Design of ramp-stress accelerated life test plans for a parallel system with two independent components using masked data

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Abstract: In this paper, we have formulated optimum Accelerated Life Test (ALT) plan for a parallel system with two independent components using masked data with ramp-stress loading scheme and Type-I censoring. Consider a system of two independent and non-identical components connected in parallel. Such a system fails whenever all of its components has failed. The exact component that causes the system to fail is often unknown due to cost and time constraint. For each parallel system at test, we observe its system's failure time and a set of component that includes the component actually causing the system to fail. The stress-life relationship is modelled using inverse power law, and cumulative exposure model is assumed to model the effect of changing stress. The optimal plan consists in finding out the optimum stress rate using D-optimality criterion. The method developed has been explained using a numerical example and sensitivity analysis carried out.

Key Words: Accelerated Life Test, D-optimality criterion, Fisher information matrix, Masked data, Ramp stress

1. INTRODUCTION AND MOTIVATION

Accelerated Life Tests help in inducing early failures in high reliability items that are likely to last for several years; by subjecting them to higher than normal stress conditions, say higher temperature, voltage, pressure etc.. Stress under accelerated condition can be applied using constant-stress, step-stress, progressive-stress, cyclic stress, random-stress, or combinations of such loadings. The choice of a stress loading depends on how the product or unit is used in service and other practical and theoretical limitations (Nelson (1980), Elsayed (1996)). Life tests under accelerated environmental conditions may be fully accelerated or partially accelerated. In fully accelerated life testing all the test units are run at accelerated condition, while in partially accelerated life testing they are run at

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both normal and accelerated conditions. The term fully accelerated life test has been coined by Bhattacharyya and Soejoeti (1989) and the term partially accelerated life test is due to DeGroot and Goel (1979). The terms accelerated life tests and fully accelerated life tests are used interchangeably in the literature (Nelson (1990)) has discussed the statistical models, test plans and methods of data analyses for the ALT. Design of accelerated life tests plans under different stress loading schemes has been studied extensively in the literature. See for example, Srivastava and Gupta (2015) and the references therein. However, all these tests are suitable for one unit or component or subsystem of a system.

Srivastava and Savita (2016) have described the optimal accelerated life test plan for a parallel system with two dependent components under ramp-stress loading using time-censored data, and the optimal plan consists in finding out the optimum stress rate using D-optimality criterion. Often, due to certain environmental conditions, the exact cause of system failure might be unknown. Instead, it can be ascertained that the cause of system failure is due to the component which belongs to some subset of the J components of the system. Such type of observation is referred to as being masked, see Usher and Hodgson (1998). Estimation of component reliability in a parallel system using masked life data has been studied by Sarhan and Ei-Bassiouny (2003). The method developed has been explained using a numerical example and sensitivity analysis carried out.

**Acronyms**

- ALT: accelerated life test
- Avar: asymptotic variance
- ML: maximum likelihood
- Cdf: cumulative distribution function
- Pdf: probability distribution function
- GAV: generalized asymptotic variance
- IWD: Inverse Weibull distribution

**Notation**

- \( n \): Total number of test units
- \( S_i \): Set of component that may cause the system failure
- \( T_i \): The life time of system \( i \)
- \( T_{ij} \): The life time of component \( j \) in system \( i \)
- \( f_j(t) \): probability distribution function of component \( j, j=1,2 \)
- \( F_j(t) \): cumulative distribution function of component \( j, j=1,2 \)
- \( ^\wedge \): Maximum Likelihood estimate
- \( \varepsilon(t) \): Cumulative Exposure at time \( t \)
- \( \gamma_0, \gamma_1 \): Parameters of the inverse power law, \( \gamma_1 > 0, \ -\infty < \gamma_0 < \infty \)
- \( n_1 \): The number of system failures for which the component 1 where
2. Reliability PARALLEL STRUCTURE AND ITS RELIABILITY

2.1 Parallel System Configuration
Consider a parallel system that consists of $J$ independent but non-identical components. This system will fail if all of its components failed. The graphical representation of a parallel system with two components also referred to as a parallel system of order 2 is described in Figure 1.

![Figure 1. Parallel System](image)

2.2 Reliability Function of a Parallel System of order 2
The life time of $i^{th}$ parallel system, $i = 1, 2, ..., n$, with two independent and identical components is:

$$T_i = \max(T_{i1}, T_{i2}),$$

(1)
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where $T_{11}$ and $T_{12}$ represent life times of its components.

The reliability of a parallel system of order 2 is:

$$F_0(t) = 1 - F_1(t)F_2(t).$$

(2)

3. THE MODEL

Assumptions

The following assumptions are made:

a) The system consists of two independent components connected in parallel.

b) It is assumed that the lifetimes of the components have a Weibull distribution with the failure density:

$$g_X(x; \eta, \beta) = \frac{\beta}{\eta} \left( \frac{x}{\eta} \right)^{\beta-1} \exp\left(-\left(\frac{x}{\eta}\right)^{\beta}\right).$$

Here $\beta > 0$ and $\eta > 0$ are the shape and scale parameters. In other words, $X \sim WEI(\beta, \eta)$

c) There are no any two components failing at the same time.

d) Failed parallel systems are not replaced during the test.

e) The occurrence of Masking is independent of the failure cause and time.

f) The cumulative exposure model ([5]) holds for the effect of changing stress.

g) The inverse power law holds for stress-life relationship, i.e.,

$$\eta(s(t)) = e^{\gamma_0} \left( \frac{s_0}{s(t)} \right)^{\gamma_1},$$

where parameters $\gamma_0$ and $\gamma_1$ are the characteristics of the product, $s(t)$ is a linear function of time in ramp-stress, $s(t) = kt$, where $k$ is the stress rate.

h) The stress applied to test units is continuously increased with constant rate $k$, from zero.

3.1 Inverse Weibull (IW) Distribution Under Ramp-Stress

The random variable $T = \frac{\beta^2}{X}$ has the IW distribution. The life distribution and failure density of $T$ obtained using (3) are:

$$G_T(t; \eta, \beta) = e^{-\left(\frac{t}{\eta}\right)^{\beta}},$$

and

$$g_T(t; \alpha, \beta) = \frac{\beta}{\eta} \left( \frac{t}{\eta} \right)^{-(\beta+1)} \exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right),$$

respectively. Since the components of a parallel system are assumed to be independent, therefore the joint life distribution of the $i^{th}$ system is given by:
The joint pdf of \( i \)th system is given as:

\[
g_{T_{i1}, T_{i2}}(t_{1i}, t_{2i}) = \frac{\beta_i}{\eta_1} \left( \frac{t_{1i}}{\eta_1} \right)^{(\beta_{i1} + 1)} \left( \frac{t_{2i}}{\eta_2} \right)^{(\beta_{i2} + 1)} e^{-\left( \frac{t_{1i}}{\eta_1} \right)^{\beta_{i1}} - \left( \frac{t_{2i}}{\eta_2} \right)^{\beta_{i2}}}.
\]  

The joint cdf and pdf of the lifetime of \( i \)th system under ramp-stress loading are:

\[
F_{T_{i1}, T_{i2}}(t_{1i}, t_{2i}) = G_{T_{i1}, T_{i2}}(e(t_{1i}), e(t_{2i}))
\]

where \( G(\cdot, \cdot) \) is the assumed joint inverse cumulative Weibull life distribution function with scale parameter set equal to one, and

\[
e(t) = \int_0^t \frac{1}{\eta(s(u))} \, du
\]

represents the cumulative exposure (damage) model at \( t \).

Hence, the cdf and pdf of the lifetime of the \( i \)th system under ramp-stress loading are:

\[
F_{T_{i1}, T_{i2}}(t_{1i}, t_{2i}) = \exp\left[ -\left( \frac{t_{1i}}{\alpha_1} \right)^{\beta_{i1}} - \left( \frac{t_{2i}}{\alpha_1} \right)^{\beta_{i2}} \right],
\]

\[
f_{T_{i1}, T_{i2}}(t_{1i}, t_{2i}) = \frac{\beta_{i1}}{\alpha_1} \left( \frac{t_{1i}}{\alpha_1} \right)^{\beta_{i1} - 1} \exp\left( -\left( \frac{t_{1i}}{\alpha_1} \right)^{\beta_{i1}} \right)
\]

respectively, where \( \alpha_1 = (1 + \gamma_1) \) is the scale parameter, \( \beta_1 = \beta_i (1 + \gamma_1) \) and \( i_2 = \frac{1}{2}(1 + \gamma_1) \) are the shape parameters.
The test procedure consists in putting 'n' independent and identical 2-component parallel systems (parallel systems of order 2) to ramp-stress accelerated test using time-censored (Type-I censored) data. The two components are independent. The log-likelihood function of n parallel systems of order 2 is obtained as follows:

The system will fail if all of its components fail. It means that the system fails at time $t_i$ if one of its components fails at $t_i$ and the rest of the components failed before the time $t_i$. Hence the pdf of lifetime $T_i$ of $i^{th}$ system when a component $j$, $j = 1, 2$, in the system fails at time $t_i$ and the rest of the components failed before $t_i$ is obtained as:

$$P[T_i \leq t_i | T_i \leq t_i + \Delta t_i] = \lim_{\Delta t_i \to 0} \frac{P[T_i \leq t_i]}{\Delta t_i}$$

where $j \neq k$. Since the components are independent, therefore

$$P[T_i \leq t_i, T_i \leq t_i + \Delta t_i] = \prod_{j=1}^{2} P[T_i \leq t_i | T_i \leq t_i + \Delta t_i]$$

Thus,

$$P[T_i \leq t_i, T_i \leq t_i + \Delta t_i] = \prod_{j=1}^{2} P[T_i \leq t_i | T_i \leq t_i + \Delta t_i]$$

Using (15), the likelihood function of n parallel systems is:

$$L = \prod_{i,S_1=[1]}^{n_1} f_{i1}(t_i) F_{i1}(t_i) \prod_{i,S_1=[1]}^{n_1} f_{i2}(t_i) F_{i2}(t_i) \prod_{i,S_1=[1]}^{n_2} \left\{ f_{i1}(t_i) F_{i1}(t_i) + f_{i2}(t_i) F_{i2}(t_i) \right\}$$

$$L = \prod_{i,S_1=[1]}^{n_1} \left( \frac{\beta_{i1}}{\alpha_1} \left( \frac{t_i}{\alpha_1} \right) \right)^{-\beta_{i1}} e^{-\frac{t_i}{\alpha_1}} \prod_{j=1}^{n_2} \left( \frac{\beta_{j1}}{\alpha_1} \left( \frac{t_i}{\alpha_1} \right) \right)^{-\beta_{j1}} e^{-\frac{t_i}{\alpha_1}}$$

$$= \prod_{j=1}^{n_1} \left( \frac{\beta_{i1}}{\alpha_1} \left( \frac{t_i}{\alpha_1} \right) \right)^{-\beta_{i1}} e^{-\frac{t_i}{\alpha_1}} \prod_{j=1}^{n_2} \left( \frac{\beta_{j1}}{\alpha_1} \left( \frac{t_i}{\alpha_1} \right) \right)^{-\beta_{j1}} e^{-\frac{t_i}{\alpha_1}}$$

$$\Rightarrow \log\text{-likelihood function is:}$$

$$= \sum_{i=1}^{n_1} \frac{t_i}{\alpha_1} - \sum_{j=1}^{n_2} \frac{t_i}{\alpha_1}$$
\[
\log L = \frac{n_1}{\log(1)} \log \left( \frac{\beta_1(t_i)}{\alpha_1} \right)^{(\beta_1+1)} - \frac{2}{\sum_{j=1}^{n_2} \frac{t_j}{\alpha_1}} \beta_{ij} \right) + \frac{n_2}{\log(1)} \log \left( \frac{\beta_2(t_i)}{\alpha_1} \right)^{(\beta_2+1)} - \frac{2}{\sum_{j=1}^{n_2} \frac{t_j}{\alpha_1}} \beta_{ij} \right) \\
+ \frac{n_1}{\log(1)} \log \left( \frac{\beta_1(t_i)}{\alpha_1} \right)^{(\beta_1+1)} - \frac{2}{\sum_{j=1}^{n_2} \frac{t_j}{\alpha_1}} \beta_{ij} \right) ,
\]

where \( n = n_1 + n_2 + n_12 \) and \( n \) is fixed.

3.3 Parameter Estimation
The MLEs of \( \gamma_0, \gamma_1, \beta_1 \) and \( \beta_2 \) are obtained by using \( \textit{NMaximize} \) option of \textit{Mathematica} 10.0.

3.4 Fisher Information Matrix
It is the \( 4 \times 4 \) symmetric matrix of expectation of negative second order partial derivatives of the log likelihood function with respect to \( \gamma_0, \gamma_1, \beta_1 \) and \( \beta_2 \).

Thus, the Fisher information matrix \( F \) is:

\[
F = F(\gamma_0, \gamma_1, \beta_1, \beta_2) = \\
\begin{bmatrix}
E\left(\frac{\partial^2 L}{\partial \gamma_0^2}\right) & E\left(\frac{\partial^2 L}{\partial \gamma_0 \partial \gamma_1}\right) & E\left(\frac{\partial^2 L}{\partial \gamma_0 \partial \beta_1}\right) & E\left(\frac{\partial^2 L}{\partial \gamma_0 \partial \beta_2}\right) \\
E\left(\frac{\partial^2 L}{\partial \gamma_1 \partial \gamma_0}\right) & E\left(\frac{\partial^2 L}{\partial \gamma_1^2}\right) & E\left(\frac{\partial^2 L}{\partial \gamma_1 \partial \beta_1}\right) & E\left(\frac{\partial^2 L}{\partial \gamma_1 \partial \beta_2}\right) \\
E\left(\frac{\partial^2 L}{\partial \beta_1 \partial \gamma_0}\right) & E\left(\frac{\partial^2 L}{\partial \beta_1 \partial \gamma_1}\right) & E\left(\frac{\partial^2 L}{\partial \beta_1^2}\right) & E\left(\frac{\partial^2 L}{\partial \beta_1 \partial \beta_2}\right) \\
E\left(\frac{\partial^2 L}{\partial \beta_2 \partial \gamma_0}\right) & E\left(\frac{\partial^2 L}{\partial \beta_2 \partial \gamma_1}\right) & E\left(\frac{\partial^2 L}{\partial \beta_2 \partial \beta_1}\right) & E\left(\frac{\partial^2 L}{\partial \beta_2^2}\right)
\end{bmatrix}
\]

The calculation of the element of the \( F \) in (18) is shown in the Appendix. These elements have been obtained using the following results:
(i) \( n_i \sim \text{Bin} (n, p_i) \),

where using \( (15) \)

\[
p_i = \int_0^\infty f_{1i}(t_i)F_{12}(t_i)dt_i \\
\Rightarrow E[n_i] = n \int_0^\infty f_1(t_i)F_2(t_i)dt_i
\]
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(ii) \( n_2 \sim \text{Bin}(n, p_2) \),
where using (15)
\[
p_2 = \int_0^{\infty} f_{i2}(t_i) F_i(t_i) \, dt_i .
\]
\[\implies E[n_2] = n \int_0^{\infty} f_2(t_i) F_i(t_i) \, dt_i . \tag{20}\]

(iii) \( n_{12} \sim \text{Bin}(n, p_{12}) \),
where using (15)
\[
p_{12} = \int_0^{\infty} \left(f_{i1}(t_i) F_{i2}(t_i) + f_{i2}(t_i) F_{i1}(t_i)\right) \, dt_i .
\]
\[\implies E[n_{12}] = n \int_0^{\infty} \left(f_1(t_i) F_2(t_i) + f_2(t_i) F_1(t_i)\right) \, dt_i . \tag{21}\]

Since for some set of parameters \( \{ \gamma_0, \gamma_1, s_0, \beta_1, \beta_2, k \} \); \( |F| \) or variance function may be negative, therefore, we choose only that parametric set for which \( |F| > 0 \) and variance function is positive.

4. OPTIMUM PLAN

The optimal plan consists in finding the optimal stress rate using D-optimality criterion wherein the reciprocal of the determinant of \( F \) is minimized. A smaller value of the determinant would correspond to a higher (joint) precision of the estimators of parameters (Escobar and Meeker (1995)).

Thus, the optimization problem is:

\[
\text{Minimize } \frac{1}{|F|}
\]

subject to \( k > 0 \).

Optimal \( k \) is obtained using the \texttt{NMinimize} option of \texttt{Mathematica 10.0}.

5. A NUMERICAL ILLUSTRATION AND SENSITIVITY ANALYSIS

5.1 A Numerical Example

In this section, a hypothetical ramp-stress ALT experiment for a parallel system with two independent components is considered to illustrate the method described in this paper with the following data set:

\( n = 30, s_0 = 10, \gamma_0 = -1.86, \gamma_1 = 0.55, \beta_1 = 2.12, \beta_2 = 1.12 \).

The optimal value of \( k \) is obtained as \( k^* = 0.316 \).
5.2 Sensitivity Analysis

To use an optimum test plan, one needs estimates of the design parameters $\gamma_0, \gamma_1, \beta_1$ and $\beta_2$. These estimates sometimes may significantly affect the values of the resulting decision variables; therefore, their incorrect choice may give a poor estimate of the quantile at design constant stress. Hence, it is important to conduct sensitivity analysis to evaluate the robustness of the resulting ALT plan.

The percentage deviations of the optimal settings are measured by

$$PD = \left( \frac{|Z^{**} - Z^*|}{Z^*} \right) \times 100,$$

where $Z^*$ is the setting obtained with the given design parameters, and $Z^{**}$ is the one obtained when the parameter is mis-specified. Table 1 shows the optimal test plans for various deviations from the design parameter estimates. The results show that the optimal setting of $Z$ is robust to the small deviations from baseline parameter estimates.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% change</th>
<th>$k$</th>
<th>$Z^{**}$</th>
<th>PD %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>+1%</td>
<td>0.319522</td>
<td>0.0000657636</td>
<td>0.950</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-1%</td>
<td>0.312967</td>
<td>0.0000644979</td>
<td>0.992</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>+1%</td>
<td>0.316253</td>
<td>0.0000644245</td>
<td>1.104</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-1%</td>
<td>0.316264</td>
<td>0.000065223</td>
<td>0.120</td>
</tr>
<tr>
<td>$\gamma_0$</td>
<td>+1%</td>
<td>0.305744</td>
<td>0.0000651442</td>
<td>0</td>
</tr>
<tr>
<td>$\gamma_0$</td>
<td>-1%</td>
<td>0.328012</td>
<td>0.0000651448</td>
<td>0.001</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>+1%</td>
<td>0.329534</td>
<td>0.000065594</td>
<td>0.690</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>-1%</td>
<td>0.303283</td>
<td>0.0000646995</td>
<td>0.683</td>
</tr>
</tbody>
</table>

6. CONCLUSION

Ramp-stress ALT is characterized by stress that increases linearly with time. This paper deals with design of optimal ramp-stress ALT plan for a parallel system comprising two independent components each with Weibull lifetime. The plan is devised using masked data wherein an observation is said to be masked if the exact component that caused the system to fail is not known due to time and cost constraints. However, it can be ascertained that the cause of system failure is due to the component which belongs to some subset of the system's components. The experiment is terminated when all the test specimens put to test fail or until a pre-specified time whichever is early, i.e., the test is conducted using time-censoring (Type-I censoring) scheme. The optimal plan consists in finding optimal stress rate using D-optimality criterion that is based on minimization of
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The reciprocal of the determinant of Fisher information matrix. The method develop has been explained using a numerical example and sensitivity analysis carried out. The results of sensitivity analysis show that optimum plan is robust for small deviations in the true values of the model parameters.

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REFERENCES


APPENDIX

Calculations of derivatives of the log-likelihood and the elements of Fisher information matrix given in section 3.4 respectively, has been shown following.

The first partial derivatives are,

\[ \frac{\partial L}{\partial \beta_1} = \sum_{i=1}^{n_1} \frac{1}{\beta_{11}} \left[ (-1 + \frac{t_i}{\alpha_1}) \log \left( \frac{t_i}{\alpha_1} \right) + \frac{n_2}{\alpha_1} \right] \log \left( \frac{t_i}{\alpha_1} \right) 
+ \sum_{i=1}^{n_2} \frac{t_i}{\alpha_1} \beta_{11} \beta_{12} \beta_{11} + \frac{t_i}{\alpha_1} \beta_{12} \beta_{11} \beta_{12} \log \left( \frac{t_i}{\alpha_1} \right) \] (A.1)

\[ \frac{\partial L}{\partial \beta_2} = \sum_{i=1}^{n_1} \frac{1}{\beta_{12}} \left[ (-1 + \frac{t_i}{\alpha_1}) \log \left( \frac{t_i}{\alpha_1} \right) + \frac{n_2}{\alpha_1} \right] \log \left( \frac{t_i}{\alpha_1} \right) 
+ \sum_{i=1}^{n_2} \frac{t_i}{\alpha_1} \beta_{11} \beta_{12} \beta_{12} \beta_{11} \log \left( \frac{t_i}{\alpha_1} \right) \] (A.2)

\[ \frac{\partial L}{\partial \gamma_1} = \sum_{i=1}^{n_1} \left( m_1(t_i) + m_2(t_i) \cdot m_3(t_i) \right) + \sum_{i=1}^{n_2} \left( m_4(t_i) + m_5(t_i) \cdot m_6(t_i) \right) \] (A.3)

\[ \frac{\partial L}{\partial \gamma_0} = \sum_{i=1}^{n_1} m_1(t_i) + \sum_{i=1}^{n_2} m_3(t_i) \] (A.4)

The likelihood equations are obtained by setting (A.1) – (A.4) to zero. The parameter values that solve “these equations summed over all test units” are the Maximum Likelihood estimates. As the system of likelihood equations has no closed form solution in \( \gamma_0, \gamma_1, \beta_1 \) and \( \beta_2 \), therefore the maximum likelihood estimates \( \gamma_0, \gamma_1, \beta_1 \) and \( \beta_2 \) are obtained by maximizing (17) using NMaximize option of Mathematica 10.

\[ \frac{\partial^2 L}{\partial \beta_1^2} = \left( \sum_{i=1}^{n_1} m_{11}(t_i) + \sum_{i=1}^{n_2} m_{12}(t_i) \right) \frac{1}{1 + \gamma_1^2}, \] (A.5)
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\[ \frac{\partial^2 L}{\partial \beta_2^2} = \sum_{i=1}^{n_1} \frac{1}{\beta_{12}} \left( \frac{1}{\alpha_1} + \frac{1}{\gamma_{11}} \right) \log \left( \frac{\beta_{12}}{\alpha_1} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{\alpha_1} \right)^2 \left( \frac{1}{\alpha_1} + \frac{1}{\gamma_{12}} \right) \log \left( \frac{\beta_{12}}{\alpha_1} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \beta_3^2} = \sum_{i=1}^{n_1} \frac{1}{\beta_{13}} \left( \frac{1}{\alpha_1} + \frac{1}{\gamma_{11}} \right) \log \left( \frac{\beta_{13}}{\alpha_1} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{\alpha_1} \right)^2 \left( \frac{1}{\alpha_1} + \frac{1}{\gamma_{12}} \right) \log \left( \frac{\beta_{13}}{\alpha_1} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \gamma_{11}^2} = \sum_{i=1}^{n_1} \left( \frac{1}{1 + \gamma_{11}} \right)^2 \left( 1 + (1 + \gamma_{11}) \log \left( \frac{\beta_{11}}{\gamma_{11}} \right) \right) \log \left( \frac{\beta_{11}}{\gamma_{11}} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{1 + \gamma_{11}} \right)^2 \left( 1 + (1 + \gamma_{11}) \log \left( \frac{\beta_{12}}{\gamma_{11}} \right) \right) \log \left( \frac{\beta_{12}}{\gamma_{11}} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \gamma_{12}^2} = \sum_{i=1}^{n_1} \left( \frac{1}{1 + \gamma_{12}} \right)^2 \left( 1 + (1 + \gamma_{12}) \log \left( \frac{\beta_{12}}{\gamma_{12}} \right) \right) \log \left( \frac{\beta_{12}}{\gamma_{12}} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{1 + \gamma_{12}} \right)^2 \left( 1 + (1 + \gamma_{12}) \log \left( \frac{\beta_{11}}{\gamma_{12}} \right) \right) \log \left( \frac{\beta_{11}}{\gamma_{12}} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \gamma_{13}^2} = \sum_{i=1}^{n_1} \left( \frac{1}{1 + \gamma_{13}} \right)^2 \left( 1 + (1 + \gamma_{13}) \log \left( \frac{\beta_{13}}{\gamma_{13}} \right) \right) \log \left( \frac{\beta_{13}}{\gamma_{13}} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{1 + \gamma_{13}} \right)^2 \left( 1 + (1 + \gamma_{13}) \log \left( \frac{\beta_{12}}{\gamma_{13}} \right) \right) \log \left( \frac{\beta_{12}}{\gamma_{13}} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \gamma_{11} \partial \gamma_{12}} = \sum_{i=1}^{n_1} \left( \frac{1}{1 + \gamma_{11}} \right) \left( 1 + (1 + \gamma_{11}) \log \left( \frac{\beta_{12}}{\gamma_{11}} \right) \right) \log \left( \frac{\beta_{12}}{\gamma_{11}} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{1 + \gamma_{11}} \right) \left( 1 + (1 + \gamma_{11}) \log \left( \frac{\beta_{11}}{\gamma_{11}} \right) \right) \log \left( \frac{\beta_{11}}{\gamma_{11}} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \gamma_{12} \partial \gamma_{13}} = \sum_{i=1}^{n_1} \left( \frac{1}{1 + \gamma_{12}} \right) \left( 1 + (1 + \gamma_{12}) \log \left( \frac{\beta_{13}}{\gamma_{12}} \right) \right) \log \left( \frac{\beta_{13}}{\gamma_{12}} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{1 + \gamma_{12}} \right) \left( 1 + (1 + \gamma_{12}) \log \left( \frac{\beta_{11}}{\gamma_{12}} \right) \right) \log \left( \frac{\beta_{11}}{\gamma_{12}} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \gamma_{11} \partial \gamma_{13}} = \sum_{i=1}^{n_1} \left( \frac{1}{1 + \gamma_{13}} \right) \left( 1 + (1 + \gamma_{13}) \log \left( \frac{\beta_{13}}{\gamma_{13}} \right) \right) \log \left( \frac{\beta_{13}}{\gamma_{13}} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{1 + \gamma_{13}} \right) \left( 1 + (1 + \gamma_{13}) \log \left( \frac{\beta_{12}}{\gamma_{13}} \right) \right) \log \left( \frac{\beta_{12}}{\gamma_{13}} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \gamma_{11} \partial \gamma_{12} \partial \gamma_{13}} = \sum_{i=1}^{n_1} \left( \frac{1}{1 + \gamma_{11}} \right)^2 \left( 1 + (1 + \gamma_{11}) \log \left( \frac{\beta_{12}}{\gamma_{11}} \right) \right) \log \left( \frac{\beta_{12}}{\gamma_{11}} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{1 + \gamma_{11}} \right)^2 \left( 1 + (1 + \gamma_{11}) \log \left( \frac{\beta_{11}}{\gamma_{11}} \right) \right) \log \left( \frac{\beta_{11}}{\gamma_{11}} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \gamma_{11} \partial \gamma_{12} \partial \gamma_{13}} = \sum_{i=1}^{n_1} \left( \frac{1}{1 + \gamma_{12}} \right)^2 \left( 1 + (1 + \gamma_{12}) \log \left( \frac{\beta_{13}}{\gamma_{12}} \right) \right) \log \left( \frac{\beta_{13}}{\gamma_{12}} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{1 + \gamma_{12}} \right)^2 \left( 1 + (1 + \gamma_{12}) \log \left( \frac{\beta_{11}}{\gamma_{12}} \right) \right) \log \left( \frac{\beta_{11}}{\gamma_{12}} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \gamma_{11} \partial \gamma_{12} \partial \gamma_{13}} = \sum_{i=1}^{n_1} \left( \frac{1}{1 + \gamma_{13}} \right)^2 \left( 1 + (1 + \gamma_{13}) \log \left( \frac{\beta_{13}}{\gamma_{13}} \right) \right) \log \left( \frac{\beta_{13}}{\gamma_{13}} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{1 + \gamma_{13}} \right)^2 \left( 1 + (1 + \gamma_{13}) \log \left( \frac{\beta_{12}}{\gamma_{13}} \right) \right) \log \left( \frac{\beta_{12}}{\gamma_{13}} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \gamma_{11} \partial \gamma_{12} \partial \gamma_{13}} = \sum_{i=1}^{n_1} \left( \frac{1}{1 + \gamma_{11}} \right)^2 \left( 1 + (1 + \gamma_{11}) \log \left( \frac{\beta_{12}}{\gamma_{11}} \right) \right) \log \left( \frac{\beta_{12}}{\gamma_{11}} \right)^2 + \sum_{i=1}^{n_2} \left( \frac{1}{1 + \gamma_{11}} \right)^2 \left( 1 + (1 + \gamma_{11}) \log \left( \frac{\beta_{11}}{\gamma_{11}} \right) \right) \log \left( \frac{\beta_{11}}{\gamma_{11}} \right)^2 \]
\[ \frac{\partial^2 L}{\partial \beta \gamma_i} = \sum_{i=1}^{n} \left( \frac{1}{1 + \gamma_i} \left( t_i \alpha_i - \log \left( \frac{S_{0}i}{k} \right) - \log \left( e^{\gamma_i} \frac{S_{0}i}{k} \right) \right) \right) + \sum_{i=1}^{n} \left( \frac{1}{1 + \gamma_i} \left( t_i \alpha_i - \log \left( \frac{S_{0}i}{k} \right) - \log \left( e^{\gamma_i} \frac{S_{0}i}{k} \right) \right) \right) \]

(A.9)

\[ \frac{\partial^2 L}{\partial \beta \gamma_i} = \sum_{i=1}^{n} \left( \frac{1}{1 + \gamma_i} \left( t_i \alpha_i - \log \left( \frac{S_{0}i}{k} \right) - \log \left( e^{\gamma_i} \frac{S_{0}i}{k} \right) \right) \right) + \sum_{i=1}^{n} \left( \frac{1}{1 + \gamma_i} \left( t_i \alpha_i - \log \left( \frac{S_{0}i}{k} \right) - \log \left( e^{\gamma_i} \frac{S_{0}i}{k} \right) \right) \right) + \sum_{i=1}^{n} \left( \frac{1}{1 + \gamma_i} \left( t_i \alpha_i - \log \left( \frac{S_{0}i}{k} \right) - \log \left( e^{\gamma_i} \frac{S_{0}i}{k} \right) \right) \right) \]

(A.10)

\[ \frac{\partial^2 L}{\partial \beta \gamma_i} = \sum_{i=1}^{n} \left( \frac{1}{1 + \gamma_i} \left( t_i \alpha_i - \log \left( \frac{S_{0}i}{k} \right) - \log \left( e^{\gamma_i} \frac{S_{0}i}{k} \right) \right) \right) + \sum_{i=1}^{n} \left( \frac{1}{1 + \gamma_i} \left( t_i \alpha_i - \log \left( \frac{S_{0}i}{k} \right) - \log \left( e^{\gamma_i} \frac{S_{0}i}{k} \right) \right) \right) + \sum_{i=1}^{n} \left( \frac{1}{1 + \gamma_i} \left( t_i \alpha_i - \log \left( \frac{S_{0}i}{k} \right) - \log \left( e^{\gamma_i} \frac{S_{0}i}{k} \right) \right) \right) \]

(A.11)
Design of ramp-stress accelerated life test plans for a parallel system

\[ \frac{\partial^2 L}{\partial \beta_1^2} = \sum_{i=1}^{n_1} \left( \frac{-1}{1 + \gamma_1} \beta_1 \beta_2 (t_i \alpha_i^{-1})^{\beta_1} \beta_2 (t_i \alpha_i^{-1})^{\beta_2} + \beta_2 (t_i \alpha_i^{-1})^{\beta_1} \right) \left( \frac{m_{20}(t_i) m_{21}(t_i) m_{22}(t_i) m_{23}(t_i)}{(1 + \gamma_1) (\beta_1 (t_i \alpha_i^{-1})^{\beta_1} + \beta_2 (t_i \alpha_i^{-1})^{\beta_2})^2} \right) \]

(A.12)

\[ \frac{\partial^2 L}{\partial \gamma_1^2} = \sum_{i=1}^{n_1} \left( \frac{-1}{1 + \gamma_1} \beta_1 \beta_2 (t_i \alpha_i^{-1})^{\beta_1} \beta_2 (t_i \alpha_i^{-1})^{\beta_2} + \beta_2 (t_i \alpha_i^{-1})^{\beta_1} \right) \left( \frac{m_{20}(t_i) m_{21}(t_i) m_{22}(t_i) m_{23}(t_i)}{(1 + \gamma_1) (\beta_1 (t_i \alpha_i^{-1})^{\beta_1} + \beta_2 (t_i \alpha_i^{-1})^{\beta_2})^2} \right) \]

(A.13)

\[ \frac{\partial^2 L}{\partial \beta_2^2} = \sum_{i=1}^{n_1} \left( \frac{-1}{1 + \gamma_1} \beta_1 \beta_2 (t_i \alpha_i^{-1})^{\beta_1} \beta_2 (t_i \alpha_i^{-1})^{\beta_2} + \beta_2 (t_i \alpha_i^{-1})^{\beta_1} \right) \left( \frac{m_{20}(t_i) m_{21}(t_i) m_{22}(t_i) m_{23}(t_i)}{(1 + \gamma_1) (\beta_1 (t_i \alpha_i^{-1})^{\beta_1} + \beta_2 (t_i \alpha_i^{-1})^{\beta_2})^2} \right) \]

(A.14)

The elements of the Fisher information matrix for an observation are the negative expectations of these second partial derivatives:

\[ \mathbb{E}[\frac{\partial^2 L}{\partial \beta_k^2}] = \left( \mathbb{E}[n_1] \int_0^\infty m_{15}(t_i) + \mathbb{E}[n_2] \int_0^\infty m_{16}(t_i) + \mathbb{E}[n_3] \int_0^\infty \frac{m_{23}(t_i) - (m_{18}(t_i))}{(m_{18}(t_i))^2} (1 + \gamma_1)^\gamma \right), \]

(A.15)
\[ E[\tilde{\beta}^2 / \beta_0^2] = E[n_1] \sum_{i=1}^{n_\beta}(m_{22}(t_i)(m_{23}(t_i))^2 + \frac{\beta_1 - (L_{i1})^{-\gamma_1} \beta_1}{\alpha_1} \cdot g(t)dt_i \]

\[ + E[n_2] \int_0^\infty (m_{26}(t_i)e^{\frac{\beta_2}{k}}(m_{23}(t_i))^2 + \frac{\beta_1 - (L_{i1})^{-\gamma_1} \beta_1}{\alpha_1} \cdot g(t)dt_i \]

\[ + E[n_1] \sum_{i=1}^{n_\beta}(m_{27}(t_i) \beta_1^{-\gamma_1} (m_{25}(t_i) + \beta_1 \beta_1^2 (m_{23}(t_i)) - m_{33}(t_i) m_{23}(t_i) + m_{32}(t_i)dt_i, \]

\[ (A.17) \]

\[ E[\tilde{\beta}^2 / \beta_0^2] = E[n_1] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + \beta_1 \log(t_1 \alpha_1^{-1})) + E[n_2] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + t_1 \alpha_1^{-\gamma_1})^2 + \beta_1 \log(t_1 \alpha_1^{-1})) \]

\[ + E[n_2] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + \beta_1 \log(t_1 \alpha_1^{-1})) + E[n_1] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + t_1 \alpha_1^{-\gamma_1})^2 + \beta_1 \log(t_1 \alpha_1^{-1})) \]

\[ + E[n_2] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + \beta_1 \log(t_1 \alpha_1^{-1})) + E[n_1] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + t_1 \alpha_1^{-\gamma_1})^2 + \beta_1 \log(t_1 \alpha_1^{-1})) \]

\[ (A.18) \]

\[ E[\tilde{\beta}^2 / \beta_0^2] = E[n_1] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + \beta_1 \log(t_1 \alpha_1^{-1})) + E[n_2] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + t_1 \alpha_1^{-\gamma_1})^2 + \beta_1 \log(t_1 \alpha_1^{-1})) \]

\[ + E[n_1] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + \beta_1 \log(t_1 \alpha_1^{-1})) + E[n_2] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + t_1 \alpha_1^{-\gamma_1})^2 + \beta_1 \log(t_1 \alpha_1^{-1})) \]

\[ + E[n_2] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + \beta_1 \log(t_1 \alpha_1^{-1})) + E[n_1] \int_0^\infty (t_1 \alpha_1^{-\gamma_1} (1 + t_1 \alpha_1^{-\gamma_1})^2 + \beta_1 \log(t_1 \alpha_1^{-1})) \]

\[ (A.19) \]
Design of ramp-stress accelerated life test plans for a parallel system

\[ E(\partial^2 L / \partial \beta_2 \gamma_1) = E[\int_0^\infty \left( \frac{1}{1 + \gamma_1} (t_1 \alpha_1^{-1})^{\beta_2} \log \left( \frac{S_0}{k} \right) + \log (e^{\gamma_0} \gamma_1^{-1} (1 + \gamma_1) - \log (t_1 \alpha_1^{-1}) - \gamma_1 \log (t_1 \alpha_1^{-1}) (-1 + \beta_1 \log (t_1 \alpha_1^{-1})) \right) dt_1 \] 

\[ + E[\int_0^\infty \left( \frac{1}{1 + \gamma_1} (t_1 \alpha_1^{-1})^{\beta_2} \log (e^{\gamma_0} \gamma_1^{-1} (1 + \gamma_1) - \log (t_1 \alpha_1^{-1}) - \gamma_1 \log (t_1 \alpha_1^{-1}) (-1 + \beta_1 \log (t_1 \alpha_1^{-1})) \right) dt_1 \] 

\[ + E[\int_0^\infty \left( (t_1 \alpha_1^{-1})^{\beta_2} (1 + \gamma_1) \log \left( \frac{S_0}{k} \right) - \log (e^{\gamma_0} \gamma_1^{-1} (1 + \gamma_1) - \log (t_1 \alpha_1^{-1}) - \gamma_1 \log (t_1 \alpha_1^{-1}) (-1 + \beta_1 \log (t_1 \alpha_1^{-1})) \right) dt_1 \] 

\[ (1 + \beta_1 \log (t_1 \alpha_1^{-1})) dt_1 \] 

\[ + E[\int_0^\infty (1 + \gamma_1) \beta_2 (t_1 \alpha_1^{-1})^{\beta_2} \left( -1 + (t_1 \alpha_1^{-1})^{\beta_2} \right) \cdot (1 + \beta_1 \log (t_1 \alpha_1^{-1})) dt_1 \] 

\[ + E[\int_0^\infty \left( (1 + \gamma_1) \beta_2 (t_1 \alpha_1^{-1})^{\beta_2} \right) \cdot (1 + \beta_1 \log (t_1 \alpha_1^{-1})) dt_1 \] 

(A.20)

\[ E(\partial^2 L / \partial \beta_2 \gamma_1) = E[\int_0^\infty \left( \frac{1}{1 + \gamma_1} (t_1 \alpha_1^{-1})^{\beta_2} \log \left( \frac{S_0}{k} \right) + \log (e^{\gamma_0} \gamma_1^{-1} (1 + \gamma_1) - \log (t_1 \alpha_1^{-1}) - \gamma_1 \log (t_1 \alpha_1^{-1}) (-1 + \beta_2 \log (t_1 \alpha_1^{-1})) \right) dt_1 \] 

\[ + E[\int_0^\infty \left( \frac{1}{1 + \gamma_1} (t_1 \alpha_1^{-1})^{\beta_2} \log (e^{\gamma_0} \gamma_1^{-1} (1 + \gamma_1) - \log (t_1 \alpha_1^{-1}) - \gamma_1 \log (t_1 \alpha_1^{-1}) (-1 + \beta_2 \log (t_1 \alpha_1^{-1})) \right) dt_1 \] 

\[ + E[\int_0^\infty \left( (t_1 \alpha_1^{-1})^{\beta_2} (1 + \gamma_1) \log \left( \frac{S_0}{k} \right) - \log (e^{\gamma_0} \gamma_1^{-1} (1 + \gamma_1) - \log (t_1 \alpha_1^{-1}) - \gamma_1 \log (t_1 \alpha_1^{-1}) (-1 + \beta_2 \log (t_1 \alpha_1^{-1})) \right) dt_1 \] 

\[ (1 + \gamma_1) \beta_2 (t_1 \alpha_1^{-1})^{\beta_2} \left( -1 + (t_1 \alpha_1^{-1})^{\beta_2} \right) \cdot (1 + \beta_2 \log (t_1 \alpha_1^{-1})) dt_1 \] 

(A.21)

\[ E(\partial^2 L / \partial \gamma_1) = E[\int_0^\infty \left( \frac{1}{1 + \gamma_1} (t_1 \alpha_1^{-1})^{\beta_1} \log \left( \frac{S_0}{k} \right) + \log (e^{\gamma_0} \gamma_1^{-1} (1 + \gamma_1) - \log (t_1 \alpha_1^{-1}) - \gamma_1 \log (t_1 \alpha_1^{-1}) \right) dt_1 \] 

\[ + E[\int_0^\infty \left( \frac{1}{1 + \gamma_1} (t_1 \alpha_1^{-1})^{\beta_1} \log (e^{\gamma_0} \gamma_1^{-1} (1 + \gamma_1) - \log (t_1 \alpha_1^{-1}) - \gamma_1 \log (t_1 \alpha_1^{-1}) \right) dt_1 \] 

\[ + E[\int_0^\infty \left( (t_1 \alpha_1^{-1})^{\beta_1} (1 + \gamma_1) \log \left( \frac{S_0}{k} \right) - \log (e^{\gamma_0} \gamma_1^{-1} (1 + \gamma_1) - \log (t_1 \alpha_1^{-1}) - \gamma_1 \log (t_1 \alpha_1^{-1}) \right) dt_1 \] 

\[ (1 + \gamma_1) \beta_1 (t_1 \alpha_1^{-1})^{\beta_1} \left( -1 + (t_1 \alpha_1^{-1})^{\beta_1} \right) \cdot (1 + \beta_1 \log (t_1 \alpha_1^{-1})) dt_1 \] 

\[ - E[\int_0^\infty \left( - (1 + \gamma_1) \beta_1 (t_1 \alpha_1^{-1})^{\beta_1} \left( -1 + (t_1 \alpha_1^{-1})^{\beta_1} \right) \cdot (1 + \beta_1 \log (t_1 \alpha_1^{-1})) dt_1 \] 

(A.22)
\[
E[\hat{\theta}_{1}^{2}L / \partial \beta_2] = E[n_2] \int_{0}^{\infty} \left( \frac{(-\frac{1}{\alpha_1})^{\beta_1+2\beta_2}(-1 + \beta_1(\frac{1}{\alpha_1})^{\beta_1})(-1 + \beta_2(\frac{1}{\alpha_1})^{\beta_2})}{(\beta_1(\frac{1}{\alpha_1})^{\beta_1} + \beta_2(\frac{1}{\alpha_1})^{\beta_2})^2} \ dt \right), \quad (A.23)
\]

\[
E[\hat{\theta}_{2} L / \partial \theta_{2}] = E[n_1] \int_{0}^{\infty} \left( \frac{1}{1+\gamma_1} (t_1 \alpha_1^{-1})^{\beta_2} (1 + (1 + \gamma_1) \log(\frac{\theta_2}{k}) - \log(\frac{\theta_2}{k})^{\gamma_1} (1 + \gamma_1)) - \log(t_1 \alpha_1^{-1}) - \gamma_1 \log(t_1 \alpha_1^{-1})(-1 + \beta_1 \log(t_1 \alpha_1^{-1})) \right) dt_i
\]

\[
+ E[n_2] \int_{0}^{\infty} \left( \frac{1}{1+\gamma_1} (t_1 \alpha_1^{-1})^{\beta_2} (1 + (1 + \gamma_1) \log(\frac{\theta_2}{k}) - \log(\frac{\theta_2}{k})^{\gamma_1} (1 + \gamma_1)) - \log(t_1 \alpha_1^{-1}) - \gamma_1 \log(t_1 \alpha_1^{-1})(-1 + (t_1 \alpha_1^{-1})^{\beta_2} + \beta_2 \log(t_1 \alpha_1^{-1})) \right) dt_i
\]

\[
+ E[n_3] \int_{0}^{\infty} \left( (1 + (1 + \gamma_1) \log(\frac{\theta_2}{k}) - \log(\frac{\theta_2}{k})^{\gamma_1} (1 + \gamma_1)) - \log(t_1 \alpha_1^{-1}) - \gamma_1 \log(t_1 \alpha_1^{-1})(-1 + (t_1 \alpha_1^{-1})^{\beta_2} + \beta_2 \log(t_1 \alpha_1^{-1})) \right) dt_i
\]

\[
+ E[n_4] \int_{0}^{\infty} \left( (1 + (1 + \gamma_1) \log(\frac{\theta_2}{k}) - \log(\frac{\theta_2}{k})^{\gamma_1} (1 + \gamma_1)) - \log(t_1 \alpha_1^{-1}) - \gamma_1 \log(t_1 \alpha_1^{-1})(-1 + (t_1 \alpha_1^{-1})^{\beta_2} + \beta_2 \log(t_1 \alpha_1^{-1})) \right) dt_i
\]

\[
(A.24)
\]
where: $m_1(t_i) = \left(\frac{1}{\beta_1}\right) + (-1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1}))\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_2$,

$$m_2(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1}) \times \left(\frac{1}{e^{\gamma}(\frac{S_0}{k})^{(1 + \gamma_1)}}\right),$$

$$m_3(t_i) = (1 + (1 + \gamma_1)\log(\frac{S_0}{k}) + \log(e^{\gamma}(\frac{S_0}{k})^{(1 + \gamma_1)})),$$

$$m_4(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1 + \left(-1 + \left(\frac{1}{\alpha_1}\right)\beta_2\right)\log(\frac{t_i}{\alpha_1})\beta_2,$$

$$m_5(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1 + \left(-1 + \left(\frac{1}{\alpha_1}\right)\beta_2\right)\log(\frac{t_i}{\alpha_1})\beta_2,$$

$$m_6(t_i) = (1 + (1 + \gamma_1)\log(\frac{S_0}{k}) + \log(e^{\gamma}(\frac{S_0}{k})^{(1 + \gamma_1)})),$$

$$m_7(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1 + \left(-1 + \left(\frac{1}{\alpha_1}\right)\beta_2\right)\log(\frac{t_i}{\alpha_1})\beta_2,$$

$$m_8(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1 + \left(-1 + \left(\frac{1}{\alpha_1}\right)\beta_2\right)\log(\frac{t_i}{\alpha_1})\beta_2,$$

$$m_9(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1 + \left(-1 + \left(\frac{1}{\alpha_1}\right)\beta_2\right)\log(\frac{t_i}{\alpha_1})\beta_2,$$

$$m_{10}(t_i) = (e^{\gamma}(\frac{S_0}{k})^{(1 + \gamma_1)})^{(1 + \gamma_1)} \times \left(\frac{1}{e^{\gamma}(\frac{S_0}{k})^{(1 + \gamma_1)}}\right),$$

$$m_{11}(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1 + \left(-1 + \left(\frac{1}{\alpha_1}\right)\beta_2\right)\log(\frac{t_i}{\alpha_1})\beta_2,$$

$$m_{12}(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1 + \left(-1 + \left(\frac{1}{\alpha_1}\right)\beta_2\right)\log(\frac{t_i}{\alpha_1})\beta_2,$$

$$m_{13}(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1 + \left(-1 + \left(\frac{1}{\alpha_1}\right)\beta_2\right)\log(\frac{t_i}{\alpha_1})\beta_2,$$

$$m_{14}(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1 + \left(-1 + \left(\frac{1}{\alpha_1}\right)\beta_2\right)\log(\frac{t_i}{\alpha_1})\beta_2,$$

$$m_{15}(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1 + \left(-1 + \left(\frac{1}{\alpha_1}\right)\beta_2\right)\log(\frac{t_i}{\alpha_1})\beta_2,$$

$$m_{16}(t_i) = \left(\frac{1}{\alpha_1}\right)\beta_1 + \left(\frac{1}{\alpha_1}\right)\log(\frac{t_i}{\alpha_1})\beta_1.$$


\[m_{17}(t_i) = \frac{t_i}{\alpha_1}\beta_{11} - \left(\frac{t_i}{\alpha_1}\beta_{11}\right)^2 - 2\left(\frac{t_i}{\alpha_1}\beta_{11}\right)^3 + \beta_{12}\beta_{12}\log\left(\frac{t_i}{\alpha_1}\right),\]

\[m_{18}(t_i) = \frac{t_i}{\alpha_1}^{2\beta_{11}}\beta_{11}^2 - (2 + \left(\frac{t_i}{\alpha_1}\beta_{11}\right)^2 + \beta_{12}\beta_{12}\beta_{12} + \left(\frac{t_i}{\alpha_1}\beta_{11}\right)^2)\log\left(\frac{t_i}{\alpha_1}\right)^2,\]

\[m_{19}(t_i) = \frac{t_i}{\alpha_1}\beta_{11} + \frac{t_i}{\alpha_1}\beta_{11}\beta_{12},\]

\[m_{20}(t_i) = (t_1\alpha_1^{-1})^{-\beta_{11}+\beta_{12}}, m_{21}(t_i) = \beta_{11}^2(t_1\alpha_1^{-1})\beta_{11} + \beta_{12}^4(t_1\alpha_1^{-1})\beta_{12} + \beta_{12}^2,\]

\[m_{22}(t_i) = 2(t_1\alpha_1^{-1})^{-\beta_{11}+\beta_{11}+\beta_{12}} + (t_1\alpha_1^{-1})^{-\beta_{11}+\beta_{12}} - \beta_{12}^2(t_1\alpha_1^{-1})^{-\beta_{11}+\beta_{12}},\]

\[m_{23}(t_i) = (-2 + (t_1\alpha_1^{-1})\beta_{12}) - \beta_{12}^2(t_1\alpha_1^{-1})\beta_{12} + \beta_{12}^2 + \beta_{12}^2(-2 + (t_1\alpha_1^{-1})\beta_{12})(1 + (1 + \gamma_1)\log\left(\frac{S_0}{k}\right) - \log(e^{\gamma_1}(\frac{S_0}{k}))^{1}(1 + \gamma_1)),\]

\[m_{24}(t_i) = -\frac{\beta_{11} - \left(\frac{t_i}{\alpha_1}\beta_{11}\right)\beta_{11} + \left(\frac{t_i}{\alpha_1}\beta_{11}\right)^2\beta_{12} - \beta_{12}\beta_{12}}{\alpha_1^2},\]

\[m_{25}(t_i) = \left(e^{\gamma_1}(\frac{S_0}{k})^{1}(e^{\gamma_1}(\frac{S_0}{k})^{1}) (1 + \gamma_1)\right)^{-1},\]

\[m_{26}(t_i) = -\frac{\left(\frac{t_i}{\alpha_1}\beta_{11}\beta_{11} + \left(\frac{t_i}{\alpha_1}\beta_{11}\right)^2\beta_{12} - \beta_{12}\beta_{12}}{\alpha_1^2} ,\]

\[m_{27}(t_i) = \frac{1}{\alpha_1^2}\left(\frac{t_i}{\alpha_1}\beta_{11} + \left(\frac{t_i}{\alpha_1}\beta_{11}\right)^2\beta_{12} - \beta_{12}\beta_{12}\right),\]

\[m_{28}(t_i) = \frac{t_i}{\alpha_1}^{-\beta_{12}^2}\beta_{11}^2 - \left(\frac{t_i}{\alpha_1}\beta_{11}\beta_{12} - (1 + \frac{t_i}{\alpha_1}\beta_{12} + \beta_{12}\beta_{12})\right), m_{29}(t_i) = \frac{3}{2} - \frac{t_i}{\alpha_1}\beta_{12} + \beta_{12}\beta_{12},\]

\[m_{30}(t_i) = (3 - \frac{t_i}{\alpha_1}\beta_{12}^2 + \beta_{12}^2 + \left(\frac{t_i}{\alpha_1}\beta_{12}^2 + \beta_{12}^2 + 2\beta_{12}\beta_{12}\beta_{12}\beta_{12}\right),\]

\[m_{31}(t_i) = \frac{t_i}{\alpha_1}\beta_{11} + \beta_{12}\beta_{12}\beta_{12}\beta_{12},\]

\[m_{32}(t_i) = \frac{\left(-1 + \left(\frac{t_i}{\alpha_1}\beta_{11}\right)^2 + \left(\frac{t_i}{\alpha_1}\beta_{11}\right)^2\beta_{11} - 2\beta_{11}\beta_{12} - (1 + \frac{t_i}{\alpha_1}\beta_{11}\beta_{12})\right)\beta_{11}\beta_{12}}{\alpha_1\left(\frac{t_i}{\alpha_1}\beta_{12}\beta_{11} + \frac{t_i}{\alpha_1}\beta_{12}\beta_{12}\right)}.\]