A POLAR REPRESENTATION OF A REGULARITY OF A DUAL QUATERNIONIC FUNCTION IN CLIFFORD ANALYSIS

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Abstract. The paper gives the regularity of dual quaternionic functions and the dual Cauchy-Riemann system in dual quaternions. Also, the paper researches the polar representation and properties of a dual quaternionic function and their regular quaternionic functions.

1. Introduction

A dual number $z$ is consisted of real numbers $x$ and $y$ associated with a real unit $1$ and the dual unit $\varepsilon$, where $\varepsilon^2 = 0$. A dual number is denoted in the form $z = x + \varepsilon y$. Thus, the dual numbers are elements of the two dimensional real algebra

$$D = R[\varepsilon] = \{ z = x + \varepsilon y \mid x, y \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0 \}$$

generated by 1 and $\varepsilon$ (see [17]).

The algebra of dual numbers has been studied by Clifford [1] and its applications to mechanics are due to Study [20]. Dual algebra has been often used for the field of displacement analysis, kinematic synthesis and dynamic analysis of spatial mechanisms. Dual numbers can be represented as follows ([3]):

1. Gaussian representation: $z = x + \varepsilon y$,
2. Polar representation: $z = r(1 + \varepsilon \phi)$,
3. Exponential representation: $z = r \exp(\varepsilon \phi)$, where $r = x$ ($x \neq 0$), $\phi = \frac{y}{x}$ and $\exp(\varepsilon \phi) = 1 + \varepsilon \phi$.

The dual number has a geometrical property which is investigated detail in [4, 17].

Clifford [1] also has studied the following algebra

$$\mathbb{H} = \{ p = z_1 + z_2 j \mid z_1 = x_0 + x_1 i, z_2 = x_2 + x_3 i, x_r \in \mathbb{R} \ (r = 0, 1, 2, 3) \}$$

Received March 8, 2016; Revised June 14, 2016.

2010 Mathematics Subject Classification. 32W50, 32A99, 30G35, 11E88.

Key words and phrases. quaternion, dual number, polar representation, regularity, Clifford analysis.

This work was supported by the Dongguk University Research Fund of 2017.

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called the set of quaternions. Here imaginary basis elements $i$, $j$ and $k$ satisfy the following conditions:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$ 

For two quaternions $p = z_1 + z_2 j$ and $q = w_1 + w_2 j$, they are given the rules of the addition and multiplication as follows, respectively,

$$p + q = (z_1 + w_1) + (z_2 + w_2)j$$

and

$$pq = (z_1 w_1 - z_2 w_2) + (z_1 w_2 + z_2 w_1)j,$$

where $w_k = y_{k0} - y_{k1}i$ for $w_k = y_{k0} + y_{k1}i$, $y_{k} \in \mathbb{R}$, $k = 1, 2$, $j = 0, 1$. Kajiwara et al. [5, 6] applied the theory on a closed densely defined operator and a priori estimate for the adjoint operator in a Hilbert space and biconvex domains. We [9, 10, 11, 12] researched corresponding Cauchy-Riemann systems and properties of functions with values in special quaternions such as reduced quaternions, split quaternions and dual split quaternions. We [13, 14, 15] investigated a regular functions defined by the differential operators of special quaternions, dual quaternionic functions by using a dual Cauchy-Riemann system and their regularity of that functions in dual quaternions. Porter [19] gave an explicit solution to the linear equation in the quaternions $\mathbb{H}$.

This paper gives expressions of the differential operators and the exponential functions in dual quaternions. The paper researches the polar representation of dual quaternionic functions by using a dual Cauchy-Riemann system and their regularity of that functions in dual quaternions.

2. Preliminaries

For $p = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, we denote by $Sc(p)$ the scalar part, and by $Vec(p)$ the spatial vector part:

$$p = Sc(p) + Vec(p),$$

where $Sc(p) = x_0$ and $Vec(p) = x_1 i + x_2 j + x_3 k$ with $x_r \in \mathbb{R}$ ($r = 0, 1, 2, 3$).

Then for $p, q \in \mathbb{H}$, we have

$$p + q = Sc(p) + Sc(q) + Vec(p) + Vec(q),$$

and

$$pq = Sc(p)Sc(q) - Vec(p) \cdot Vec(q)$$

$$+ Sc(p)Vec(q) + Vec(p)Sc(q) + Vec(p) \times Vec(q),$$

where $Sc(q) = y_0, Vec(q) = y_1 i + y_2 j + y_3 k$ with $y_r \in \mathbb{R}$ ($r = 0, 1, 2, 3$), the symbol $\cdot$ is a usual inner product,

$$Vec(p) \cdot Vec(q) = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

and the symbol $\times$ is a usual outer product,

$$Vec(p) \times Vec(q) = (x_2 y_3 - x_3 y_2)i - (x_1 y_3 - x_3 y_1)j + (x_1 y_2 - x_2 y_1)k.$$ 

The norm for a quaternion is

$$|p|^2 := pp^* = Sc(p)^2 + Vec(p) \cdot Vec(p),$$

where $p^*$ is the conjugate of $p$. This norm is positive definite over $\mathbb{H}$.
where \( p^* = \text{Sc}(p) - \text{Vec}(p) \), and the inverse of \( p \) is
\[
p^{-1} = \frac{p^*}{|p|^2}.
\]
For a unit quaternion, \(|p| = 1\), it is given by:
\[
p = (\cos(\theta/2), n \sin(\theta/2))
\]
and
\[
x_0 = \cos(\theta/2), \quad x_1 = n_1 \sin(\theta/2),
\]
\[
x_2 = n_2 \sin(\theta/2), \quad x_3 = n_3 \sin(\theta/2),
\]
where an angle \( \theta \) and axis \( n = (n_1, n_2, n_3) \) of rotation (see [7]).

We consider the following form:
\[
\mathbb{D}_q = \{ Z = p_1 + \varepsilon p_2 \mid p_r \in \mathbb{H}, \varepsilon^2 = 0, \ r = 1, 2 \} \cong \mathbb{H}^2 \cong \mathbb{R}^8,
\]
where \( \varepsilon \) is the dual unit that commutes with \( i, j \) and \( k \). The dual quaternion \( Z = p_1 + \varepsilon p_2 \in \mathbb{D}_q \) is also written as a linear combination of a scalar, denoted by \( \text{Sc}(Z) \), and a spatial vector, denoted by \( \text{Vec}(Z) \) (see [7, 8]):
\[
Z = \text{Sc}(Z) + \text{Vec}(Z) = \text{Sc}(p_1) + \text{Vec}(p_1) + \varepsilon \{ \text{Sc}(p_2) + \text{Vec}(p_2) \},
\]
where
\[
\text{Sc}(Z) = \text{Sc}(p_1) + \varepsilon \text{Sc}(p_2), \quad \text{Vec}(Z) = \text{Vec}(p_1) + \varepsilon \text{Vec}(p_2)
\]
with \( p_1, p_2 \in \mathbb{H} \).

For two elements \( Z \) and \( W = \text{Sc}(W) + \text{Vec}(W) = \text{Sc}(q_1) + \text{Vec}(q_1) + \varepsilon \{ \text{Sc}(q_2) + \text{Vec}(q_2) \} \) of \( \mathbb{D}_q \), we give the addition and the multiplication on \( \mathbb{D}_q \) as follows:
\[
Z + W = \text{Sc}(Z) + \text{Sc}(W) + \varepsilon \{ \text{Vec}(Z) + \text{Vec}(W) \}
\]
and
\[
ZW = \text{Sc}(Z)\text{Sc}(W) - \text{Vec}(Z) \cdot \text{Vec}(W) + \text{Sc}(Z)\text{Vec}(W)
+ \text{Sc}(W)\text{Vec}(Z) + \text{Vec}(Z) \times \text{Vec}(W),
\]
where
\[
\text{Vec}(Z) \cdot \text{Vec}(W) = \text{Vec}(p_1) \cdot \text{Vec}(q_1) + \varepsilon \{ \text{Vec}(p_1) \cdot \text{Vec}(q_2) + \text{Vec}(p_2) \cdot \text{Vec}(q_1) \}
\]
and
\[
\text{Vec}(Z) \times \text{Vec}(W) = \text{Vec}(p_1) \times \text{Vec}(q_1) + \varepsilon \{ \text{Vec}(p_1) \times \text{Vec}(q_2) + \text{Vec}(p_2) \times \text{Vec}(q_1) \}.
\]

We give the complex conjugate element of \( \mathbb{D}_q \):
\[
Z^* = \text{Sc}(p_1) - \text{Vec}(p_1) + \varepsilon \{ \text{Sc}(p_2) - \text{Vec}(p_2) \}.
\]
It is also written as
\[
Z^* = \text{Sc}(Z) - \text{Vec}(Z),
\]
and the modulus of \( Z \), denoted by \(|Z|\), is described by
\[
|Z|^2 := \text{Sc}(Z)\text{Sc}(Z^*) + \text{Vec}(Z) \cdot \text{Vec}(Z^*) = \{ \text{Sc}(p_1) \}^2 + \text{Vec}(p_1) \cdot \text{Vec}(p_1) = |p_1|^2.
\]
Since every element of the set \( \{ \varepsilon p | p \in \mathbb{H} \} \) has no inverse, the inverse of a dual quaternion is given by

\[
Z^{-1} = \frac{Z^{\dagger}}{|p_1|^2} \in \mathbb{D}_q \quad (p_1 \neq 0),
\]

where

\[
Z^{\dagger} = (|p_1|^2 - \varepsilon p_2 p_1^*) p_1^{-1},
\]

where \( p_1^{-1} = \frac{p_1^*}{|p_1|^2} \), called the dual conjugate of \( Z \) with \( ZZ^{\dagger} = Z^{\dagger} Z = p_1 p_1^* = |p_1|^2 \).

Plucker [18] gave screw coordinates so that we can rewrite dual quaternions in a form of the spherical linear interpolation. Screw parameters have the form \((\theta, d, \mathbf{l}, \mathbf{m})\), where

\[
\begin{align*}
\theta & \quad \text{is the angle of rotation,} \\
d & \quad \text{is the translation along the axis,} \\
\mathbf{l} & \quad \text{is the vector line direction,} \\
\mathbf{m} = \mathbf{p} \times \mathbf{l} & \quad \text{is the line moment with} \ \mathbf{p} \ \text{is a point on a given line.}
\end{align*}
\]

From the above components, Daniilidis [2] converted a unit dual quaternion to screw coordinates as follows:

\[
\begin{align*}
(2.1) & \quad \text{Sc}(p_1) = \cos(\theta/2), \ Vec(p_1) = \mathbf{l} \sin(\theta/2), \ Sc(p_2) = -\frac{d}{2} \sin(\theta/2), \\
(2.2) & \quad Vec(p_2) = \frac{d}{2} \cos(\theta/2) + \frac{d}{2} \sin(\theta/2).
\end{align*}
\]

Referring [2], we can write the following representation of a unit dual quaternion

\[
Z = \cos\left( \frac{\theta + \varepsilon d}{2} \right) + (\mathbf{l} + \varepsilon \mathbf{m}) \sin\left( \frac{\theta + \varepsilon d}{2} \right) = \cos(\phi) + \mathbf{v} \sin(\phi),
\]

where \( \mathbf{v} = \mathbf{l} + \varepsilon \mathbf{m} \) and \( \phi = \frac{\theta + \varepsilon d}{2} \). By the properties of trigonometric functions, we have the representation of a unit dual quaternion

\[
Z = \rho \cos(\phi) + \mathbf{v} \rho \sin(\phi)
\]

\[
= \cos\left( \frac{\theta}{2} \right) \cos\left( \frac{\varepsilon d}{2} \right) - \sin\left( \frac{\theta}{2} \right) \sin\left( \frac{\varepsilon d}{2} \right)
\]

\[
+ \mathbf{v} \left\{ \sin\left( \frac{\theta}{2} \right) \cos\left( \frac{\varepsilon d}{2} \right) + \cos\left( \frac{\theta}{2} \right) \sin\left( \frac{\varepsilon d}{2} \right) \right\}.
\]

From the representation of a Taylor series, since \( \cos\left( \frac{\varepsilon d}{2} \right) = 1 \) and \( \sin\left( \frac{\varepsilon d}{2} \right) = \varepsilon d \), we have

\[
Z = \cos\left( \frac{\theta}{2} \right) - \sin\left( \frac{\theta}{2} \right) \varepsilon d + \mathbf{v} \left\{ \sin\left( \frac{\theta}{2} \right) + \cos\left( \frac{\theta}{2} \right) \varepsilon d \right\},
\]

where \( \rho = |Z|^2 \). From the equations (2.1), we have

\[
Z = \text{Sc}(p_1) + \varepsilon \text{Sc}(p_2) + \mathbf{v} \left\{ -\frac{d}{2} \text{Sc}(p_2) + \varepsilon \text{Sc}(p_1) \frac{d}{2} \right\} = p + \mathbf{v} \mathbf{q},
\]
where
\[ p = Sc(p_1) + \varepsilon Sc(p_2), \quad q = -\frac{2}{d} Sc(p_2) + \varepsilon Sc(p_1) \frac{d}{2}. \]

Since we have
\[ \mathbf{v}^2 = I^2 = -1, \]
we obtain a corresponding Euler’s formula for a unit dual quaternion:
\[ \exp(\mathbf{v}\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{v}\phi)^n = \cos(\phi) + \mathbf{v} \sin(\phi). \]

**Proposition 2.1.** For any unit dual quaternion, we have
1. \( \exp(\mathbf{v}\phi_1) \exp(\mathbf{v}\phi_2) = \exp(\mathbf{v}(\phi_1 + \phi_2)) \),
2. \( \frac{\exp(\mathbf{v}\phi_1)}{\exp(\mathbf{v}\phi_2)} = \exp(\mathbf{v}(\phi_1 - \phi_2)). \)

**Proof.** From the corresponding Euler’s formula for a dual quaternion, we have
\[
\begin{align*}
\exp(\mathbf{v}\phi_1) \exp(\mathbf{v}\phi_2) &= \{\cos(\phi_1) + \mathbf{v} \sin(\phi_1)\} \{\cos(\phi_2) + \mathbf{v} \sin(\phi_2)\} \\
&= \cos(\phi_1 + \phi_2) + \mathbf{v} \sin(\phi_1 + \phi_2) \\
&= \exp(\mathbf{v}(\phi_1 + \phi_2))
\end{align*}
\]
and
\[
\begin{align*}
\frac{\exp(\mathbf{v}\phi_1)}{\exp(\mathbf{v}\phi_2)} &= \{\cos(\phi_1) + \mathbf{v} \sin(\phi_1)\} \{\cos(\phi_2) - \mathbf{v} \sin(\phi_2)\} \\
&= \cos(\phi_1 - \phi_2) + \mathbf{v} \sin(\phi_1 - \phi_2) \\
&= \exp(\mathbf{v}(\phi_1 - \phi_2)).
\end{align*}
\]
Therefore, we obtain the results. \( \square \)

**Proposition 2.2.** Let \( Z = \cos(\phi) + \mathbf{v} \sin(\phi) \) be a unit dual quaternion. Then we have
\[
(2.3) \quad Z^n = (\cos(\phi) + \mathbf{v} \sin(\phi))^n = \cos(n\phi) + \mathbf{v} \sin(n\phi)
\]
for all integer \( n \).

**Proof.** From the induction for integers \( n \), the equation (2.3) is obtained. \( \square \)

### 3. Hyperholomorphic function in dual quaternions

Let \( \Omega \) be a bounded open set in \( \mathbb{H}^2 \). A function \( F \) is given by
\[
F : \Omega \to \mathbb{D}_q; \quad F(Z) = f_1(p_1, p_2) + \varepsilon f_2(p_1, p_2),
\]
where
\[
\begin{align*}
f_1 &= g_1(z_1, z_2, w_1, w_2) + g_2(z_1, z_2, w_1, w_2) j \\
f_2 &= h_1(z_1, z_2, w_1, w_2) + h_2(z_1, z_2, w_1, w_2) j
\end{align*}
\]
are quaternionic functions, \( g_r \) and \( h_r \) \((r = 1, 2)\) are complex-valued functions.
Definition. A function $F$ is said to be hyperholomorphic on $\Omega = D \cap L$, where $D$ is an open subset of $\mathbb{D}_q$ and $L = \mathbb{H} \setminus \{0\} + \varepsilon \mathbb{H}$, with values in $\mathbb{D}_q$ if the limit \begin{equation}
abla \frac{dF(Z)}{dZ} := \lim_{\zeta \to 0} \{ F(Z + \zeta) - F(Z) \} \zeta^{-1} = \lim_{\zeta \to 0} \frac{[F(Z + \zeta) - F(Z)]\zeta^*}{\eta_1 \eta_1^*} \quad (\eta_1 \neq 0)
abla \end{equation}
exists, where $\zeta = \eta_1 + \varepsilon \eta_2 \to 0$ means $\eta_1 \to 0$ and $\eta_2 \to 0$ which are referred by [16].

Theorem 3.1. A function $F$ is hyperholomorphic on $\Omega$ with values in $\mathbb{D}_q$ if and only if the following conditions are held:

\begin{equation}
\lim_{\zeta \to 0} \{ F(Z + \zeta) - F(Z) \} \eta_1^{-1} = \lim_{\eta_1 \to 0, \eta_2 \to 0} \frac{[F(Z + \zeta) - F(Z)](\eta_1^* - \varepsilon \eta_2^*)}{\eta_1 \eta_1^*} \quad (\eta_1 \neq 0)
\end{equation}

Proof. Since the dual part of a dual quaternion has no inverse elements, we use the dual conjugation of $Z$ as follows:

\begin{align*}
\lim_{\zeta \to 0} \{ F(Z + \zeta) - F(Z) \} \eta_1^{-1} &= \lim_{\eta_1 \to 0, \eta_2 \to 0} \frac{[F(Z + \zeta) - F(Z)](\eta_1^* - \varepsilon \eta_2^*)}{\eta_1 \eta_1^*} \\
&= \lim_{\eta_1 \to 0, \eta_2 \to 0} \{ f_1(\eta_1 + \eta_1, \eta_2 + \eta_2) - f_1(\eta_1, \eta_2) \} \eta_1^{-1} \left( \eta_2 \eta_1^{-1} \right)^2.
\end{align*}

For the existence of the above limit, the limit
\begin{equation}
\lim_{\eta_1 \to 0, \eta_2 \to 0} \{ f_1(\eta_1 + \eta_1, \eta_2 + \eta_2) - f_1(\eta_1, \eta_2) \} \eta_1^{-1}
\end{equation}
has to be independent to $(\eta_2 \eta_1^{-1})^2$. Thus, we obtain the following equation:
\begin{equation}
\lim_{\eta_1 \to 0, \eta_2 \to 0} \{ f_1(\eta_1 + \eta_1, \eta_2 + \eta_2) - f_1(\eta_1, \eta_2) \} \eta_1^{-1} = 0.
\end{equation}

Conversely, if the conditions (3.5) are satisfied for the function $F$, then the limit
\begin{equation}
\lim_{\zeta \to 0} \{ F(Z + \zeta) - F(Z) \} \zeta^{-1}
\end{equation}
exists. From the definition of a hyperholomorphic function in $\mathbb{D}_q$, the function $F$ is hyperholomorphic. \hfill \Box

We give the left differential operators in $\mathbb{D}_q$:
\begin{align*}
D_1 := \frac{\partial}{\partial z_1} - j \frac{\partial}{\partial z_2} \quad \text{and} \quad D_1^* = \frac{\partial}{\partial \bar{z}_1} + j \frac{\partial}{\partial \bar{z}_2},
\end{align*}
where $\frac{\partial}{\partial z_r}$ and $\frac{\partial}{\partial y_r}$ ($r = 1, 2$) are usual complex differential operators and $j$ is an imaginary basis element in $\mathbb{H}$.

**Remark 3.2.** From the representation of differential operators in $\mathbb{D}_q$, we have

$$F D_1 = \{g_1 + g_2 j + \varepsilon (h_1 + h_2 j)\} \left(\frac{\partial}{\partial z_1} - j \frac{\partial}{\partial z_2}\right)$$

$$= \left\{\frac{\partial g_1}{\partial z_1} + \frac{\partial g_2}{\partial z_2} + \left(\frac{\partial g_2}{\partial z_1} - \frac{\partial g_1}{\partial z_2}\right) j\right\} + \varepsilon \left\{\frac{\partial h_1}{\partial z_1} + \frac{\partial h_2}{\partial z_2} + \left(\frac{\partial h_2}{\partial z_1} - \frac{\partial h_1}{\partial z_2}\right) j\right\},$$

and

$$F D_1^* = \{g_1 + g_2 j + \varepsilon (h_1 + h_2 j)\} \left(\frac{\partial}{\partial z_1} + j \frac{\partial}{\partial z_2}\right)$$

$$= \left\{\frac{\partial g_1}{\partial z_1} - \frac{\partial g_2}{\partial z_2} + \left(\frac{\partial g_2}{\partial z_1} + \frac{\partial g_1}{\partial z_2}\right) j\right\} + \varepsilon \left\{-\frac{\partial h_2}{\partial z_2} + \frac{\partial h_1}{\partial z_1} + \left(\frac{\partial h_1}{\partial z_2} + \frac{\partial h_2}{\partial z_1}\right) j\right\}.$$
Proof. From the definition of the hyperholomorphic function \( F \) on \( \Omega \) with values in \( D_q \) and Definition 3, if the following limit

\[
\lim_{\lambda_1, \lambda_2 \to 0} \frac{\text{Sc}(f_1) + \varepsilon \text{Sc}(f_2) + v\left(\frac{\varepsilon}{d} \text{Sc}(f_2) + \varepsilon \frac{d}{2} \text{Sc}(f_1)\right)}{\lambda_1 + \varepsilon \lambda_2 + v\left(\frac{\varepsilon}{d} \lambda_2 + \varepsilon \frac{d}{2} \lambda_1\right)}
\]

exists, then the function \( F \) is hyperholomorphic, where \( \lambda_1 = \text{Sc}(\eta_1) \) and \( \lambda_2 = \text{Sc}(\eta_2) \). By the definition of the existence of the limit and calculating of the complex conjugation of a dual quaternion, we have

\[
\frac{\partial \text{Sc}(f_1)(1 + \varepsilon \frac{d}{2}) + \partial \text{Sc}(f_2)(\varepsilon - \frac{2}{d}v)}{\partial \text{Sc}(p_2)(\varepsilon - \frac{d}{2}v)} \cdot \frac{\partial \text{Sc}(f_1)(1 + \varepsilon \frac{d}{2}) + \partial \text{Sc}(f_2)(\varepsilon - \frac{2}{d}v)}{\partial \text{Sc}(p_2)(\varepsilon - \frac{d}{2}v)}
\]

By arranging the above equation, we have

\[
\left\{ \frac{\partial \text{Sc}(f_1)}{\partial \text{Sc}(p_1)} - \frac{\partial \text{Sc}(f_2)}{\partial \text{Sc}(p_2)} \right\} \left(2\varepsilon - \frac{2}{d}v\right) + \left\{ \frac{\partial \text{Sc}(f_1)}{\partial \text{Sc}(p_2)} - \frac{4}{d^2} \frac{\partial \text{Sc}(f_2)}{\partial \text{Sc}(p_1)} \right\}(1 + \varepsilon d) = 0.
\]

Therefore, we obtain the equations (3.7).

**Theorem 3.4.** Let \( \Omega \) be a bounded open set in \( \mathbb{H}^2 \) and a function \( F = M + vN \) be hyperholomorphic on \( \Omega \) with values in \( D_q \). Then the following equations hold:

\[
\rho \left(1 + \varepsilon\right) \frac{\partial M}{\partial \rho} = \frac{\partial N}{\partial \phi} \quad \text{and} \quad \rho \left(1 + \varepsilon\right) \frac{\partial N}{\partial \rho} = -\frac{\partial M}{\partial \phi}.
\]

**Proof.** From the chain rule of multi variables calculus, we have

\[
\frac{\partial M}{\partial \rho} = \frac{\partial p}{\partial \rho} \frac{\partial M}{\partial p} + \frac{\partial q}{\partial \rho} \frac{\partial M}{\partial q}, \quad \frac{\partial M}{\partial \phi} = -\rho \left(1 + \varepsilon\right) \sin \phi \frac{\partial M}{\partial p} + \rho \left(1 + \varepsilon\right) \cos \phi \frac{\partial M}{\partial q},
\]

\[
\frac{\partial N}{\partial \rho} = \frac{\partial p}{\partial \rho} \frac{\partial N}{\partial p} + \frac{\partial q}{\partial \rho} \frac{\partial N}{\partial q}, \quad \frac{\partial N}{\partial \phi} = \rho \left(1 + \varepsilon\right) \cos \phi \frac{\partial N}{\partial p} + \rho \left(1 + \varepsilon\right) \sin \phi \frac{\partial N}{\partial q}.
\]

Since we have the following equations:

\[
\frac{\partial p}{\partial \phi} = \cos \phi, \quad \frac{\partial q}{\partial \phi} = \sin \phi, \quad \frac{\partial p}{\partial \phi} = -\rho \left(1 + \varepsilon\right) \sin \phi, \quad \frac{\partial q}{\partial \phi} = \rho \left(1 + \varepsilon\right) \cos \phi,
\]

we have

\[
\frac{\partial M}{\partial \rho} = \cos \phi \frac{\partial M}{\partial p} + \sin \phi \frac{\partial M}{\partial q},
\]

\[
\frac{\partial M}{\partial \phi} = -\rho \sin \phi \left(1 + \varepsilon\right) \frac{\partial M}{\partial p} + \rho \cos \phi \left(1 + \varepsilon\right) \frac{\partial M}{\partial q},
\]

\[
\frac{\partial N}{\partial \rho} = \cos \phi \frac{\partial N}{\partial p} + \sin \phi \frac{\partial N}{\partial q}.
\]
\[
\frac{\partial N}{\partial \phi} = -\rho \sin \phi (1 + \varepsilon) \frac{\partial N}{\partial p} + \rho \cos \phi (1 + \varepsilon) \frac{\partial N}{\partial q},
\]

where
\[
\begin{align*}
\frac{\partial M}{\partial p} &= \frac{\partial \text{Sc}(f_1)}{\partial \text{Sc}(p_1)} + \varepsilon \frac{\partial \text{Sc}(f_2)}{\partial \text{Sc}(p_1)} + \frac{\partial \text{Sc}(p_2)}{\partial p} \left\{ \frac{\partial \text{Sc}(f_1)}{\partial \text{Sc}(p_2)} + \varepsilon \frac{\partial \text{Sc}(f_2)}{\partial \text{Sc}(p_2)} \right\}, \\
\frac{\partial M}{\partial q} &= \frac{\partial \text{Sc}(p_1)}{\partial \text{q}} \left\{ \frac{\partial \text{Sc}(f_1)}{\partial \text{Sc}(p_1)} + \varepsilon \frac{\partial \text{Sc}(f_2)}{\partial \text{Sc}(p_1)} \right\} - \frac{1}{2} \left\{ \frac{\partial \text{Sc}(f_1)}{\partial \text{Sc}(p_2)} + \varepsilon \frac{\partial \text{Sc}(f_2)}{\partial \text{Sc}(p_2)} \right\} , \\
\frac{\partial N}{\partial p} &= -2 \frac{\partial \text{Sc}(f_2)}{\partial \text{Sc}(p_1)} + \varepsilon \frac{d \partial \text{Sc}(f_1)}{\partial \text{Sc}(p_1)} + \frac{d \partial \text{Sc}(p_2)}{\partial p} \left\{ -2 \frac{\partial \text{Sc}(f_2)}{\partial \text{Sc}(p_2)} + \varepsilon \frac{d \partial \text{Sc}(f_1)}{\partial \text{Sc}(p_2)} \right\} , \\
\frac{\partial N}{\partial q} &= \frac{\partial \text{Sc}(p_1)}{\partial \text{q}} \left\{ -2 \frac{\partial \text{Sc}(f_2)}{\partial \text{Sc}(p_1)} + \varepsilon \frac{d \partial \text{Sc}(f_1)}{\partial \text{Sc}(p_1)} \right\} - \frac{d}{2} \left\{ -2 \frac{\partial \text{Sc}(f_2)}{\partial \text{Sc}(p_2)} + \varepsilon \frac{d \partial \text{Sc}(f_1)}{\partial \text{Sc}(p_2)} \right\} .
\end{align*}
\]

From Theorem 3.3, we have the following equations by comparing with the equations (3.7) and the derivative of \( \text{Sc}(f_1) \) and \( \text{Sc}(f_2) \) for \( \text{Sc}(p_1) \) and \( \text{Sc}(p_2) \):
\[
\frac{\partial M}{\partial q} = \frac{\partial N}{\partial p} \quad \text{and} \quad \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}.
\]

Therefore, the equations (3.8) are obtained. \( \square \)

**Example 3.5.** Let \( F(Z) = Z = \rho \cos \phi + \nu \rho \sin \phi \) on \( \Omega \) in \( \mathbb{H}^2 \). Then we have
\[
\frac{\partial M}{\partial \phi} = \frac{\partial (\rho \sin \phi)}{\partial \phi} = \frac{1}{2} \rho \cos \phi + \varepsilon \frac{\rho}{2} \cos \phi ,
\]
\[
\frac{\partial N}{\partial \phi} = \cos \phi , \quad \frac{\partial N}{\partial p} = \cos \phi ,
\]
and
\[
\frac{\partial M}{\partial \phi} = \frac{\partial (\rho \cos \phi)}{\partial \phi} = -\frac{1}{2} \rho \sin \phi - \frac{\varepsilon}{2} \rho \sin \phi .
\]

Therefore, the function \( F \) satisfies the equations (3.8).

**References**


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