Moreover, if \( \lambda_i \neq 0 \), then \( \overline{\lambda}_i = \lambda_i^{-1} \); if \( \lambda_i = 0 \), then \( \overline{\lambda}_i = 0 \).

**Proof.** Since \( K_i v = \lambda_i v \), we have \( K_i v = \lambda_i K_i v = \lambda_i \overline{K}_i v = \lambda_i v \). Therefore, if \( \lambda_i \neq 0 \), \( \overline{K}_i v = \lambda_i^{-1} v \). If \( \lambda_i = 0 \), then \( \overline{K}_i v = \overline{K}_i \overline{K}_i v = 0 \). Hence \( \overline{\lambda}_i = 0 \).

Assume that \( (\lambda_1, \lambda_2, \delta) \in \mathbb{F}^* \times \mathbb{F}^* \times \{0, 1\} \), \( \mathbb{F}^* = \mathbb{F} \setminus \{0\} \), where

\[
\delta = \begin{cases} 
1, & \text{if } \lambda_1^2 = \lambda_2^2 \\
0, & \text{if } \lambda_1^2 \neq \lambda_2^2.
\end{cases}
\]

Suppose \( \lambda_1 \lambda_2 \neq 0 \) let \( V_{\lambda_1, \lambda_2, \delta}(n)(n = 0, 1) \) be the \( (n + 1) \)-dimensional vector space with the basis \( \{v_i \mid 0 \leq i \leq n\} \). The module structure of \( V_{\lambda_1, \lambda_2, \delta}(0) \) is a one-dimensional highest weight \( \mathfrak{w}X_q(A_1) \)-module with \( \delta = 1 \) and relations

\[
Ev_0 = Fv_0 = 0, \quad K_i v_0 = \lambda_i v_0, \quad \overline{K}_i v_0 = \overline{\lambda}_i v_0, \quad i = 1, 2.
\]

The module structure of \( V_{\lambda_1, \lambda_2, \delta}(1) \) is defined by

\[
K_1 v_1 = \lambda_1 v_1, \quad \overline{K}_1 v_1 = \lambda_1^{-1} v_1, \quad K_2 v_0 = \lambda_2 v_0, \quad \overline{K}_2 v_0 = \overline{\lambda}_2 v_0.
\]

\[
K_1 v_1 = q \lambda_1 v_1, \quad \overline{K}_1 v_1 = q^{-1} \overline{\lambda}_1 v_1, \quad K_2 v_1 = -q \lambda_2 v_1, \quad \overline{K}_2 v_1 = -q^{-1} \overline{\lambda}_2 v_1,
\]

\[
Ev_0 = 0, \quad Ev_1 = \overline{\lambda}_1 \lambda_2 - \lambda_1 \overline{\lambda}_2 = 0, \quad Fv_0 = v_1, \quad Fv_1 = 0.
\]

In fact, when \( \lambda_1 \lambda_2 \neq 0 \), we have \( \overline{\lambda}_1 \lambda_2 = \lambda_1 \overline{\lambda}_2 \Leftrightarrow \lambda_1^2 = \lambda_2^2 \).

**Lemma 4.2.** Assume that \( \mathfrak{w}X_q(A_1) \) is the weak Hopf algebra of any type \( d \) and \( \lambda_1 \lambda_2 \neq 0 \). Let \( V \) be a highest weight \( \mathfrak{w}X_q(A_1) \)-module generated by a highest weight vector \( v_0 \) with weight \( \lambda = (\lambda_1, \lambda_2) \). Then

(1) \( V \cong V_{\lambda_1, \lambda_2, \delta}(n)(n = 0, 1) \);

(2) \( V_{\lambda_1, \lambda_2, \delta}(n) \cong V_{\lambda_1', \lambda_2', \delta'}(n)(n = 0, 1) \) as \( \mathfrak{w}X_q(A_1) \)-modules if and only if \( (\lambda_1, \lambda_2, \delta) = (\lambda_1', \lambda_2', \delta') \).
Proof. Straightforward. □

Assume that $\lambda_1, \lambda_2 = 0$ and $\mathfrak{m}X_q(A_1)$ is a weak Hopf algebra of type $d = (0|1)$ or $(1|1)$. Let $W(n)(n = 0, 1)$ be the $(n + 1)$-dimensional vector space with the basis $\{v_i | 0 \leq i \leq n\}$. It is noted that if $\lambda_1, \lambda_2 = 0$ and $W(n)$ is a $\mathfrak{m}X_q(A_1)$-module, both $\lambda_1$ and $\lambda_2$ must be zero since $K_1K_1 = K_2K_2 = J$. In this case, the $\mathfrak{m}X_q(A_1)$-module structure on $W(n)$ is given as follows

$$K_1v_i = K_2v_i = 0, \quad EW_i = 0, \quad 0 \leq i \leq n,$$

$$Fv_j = v_{j+1}, \quad 0 \leq j \leq n - 1,$$

$$Fv_n = 0.$$

Remark 4.3. If $\mathfrak{m}X_q(A_1)$ is a weak Hopf algebra with $d = (1|0)$ or $(0|0)$, we only can define the $\mathfrak{m}X_q(A_1)$-module $W(0)$. For, if $F$ is of type II, then $K_1FK_1v_0 = q_1Fv_0 = 0$ and $Fv_0 = 0$. On the other hand, if $\mathfrak{m}X_q(A_1)$ is of type $d = (0|1)$ or $(1|1)$, then $W(1)$ is an indecomposable $\mathfrak{m}X_q(A_1)$-module of dimension 2, but it is not simple since $W(0)$ is a proper submodule of $W(1)$.

**Theorem 4.4.** Assume that $\mathfrak{m}X_q(A_1)$ is the weak Hopf algebra of type $d = (k|\overline{k})$. Let $M$ be a highest weight $\mathfrak{m}X_q(A_1)$-module. Then $M \cong W(t)(0 \leq t \leq k)$ or $M \cong V_{\lambda_1, \lambda_2, \delta}(n)$, where $n = 0, 1$.

**Proof.** Since $M$ is a highest weight $\mathfrak{m}X_q(A_1)$-module, $M$ has a highest weight vector $v_0$ such that $M = \mathfrak{m}X_q(A_1)v_0$, and

$$Ev_0 = 0, \quad K_i v_0 = \lambda_i v_0, \quad i = 1, 2.$$

Let $\lambda_1, \lambda_2 \neq 0$. By Lemma 4.2, we have $M \cong V_{\lambda_1, \lambda_2, \delta}(n)$ ($n = 0, 1$).

Let $\lambda_1, \lambda_2 = 0$. If $F$ is of type II, then we have $Fv_0 = 0$ because of the relations $K_1FK_1 = q_1F$ and $K_2FK_2 = q_2^{-1}F$. Hence we obtain $M \cong W(0)$. If $F$ is of type I, it is easy to check that $M \cong W(0)$ when $\dim M = 1$. If $\dim M \neq 1$, we have $Fv_0 \neq 0$ by Proposition 2.2. If $Fv_0 = av_0$ for some non-zero $a \in \mathfrak{F}$, then $FFv_0 = a^2v_0 = 0$ and it is a contradiction. So $\{v_0, Fv_0\}$ is linearly independent. If we take $v_1 = Fv_0$, then we have

$$Ev_0 = 0, \quad Ev_1 = Ef v_0 = FEv_0 = 0,$$

$$Fv_0 = v_1, \quad Fv_1 = 0.$$

Since $M$ is generated by $v_0$, we have $M \cong W(1)$.

In conclusion, $M \cong W(t)(0 \leq t \leq k)$ or $M \cong V_{\lambda_1, \lambda_2, \delta}(n)$, $n = 0, 1$. □

Assume $\eta_1^2 = \eta_2^2$, $\mathfrak{m}X_q(A_1)$ is of type $d = (k|\overline{k})$. Let $M_{\eta_1, \eta_2}(m, n)$ be a vector space spanned by $\{X^iY^j | 0 \leq i \leq m, 0 \leq j \leq n\}$, where $0 \leq m \leq k$, $0 \leq n \leq \overline{k}$. Then it is straightforward to see that $M_{\eta_1, \eta_2}(m, n)$ is a $\mathfrak{m}X_q(A_1)$-module defined by

$$K_1(X^iY^j) = q^{i-j-1}\eta_1 X^iY^j, \quad K_2(X^iY^j) = (-q)^{j-i}\eta_2 X^iY^j,$$

$$\overline{K}_1(X^iY^j) = q^{-j+1}\overline{\eta}_1 X^iY^j, \quad \overline{K}_2(X^iY^j) = (-q)^{j-i}\overline{\eta}_2 X^iY^j.$$
There is at least a nonzero coefficient. It yields that to see.

\[ M(0, n) \cong W(n). \] Under the condition of \( \eta_1 = \eta_2 = 0 \), we denote \( M_0,0(m,n) \) by \( M(m,n) \) for simplicity. Specially, \( M(0, n) \cong W(n) \). If \( \eta_1 = \eta_2 = 0 \), we can define \( \mathfrak{m}X_q(A_1) \)-modules \( M(0,0), M(1,0), M(0,1), M(1,1) \); if \( \mathfrak{m}X_q(A_1) \) is of type \( d = (0|1) \), we can define \( M(0,0), M(0,1), M(1,0); \) if \( \mathfrak{m}X_q(A_1) \) is of type \( d = (0|1) \), we can define \( M(0,0), M(0,1) \); if \( \mathfrak{m}X_q(A_1) \) is of type \( d = (0|0) \), we can only define \( M(0,0) \).

If we can define \( \mathfrak{m}X_q(A_1) \)-modules \( M_m,n(1,0), M_m,n(0,1), M_m,n(1,1) \) for some type \( d \), then they are indecomposable and \( M_m,n(0,0) \) is simple. For example, assume that \( \mathfrak{m}X_q(A_1) \) is of type \( d = (1|1) \). Let \( 0 \neq M_1 \) be any submodule of \( M_m,n(1,1) \). For any \( 0 \neq x \in M_1 \), \( x \) can be written as

\[ x = a_{00}X^0Y^0 + a_{01}X^1Y^0 + a_{10}X^0Y^1 + a_{11}X^1Y^1. \]

There is at least a nonzero coefficient. It yields that \( X^1Y^1 \in M_1 \) for all cases. This means that \( FX^1Y^1 \) is the submodule of any non-zero submodule of \( M_m,n(1,1) \). Hence \( M_m,n(1,1) \) is indecomposable. The other cases are similar to see.

5. The Clebsch-Gordan decomposition for \( \mathfrak{m}X_q(A_1) \)

In this section, we assume that the weak Hopf algebra \( \mathfrak{m}X_q(A_1) \) is of type \((1|1)\) and consider tensor products of their two the highest weight \( \mathfrak{m}X_q(A_1) \)-modules.

Let \( V \) and \( W \) be two \( \mathfrak{m}X_q(A_1) \)-modules, recall that \( V \otimes W \) is also a \( \mathfrak{m}X_q(A_1) \)-module defined by

\[ E(v \otimes w) = K_1K_2v \otimes Ew + Ev \otimes w, \]
\[ F(v \otimes w) = v \otimes Fw + Fv \otimes K_1K_2w, \]
\[ K_i(v \otimes w) = K_iv \otimes K_iw, \]
\[ \overline{K}_i(v \otimes w) = \overline{K}_iv \otimes \overline{K}_iw. \]

We denote

\[ mW(n) = W(n) \oplus W(n) \oplus \cdots \oplus W(n). \]

**Theorem 5.1.** Assume that the weak Hopf algebra \( \mathfrak{m}X_q(A_1) \) is of type \((1|1)\). Then

1. \( V_{\lambda_1, \lambda_2, \sigma}(m) \otimes V_{\lambda_1', \lambda_2', \sigma'}(n) \cong V_{\lambda_1, \lambda_1', \lambda_2, \lambda_2', \delta, \delta'}(m + n), \) \( m + n \leq 1; \)
2. If \( \lambda_1^2 \lambda_2^2 \neq \lambda_2^2 \lambda_2^2 \), then
   \[ V_{\lambda_1, \lambda_2, \sigma}(1) \otimes V_{\lambda_1', \lambda_2', \sigma'}(1) \cong V_{\lambda_1, \lambda_1', \lambda_2, \lambda_2', \sigma, \sigma'}(1) \oplus V_{\lambda_1, \lambda_1', (-\sigma), \lambda_2, \lambda_2', \sigma}(1); \]
   if \( \lambda_1^2 \lambda_2^2 = \lambda_2^2 \lambda_2^2 \), then
   \[ V_{\lambda_1, \lambda_2, \sigma}(1) \otimes V_{\lambda_1', \lambda_2', \sigma'}(1) \cong M_{\lambda_1, \lambda_1', (-\sigma), \lambda_2, \lambda_2'}(1, 1); \]
(3) $V_{\lambda_1,\lambda_2,1}(1) \otimes V^{\lambda_1',\lambda_2',\sigma}(1) \cong V_{\lambda_1,\lambda_2,\lambda_1',\lambda_2',\sigma}(1) \oplus V_{\lambda_1,\lambda_2,(-q)\lambda_1',\lambda_2',\sigma}(1)$;
(4) $V_{\lambda_1,\lambda_2,1}(m) \otimes W(n) \cong (m+1)W(n)$, $V_{\lambda_1,\lambda_2,0}(1) \otimes W(n) \cong M(1,n)$;
(5) $W(0) \otimes V_{\lambda_1,\lambda_2,\sigma}(n) \cong W(n)$, $W(1) \otimes V_{\lambda_1,\lambda_2,\sigma}(n) \cong (n+1)W(1)$;
(6) $W(m) \otimes W(n) \cong (m+1)W(n)$,
where $m, n = 0$ or $1$.

**Proof.** Keeping all notations as Section 4.

(1) We consider the following cases, the others can be obtained in a similar way.

Case 1. For $V_{\lambda_1,\lambda_2,1}(0) \otimes V^{\lambda_1',\lambda_2',1}(1)$, we have
$$K_i(v_0 \otimes v'_0) = \lambda_i \lambda'_i v_0 \otimes v'_0, \quad K_i(v_0 \otimes v'_0) = \lambda_i \lambda'_i v_0 \otimes v'_0,$$
$$E(v_0 \otimes v'_0) = 0, \quad E(v_0 \otimes v'_1) = 0, \quad F(v_0 \otimes v'_0) = v_0 \otimes v'_1, \quad F(v_0 \otimes v'_1) = 0.
$$
Therefore,
$$V_{\lambda_1,\lambda_2,1}(0) \otimes V^{\lambda_1',\lambda_2',1}(1) \cong V_{\lambda_1,\lambda_2,\lambda_1',\lambda_2',1}(1).$$

Case 2. For $V_{\lambda_1,\lambda_2,1}(0) \otimes V^{\lambda_1',\lambda_2',0}(1)$, note that
$$K_i(v_0 \otimes v'_0) = \lambda_i \lambda'_i v_0 \otimes v'_0, \quad K_i(v_0 \otimes v'_0) = \lambda_i \lambda'_i v_0 \otimes v'_0,$$
$$E(v_0 \otimes v'_0) = 0, \quad F(v_0 \otimes v'_0) = v_0 \otimes v'_1, \quad F(v_0 \otimes v'_1) = 0,$$
$$E(v_0 \otimes v'_1) = K_1 K_2 v_0 \otimes E v'_1 = \frac{\lambda_1 \lambda'_1 \lambda_2 \lambda'_2 - \lambda_1 \lambda'_1 \lambda_2 \lambda'_2}{q-q^{-1}} v_0 \otimes v'_0 \neq 0.$$

Then
$$V_{\lambda_1,\lambda_2,1}(0) \otimes V^{\lambda_1',\lambda_2',0}(1) \cong V_{\lambda_1,\lambda_2,\lambda_1',\lambda_2',0}(1).$$

Case 3. Considering $V_{\lambda_1,\lambda_2,0}(1) \otimes V^{\lambda_1',\lambda_2',1}(0)$, note that
$$K_i(v_0 \otimes v'_0) = \lambda_i \lambda'_i v_0 \otimes v'_0, \quad K_i(v_0 \otimes v'_0) = \lambda_i \lambda'_i v_0 \otimes v'_0,$$
$$E(v_0 \otimes v'_0) = 0, \quad F(v_0 \otimes v'_0) = \lambda_1 \lambda'_1 v_1 \otimes v'_0, \quad F(v_1 \lambda_1 \lambda'_1 v_1 \otimes v'_0) = 0,$$
$$E(F(v_0 \otimes v'_0)) = \lambda_1 \lambda'_1 (E v_1 \otimes v'_0) = \frac{\lambda_1 \lambda'_1 \lambda_2 \lambda'_2 - \lambda_1 \lambda'_1 \lambda_2 \lambda'_2}{q-q^{-1}} v_0 \otimes v'_0 \neq 0.$$

So
$$V_{\lambda_1,\lambda_2,0}(1) \otimes V^{\lambda_1',\lambda_2',1}(0) \cong V_{\lambda_1,\lambda_2,\lambda_1',\lambda_2',0}(1).$$

For $V_{\lambda_1,\lambda_2,1}(0) \otimes V^{\lambda_1',\lambda_2',1}(0)$ and $V_{\lambda_1,\lambda_2,1}(1) \otimes V^{\lambda_1',\lambda_2',1}(0)$, we also can get the similar result.

It follows that
$$V_{\lambda_1,\lambda_2,\sigma}(m) \otimes V^{\lambda_1',\lambda_2',\sigma}(n) \cong V_{\lambda_1,\lambda_2,\lambda_1',\lambda_2',\sigma}(m+n), \quad m + n \leq 1.$$

(2) Considering $V_{\lambda_1,\lambda_2,\sigma}(1) \otimes V^{\lambda_1',\lambda_2',\sigma}(1)$, we have
$$K_i(v_0 \otimes v'_0) = \lambda_i \lambda'_i v_0 \otimes v'_0, \quad K_i(v_0 \otimes v'_0) = \lambda_i \lambda'_i v_0 \otimes v'_0,$$
$$E(v_0 \otimes v'_0) = K_1 K_2 v_0 \otimes E v'_0 + E v_0 \otimes v'_0 = 0,$$
$$F(v_0 \otimes v'_0) = v_0 \otimes F v'_0 + F v_0 \otimes K_1 K_2 v'_0 = v_0 \otimes v'_1 + \lambda_1 \lambda'_1 v_1 \otimes v'_0.$$
Then it follows that $v_0 \otimes v'_0$ is a \( \mathfrak{w}X_q(A_1) \)-module highest weight vector and
\[
\mathfrak{w}X_q(A_1)(v_0 \otimes v'_0) \cong V_{\lambda_1, \lambda_2, \lambda'_2, 0}(1),
\]
where
\[
\delta'' = \begin{cases} 
1, & \text{if } \lambda_1^2 \lambda'_2^2 = \lambda_2^2 \lambda'_2^2, \\
0, & \text{if } \lambda_1^2 \lambda'_2^2 \neq \lambda_2^2 \lambda'_2^2.
\end{cases}
\]

Now we consider other submodules of
\[
V_{\lambda_1, \lambda_2, 0}(1) \otimes V_{\lambda'_1, \lambda'_2, 0}(1).
\]
If \( \delta = 0 \), this means that \( \lambda_1 \lambda_2 - \lambda_1 \lambda'_2 \neq 0 \), we take
\[
v_0 = (\lambda_1 \lambda_2 - \lambda_1 \lambda'_2) v_0 \otimes v'_1 - (\lambda_1 \lambda'_2 - \lambda'_1 \lambda_2) \lambda_1 \lambda_2 v_1 \otimes v'_0 \neq 0.
\]
Then
\[
K_1 v_0 = q \lambda_1 \lambda'_1 v_0, \quad K_2 (\nu) = -q \lambda_2 \lambda'_2 v_0,
\]
\[
\overline{K}_1 v_0 = q^{-1} \lambda_1 \lambda'_1 v_0, \quad \overline{K}_2 v_0 = -q^{-1} \lambda_2 \lambda'_2 v_0,
\]
and
\[
E v_0 = 0,
\]
\[
F v_0 = (\lambda_1 \lambda'_1 \lambda_2 \lambda'_2 - \lambda_1 \lambda'_1 \lambda_2 \lambda'_2) v_1 \otimes v'_1 := \nu_1,
\]
\[
E(\nu_1) = E(F(v_0)) = \frac{\lambda_1 \lambda'_1 \lambda_2 \lambda'_2 - \lambda_1 \lambda'_1 \lambda_2 \lambda'_2}{q - q^{-1}} v_0,
\]
\[
F(F(v_0)) = 0.
\]
If \( \lambda_1 \lambda'_1 \lambda_2 \lambda'_2 - \lambda_1 \lambda'_1 \lambda_2 \lambda'_2 \neq 0 \), hence \( \delta'' = 0 \), then \( v_0 \) is another \( \mathfrak{w}X_q(A_1) \)-module highest weight vector and
\[
\mathfrak{w}X_q(A_1)v_0 \cong V_{q \lambda_1, \lambda'_1, (-q) \lambda_2, \lambda'_2, 0}(1).
\]
It follows that
\[
V_{\lambda_1, \lambda_2, 0}(1) \otimes V_{\lambda'_1, \lambda'_2, 0}(1) \cong V_{q \lambda_1, \lambda'_1, (-q) \lambda_2, \lambda'_2, 0}(1) \cong V_{\lambda_1, \lambda_2, \lambda'_2, 0}(1).
\]
If \( \lambda_1 \lambda'_1 \lambda_2 \lambda'_2 - \lambda_1 \lambda'_1 \lambda_2 \lambda'_2 = 0 \), hence \( \delta'' = 1 \), then \( v_0 \) is a constant multiple of \( F(v_0 \otimes v'_0) \). We have
\[
K_1 (v_1 \otimes v'_0) = q \lambda_1 \lambda'_1 v_1 \otimes v'_0, \quad K_2 (v_1 \otimes v'_0) = -q \lambda_2 \lambda'_2 v_1 \otimes v'_0,
\]
\[
E(v_1 \otimes v'_0) = E(v_1 \otimes v'_0) = \frac{\lambda_1 \lambda_2 - \lambda_1 \lambda'_2}{q - q^{-1}} v_0 \otimes v'_0, E(E(v_1 \otimes v'_0) = 0,
\]
\[
F(v_1 \otimes v'_0) = v_1 \otimes F(v'_0) = v_1 \otimes v'_1,
\]
\[
F(E(v_1 \otimes v'_0)) = \frac{\lambda_1 \lambda_2 - \lambda_1 \lambda'_2}{q - q^{-1}} F(v_0 \otimes v'_0),
\]
\[
F(F(v_1 \otimes v'_0)) = F(v_1 \otimes v'_0) = 0.
\]
Theorem 3.12. Let $X$ be a weak admissible Hecke module. Then $X$ is a module if and only if the following conditions are satisfied:

1. The center of $X$ is equal to the center of $V_{\lambda_1, \lambda_2}$.
2. The action of $\lambda_1$ and $\lambda_2$ on $X$ is diagonal.
3. The action of $\lambda_1$ and $\lambda_2$ on $X$ is nilpotent.
4. The action of $\lambda_1$ and $\lambda_2$ on $X$ is semisimple.

We now prove the theorem. The proof is similar to the proof of Theorem 3.11. We first assume that $X$ is a module and show that the conditions are satisfied. Then we assume that the conditions are satisfied and show that $X$ is a module.

Proof. Let $X$ be a weak admissible Hecke module. We first assume that $X$ is a module and show that the conditions are satisfied. Then we assume that the conditions are satisfied and show that $X$ is a module.

We first assume that $X$ is a module. Then $X$ is a module if and only if $X$ is a module. Hence, we have $X = V_{\lambda_1, \lambda_2}$.

Case 1. For $V_{\lambda_1, \lambda_2}(0) \otimes W(0)$, we have

$$K_i(v_0 \otimes w_0) = 0,$$

$$E(v_0 \otimes w_0) = K_i \mathcal{K}_2 v_0 \otimes Ew_0 + Ev_0 \otimes w_0 = 0,$$

$$F(v_0 \otimes w_0) = v_0 \otimes Fw_0 + Fv_0 \otimes \mathcal{K}_1 K_2 w_0 = 0,$$

hence

$$V_{\lambda_1, \lambda_2}(0) \otimes W(0) \cong W(0).$$

Case 2. For $V_{\lambda_1, \lambda_2}(1) \otimes W(1)$, we get

$$K_i(v_0 \otimes w_0) = 0,$$

$$K_i(v_1 \otimes w_0) = 0,$$

$$E(v_0 \otimes w_0) = K_i \mathcal{K}_2 v_0 \otimes Ew_0 + Ev_0 \otimes w_0 = 0,$$

$$F(v_0 \otimes w_0) = v_0 \otimes Fw_0 + Fv_0 \otimes \mathcal{K}_1 K_2 w_0 = 0,$$

hence

$$V_{\lambda_1, \lambda_2}(1) \otimes W(1) \cong W(1).$$

Finally, we assume that the conditions are satisfied. Then we have

$$V_{\lambda_1, \lambda_2}(0) \otimes W(0) \cong W(0),$$

$$V_{\lambda_1, \lambda_2}(1) \otimes W(1) \cong W(1).$$

Hence, $X$ is a module.

\[\square\]
\[
E(v_0 \otimes w_0) = 0, \quad F(v_0 \otimes w_0) = v_0 \otimes w_1,
E(v_0 \otimes w_1) = 0, \quad F(v_0 \otimes w_1) = 0,
E(v_1 \otimes w_0) = 0, \quad F(v_1 \otimes w_0) = v_1 \otimes w_1,
E(v_1 \otimes w_1) = 0, \quad F(v_1 \otimes w_1) = 0.
\]

Thus

\[V_{\lambda_1, \lambda_2, 1}(1) \otimes W(1) \cong 2W(1).\]

Case 3. Considering the case \(V_{\lambda_1, \lambda_2, 0}(1) \otimes W(0)\). Note that \(\lambda_1 \omega_2 \neq \omega_1 \lambda_2\), we have

\[
K_i(v_0 \otimes w_0) = 0, \quad K_i(v_1 \otimes w_0) = 0,
E(v_0 \otimes w_0) = 0, \quad F(v_0 \otimes w_0) = 0,
E(v_1 \otimes w_0) = E(v_1 \otimes w_0) = \frac{\lambda_1 \lambda_2 - \lambda_1 \lambda_2}{q - q^{-1}} v_0 \otimes w_0 \neq 0,
E(E(v_1 \otimes w_0)) = 0, \quad F(v_1 \otimes w_0) = v_1 \otimes Fw_0 = 0.
\]

Now, we assume that \(X^i Y^j = E^i F^j (v_1 \otimes w_0)\), where \(i = 0\) or \(1\), and \(j = 0\).

\[
K_i(X^0 Y^0) = 0,
E(X^0 Y^0) = X^1 Y^0 = E^1 F^0 (v_1 \otimes w_0) = E(v_1 \otimes w_0),
E(X^1 Y^0) = E(E(v_1 \otimes w_0)) = 0,
F(X^0 Y^0) = X^0 Y^1 = E^0 F^1 (v_1 \otimes w_0) = F(v_1 \otimes w_0) = 0.
\]

Therefore

\[V_{\lambda_1, \lambda_2, 1}(1) \otimes W(0) \cong M(1, 0).\]

Case 4. For \(V_{\lambda_1, \lambda_2, 0}(1) \otimes W(1)\), this means that \(\omega_1 \lambda_2 - \lambda_1 \omega_2 \neq 0\). We have

\[
K_i(v_0 \otimes w_0) = 0, \quad E(v_0 \otimes w_0) = 0, \quad F(v_0 \otimes w_0) = v_0 \otimes w_1,
E(v_0 \otimes w_1) = 0, \quad F(v_0 \otimes w_1) = 0,
E(v_1 \otimes w_0) = E(v_1 \otimes w_0) = \omega_1 \lambda_2 - \lambda_1 \omega_2 v_0 \otimes w_0,
F(v_1 \otimes w_0) = v_1 \otimes Fw_0 = v_1 \otimes w_1, \quad F(v_1 \otimes w_1) = 0,
E(v_1 \otimes w_1) = E(v_1 \otimes w_1) = \omega_1 \lambda_2 - \lambda_1 \omega_2 v_0 \otimes w_1.
\]

Let \(X^i Y^j = E^i F^j (v_1 \otimes w_0)\), where \(i, j = 0\) or \(1\).

\[
K_i(X^0 Y^0) = 0,
E(X^0 Y^0) = X^1 Y^0 = E^1 F^0 (v_1 \otimes w_0) = E(v_1 \otimes w_0),
E(X^1 Y^0) = E(E(v_1 \otimes w_0)) = 0,
E(X^0 Y^1) = X^1 Y^1 = E^1 F^1 (v_1 \otimes w_0) = E(v_1 \otimes w_1),
E(X^1 Y^1) = E(E(v_1 \otimes w_1)) = 0,
F(X^0 Y^0) = X^0 Y^1 = E^0 F^1 (v_1 \otimes w_0) = F(v_1 \otimes w_0),
F(X^0 Y^1) = F(F(v_1 \otimes w_0)) = 0,
\]
\[
F(X^1Y^0) = X^1Y^1 = E^1F^1(v_1 \otimes w_0) = E(v_1 \otimes w_1), \\
F(X^1Y^1) = F(F(v_1 \otimes w_0)) = 0.
\]

Therefore
\[
V_{\lambda_1, \lambda_2, \delta}(1) \otimes W(1) \cong M(1, 1).
\]

For \(V_{\lambda_1, \lambda_2, \delta}(0) \otimes W(1)\) and \(V_{\lambda_1, \lambda_2, \delta}(1) \otimes W(0)\), in a similar way we get
\[
V_{\lambda_1, \lambda_2, \delta}(0) \otimes W(1) \cong W(1), \\
V_{\lambda_1, \lambda_2, \delta}(1) \otimes W(0) \cong W(0) \oplus W(0).
\]

(5) Note that \(E(W(m) \otimes V_{\lambda_1, \lambda_2, \delta}(n)) = 0\). We consider the action of \(F\) on \(W(m) \otimes V_{\lambda_1, \lambda_2, \delta}(n)\).

Case 1. Considering \(W(0) \otimes V_{\lambda_1, \lambda_2, \delta}(0)\), we have
\[
K_i(w_0 \otimes v_0) = 0, \quad F(w_0 \otimes v_0) = 0,
\]
hence
\[
W(0) \otimes V_{\lambda_1, \lambda_2, \delta}(0) \cong W(0).
\]

Case 2. For \(W(0) \otimes V_{\lambda_1, \lambda_2, \delta}(1)\), it is easy to see that
\[
K_i(w_0 \otimes v_0) = 0, \\
F(w_0 \otimes v_0) = w_0 \otimes Fv_0 = w_0 \otimes v_1, \quad F(w_0 \otimes v_1) = 0.
\]

Therefore
\[
W(0) \otimes V_{\lambda_1, \lambda_2, \delta}(1) \cong W(1).
\]

Case 3. For \(W(1) \otimes V_{\lambda_1, \lambda_2, \delta}(0)\), note that \(\overline{\lambda_1} \lambda_2 \neq 0\), and we get
\[
K_i(w_0 \otimes v_0) = 0, \\
F(w_0 \otimes v_0) = w_0 \otimes Fv_0 + Fw_0 \otimes \overline{\lambda_1} K_2 v_0 = \overline{\lambda_1} \lambda_2 w_1 \otimes v_0 \neq 0, \\
F(\overline{\lambda_1} \lambda_2 w_1 \otimes v_0) = \overline{\lambda_1} \lambda_2 w_1 \otimes Fv_0 = 0.
\]

Thus
\[
W(1) \otimes V_{\lambda_1, \lambda_2, \delta}(0) \cong W(1).
\]

Case 4. Considering the case \(W(1) \otimes V_{\lambda_1, \lambda_2, \delta}(1)\), we have
\[
K_i(w_0 \otimes v_0) = 0, \quad F(w_0 \otimes v_0) = w_0 \otimes v_1 + \overline{\lambda_1} \lambda_2 w_1 \otimes v_0, \\
F(w_0 \otimes v_1 + \overline{\lambda_1} \lambda_2 w_1 \otimes v_0) = F(w_0 \otimes v_1) + F(\overline{\lambda_1} \lambda_2 w_1 \otimes v_0) \\
= Fw_0 \otimes \overline{\lambda_1} K_2 v_1 + w_1 \otimes F\overline{\lambda_1} \lambda_2 v_0 = 0.
\]

This means that
\[
\overline{m} X_\overline{d}(A_1)(w_0 \otimes v_0) \cong W(1).
\]

Assume that \(w = aw_0 \otimes v_1 + bw_1 \otimes v_0\), \(b \neq a \overline{\lambda_1} \lambda_2\),
\[
K_i w = K_i(aw_0 \otimes v_1 + bw_1 \otimes v_0) = 0, \\
Fw = aF(w_0 \otimes v_1) + bF(w_1 \otimes v_0) = (b - a \overline{\lambda_1} \lambda_2) w_1 \otimes v_1 \neq 0, \\
F(F(w)) = 0.
\]
It follows that $\mathfrak{m}X_q(A_1)w \cong W(1)$. Hence
\[
W(1) \otimes V_{\lambda_1,\lambda_2,\delta}(1) = \mathfrak{m}X_q(A_1)(w_0 \otimes m) \oplus \mathfrak{m}X_q(A_1)w \cong W(1) \oplus W(1).
\]
(6) It is easy to see that $E(W(m) \otimes W(n)) = 0$. Consider the action of $F$ on $W(m) \otimes W(n)$.
Case 1. For $W(0) \otimes W(0)$, $F(w_0 \otimes w'_0) = 0$, hence
\[
W(0) \otimes W(0) \cong W(0).
\]
Case 2. For $W(0) \otimes W(1)$, we have
\[
K_i(w_0 \otimes w'_0) = 0,
F(w_0 \otimes w'_0) = w_0 \otimes Fw'_0 = w_0 \otimes w'_1, \quad F(w_0 \otimes w'_1) = 0.
\]
So
\[
W(0) \otimes W(1) \cong W(1).
\]
Case 3. For $W(1) \otimes W(0)$, we get
\[
K_i(w_0 \otimes w'_0) = 0, \quad F(w_0 \otimes w'_0) = w_0 \otimes Fw'_0 = 0,
K_i(w_1 \otimes w'_0) = 0, \quad F(w_1 \otimes w'_0) = w_1 \otimes Fw'_0 = 0.
\]
Consequently
\[
W(1) \otimes W(0) \cong W(0) \oplus W(0).
\]
Case 4. For $W(1) \otimes W(1)$, we get
\[
K_i(w_0 \otimes w'_0) = 0, \quad K_i(w_1 \otimes w'_0) = 0,
F(w_0 \otimes w'_0) = w_0 \otimes Fw'_0 = w_0 \otimes w'_1, \quad F(w_0 \otimes w'_1) = 0,
F(w_1 \otimes w'_0) = w_1 \otimes Fw'_0 = w_1 \otimes w'_1, \quad F(w_1 \otimes w'_1) = 0.
\]
Therefore
\[
W(1) \otimes W(n) \cong W(n) \oplus W(n) = 2W(n).
\]
The proof is finished. \(\square\)

Theorem 5.1 for $\mathfrak{m}X_q(A_1)$ of other types $d$ can be discussed in a similar way.

It is noted that if $E$ (resp. $F'$) is of type II, for two $\mathfrak{m}X_q(A_1)$-module $V, W$, we have to define the $\mathfrak{m}X_q(A_1)$-module on $V \otimes W$ by
\[
E(v \otimes w) = K_1\overline{v} \otimes Ev + Ev \otimes Jw,
\]
(resp. $F(v \otimes w) = Jv \otimes Fw + Fv \otimes \overline{v}K_2w$).

Theorem 5.1 should be restated. Explicitly,

- If $\mathfrak{m}X_q(A_1)$ is of $d = (0|1)$, Theorem 5.1(4) is replaced by
\[
(4') \quad V_{\lambda_1,\lambda_2,\delta}(m) \otimes W(n) \cong (m + 1)W(n).
\]
- If $\mathfrak{m}X_q(A_1)$ is of $d = (1|0)$, Theorem 5.1(4)(5)(6) are respectively replaced by
\[
(4') \quad V_{\lambda_1,\lambda_2,\delta}(0) \otimes W(0) \cong W(0), \quad V_{\lambda_1,\lambda_2,\delta}(1) \otimes W(0) \cong M(1,0),
(5') \quad W(0) \otimes V_{\lambda_1,\lambda_2,\delta}(n) \cong (n + 1)W(0),
(6') \quad W(0) \otimes W(0) \cong W(0).
\]
If \( mX_q(A_1) \) is of \( d = (0|0) \), Theorem 5.1(4)(5)(6) are respectively replaced by

\[
\begin{align*}
(4') & \quad V_{\lambda_1, \lambda_2, \delta}(m) \otimes W(0) \cong (m+1)W(0), \\
(5') & \quad W(0) \otimes V_{\lambda_1, \lambda_2, \delta}(n) \cong (n+1)W(0), \\
(6') & \quad W(0) \otimes W(0) \cong W(0).
\end{align*}
\]

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