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Bivariate Dagum distribution

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Abstract: Abstract. Camilo Dagum proposed several variants of a new model for the size distribution of personal income in a series of papers in the 1970s. He traced the genesis of the Dagum distributions in applied economics and points out parallel developments in several branches of the applied statistics literature. The main aim of this paper is to define a bivariate Dagum distribution so that the marginals have Dagum distributions. It is observed that the joint probability density function and the joint cumulative distribution function can be expressed in closed forms. Several properties of this distribution such as marginals, conditional distributions and product moments have been discussed. The maximum likelihood estimates for the unknown parameters of this distribution and their approximate variance- covariance matrix have been obtained. Some simulations have been performed to see the performances of the MLEs. One data analysis has been performed for illustrative purpose.

Key Words: *Bivariate Dagum distribution, maximum likelihood estimators, product moments*

1. INTRODUCTION

Dagum (1977a, 1977b) studied the income, wage and wealth distribution using the Dagum Distributions. Dagum Distribution belongs to the family of Beta distributions. Dagum (1690) considered Dagum Distribution to model income data of several countries and found that it provides superior fit over the whole range of data. Identifying the pattern of income distribution is very important because the trend provides a guide for the assessment of living standards and level of income inequality in the population of a country. Recently, there has been an increasing interest in the exploration of parametric models for income distribution and Dagum distribution has proved to be quite useful in modeling such data.
Although univariate Dagum distribution has received considerable attention in many applications, as far as we know, a multivariate version of this distribution has not been

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introduced. The main aim of this paper is to provide a bivariate Dagum distribution using an idea similar to that of Theorem 3.2 proposed by Marshall and Olkin (1967). These authors introduced a multivariate exponential distribution whose marginals have exponential distributions and proposed a bivariate Weibull distribution.

Contribution to this idea, Sarhan and Balakrishnan (2007) introduced a bivariate distribution that is more flexible than the bivariate exponential distribution. This distribution was generalized by Kundu and Gupta (2010). Recently, Kundu and Dey (2009) have considered the maximum-likelihood estimation of the model parameters of the Marshall–Olkin bivariate Weibull distribution via the EM algorithm. Using the maximum instead of the minimum in the Marshall and Olkin scheme, Kundu and Gupta (2009,2010) and Sarhan *et al.* (2011) introduced the bivariate generalized exponential, Bivariate proportional reversed hazard and bivariate generalized linear failure rate distributions, respectively. And Muhammed (2016) introduced the bivariate inverse Weibull distribution.

The proposed Bivariate Dagum distribution (BVD) is constructed from three independent Dagum (D) distributions using a maximization process. This new distribution is a singular distribution, and it can be used quit conveniently if there are ties in the data. The BVD model can be interpreted as Competing risk, Shock, Stress and Maintenance Model.

The paper is organized as follows: In Section 2, we introduce the BVD distribution and obtain representations for the cumulative distribution function (cdf) and probability density function (pdf). The conditional and marginal distributions of our bivariate model are presented in Section3. The maximum likelihood estimation, estimated variance-covariance matrix and asymptotic confidence intervals for BVD distribution are provided in Section 4. Simulation results are presented in Section 5. An absolutely continuous BVD distribution is introduced in Section 6. For illustrative purpose an empirical application is presented in Section 7. Finally conclude the paper in Section 8.

2. BIVARIATE DAGUM DISTRIBUTION

A random variable with the Dagum distribution has a cdf and a pdf, for $x > 0$, in the following form

$$f_D\,(x;\lambda,\delta,\beta) = \lambda\,\delta\,\beta\,x^{-\delta-1}\,(1+\lambda\,x^{-\delta})^{-\beta-1},\; F_D\,(x;\lambda,\delta,\beta) = (1+\lambda\,x^{-\delta})^{-\beta}$$

Respectively, where the quantities $\lambda > 0$ is a scale parameter and $\delta > 0$ and $\beta > 0$ are shape parameters respectively. From now on it will be denoted by $D(\lambda,\delta,\beta)$.

Now, Suppose U_1, U_2 and U_3 are three independent random variables such that $U_i \sim D(\lambda,\delta,\beta_i)$ for $i = 1, 2, 3$. Define $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$, then it is said that the bivariate vector (X_1, X_2) has bivariate Dagum distribution with parameters $(\beta_1,\beta_2,\beta_3,\lambda,\delta)$, denoted by $BVD(\beta_1,\beta_2,\beta_3,\lambda,\delta)$. Then, the joint cdf of (X_1, X_2) is given as follows

$$F_{BVD}(x_1,x_2) = \prod_{i=1}^{3} F_D(x_i;\lambda,\delta,\beta_i)$$

where $x_3 = \min(x_1, x_2)$.

The following Proposition will provide the joint cumulative distribution function (cdf), joint probability density function (pdf), the marginal distributions and conditional probability density function.

Proposition 2.1. If $(X_1, X_2) \sim BVD(\beta_1, \beta_2, \beta_3, \lambda, \delta)$. Then, the joint cdf of (X_1, X_2) can be written as

$$F_{BVD}(x_1, x_2) = \begin{cases} F_1(x_1, x_2) & if \quad 0 < x_1 < x_2 < \infty \\ F_2(x_1, x_2) & if \quad 0 < x_2 < x_1 < \infty \\ F_3(x) & if \quad 0 < x_1 = x_2 = x < \infty, \end{cases} \tag{1}$$

Where

$$F_1(x_1, x_2) = F_D(x_1; \lambda, \delta, \beta_1 + \beta_3) F_D(x_2; \lambda, \delta, \beta_2)$$
$$F_2(x_1, x_2) = F_D(x_1; \lambda, \delta, \beta_1) F_D(x_2; \lambda, \delta, \beta_2 + \beta_3)$$
$$F_3(x) = F_D(x; \lambda, \delta, \beta_1 + \beta_2 + \beta_3).$$

Proposition 2.2. If $(X_1, X_2) \sim BVD(\beta_1, \beta_1, \beta_1, \lambda, \delta)$. Then, the joint pdf of (X_1, X_2) is given as

$$f_{BVD}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & if \quad 0 < x_1 < x_2 < \infty \\ f_2(x_1, x_2) & if \quad 0 < x_2 < x_1 < \infty \\ f_3(x) & if \quad 0 < x_1 = x_2 = x < \infty, \end{cases} \tag{2}$$

Where

$$f_1(x_1, x_2) = f_D(x_1; \lambda, \delta, \beta_1 + \beta_3)\, f_D(x_2; \lambda, \delta, \beta_2)$$
$$f_2(x_1, x_2) = f_D(x_1; \lambda, \delta, \beta_1)\, f_D(x_2; \lambda, \delta, \beta_2 + \beta_3)$$
$$f_3(x) = \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3}\, f_D(x; \lambda, \delta, \beta_1 + \beta_2 + \beta_3)$$

Proof. The expressions for $f_1(.,.)$ and $f_2(.,.)$ can be obtained simply by taking $\frac{\partial^2}{\partial x_1 \partial x_2} F_{X_1, X_2}(x_1, x_2)$ for $x_1 < x_2$ and $x_2 < x_1$ respectively. But $f_3(.)$ cannot be obtained in the same way. Using the fact that

$$\int_0^\infty \int_0^{x_2} f_1(x_1, x_2) dx_1 dx_2 + \int_0^\infty \int_0^{x_1} f_2(x_1, x_2) dx_2 dx_1 + \int_0^\infty f_3(x) dx = 1,$$

$$\int_0^\infty \int_0^{x_2} f_1(x_1, x_2) dx_1 dx_2 = \frac{\beta_2}{\beta_1 + \beta_2 + \beta_3}$$

$$\int_0^\infty \int_0^{x_1} f_2(x_1, x_2) dx_2 dx_1 = \frac{\beta_1}{\beta_1 + \beta_2 + \beta_3}$$

Hence, we obtain

$$\int_0^\infty f_3(x)dx = \frac{\beta_3}{\beta_1+\beta_2+\beta_3}$$

Note that

$$\int_0^\infty f_3(x)dx = \delta\,\lambda\beta_3\int_0^\infty x^{-\delta-1}(1+\lambda x^{-\delta})^{-(\beta_1+\beta_2+\beta_3)-1}dx = \frac{\beta_3}{\beta_1+\beta_2+\beta_3}.$$

Therefore, the results follow.

It should be mentioned that the BVD distribution has both an absolute continuous part and a singular part, similar to Marshall and Olkin's bivariate exponential model. The function $f_{X_1,X_2}(.,.)$ may be considered to be a density function for the BVD distribution if it is understood that the first two terms are densities with respect to two-dimensional Lebesgue measure and the third term is a density function with respect to one dimensional Lebesgue measure, see for example Bemis et al. (1972). It is well known that although in one dimension the practical use of a distribution with this property is usually pathological, but they do arise quite naturally in higher dimension. In case of BVD distribution, the presence of a singular part means that if X_1 and X_2 are BVD distribution, then $X_1 = X_2$ has a positive probability. In many practical situations it may happen that X_1 and X_2 both are continuous random variables, but $X_1 = X_2$ has a positive probability, see Marshall and Olkin (1967) in this connection. The following Proposition will provide explicitly the absolute continuous part and the singular part of the BVD distribution function.

Proposition 2.3. If $(X_1, X_2) \sim BVD(\beta_1, \beta_2, \beta_3, \lambda, \delta)$. Then,

$$F_{X_1,X_2}(x_1,x_2) = \frac{\beta_2+\beta_3}{\beta_1+\beta_2+\beta_3}F_a(x_1,x_2) + \frac{\beta_3}{\beta_1+\beta_2+\beta_3}F_s(x_3), \qquad (3)$$

where

$x_3 = \min\{x_1, x_2\}$,

$F_s(x_3) = (1+\lambda x_3^{-\delta})^{-(\beta_1+\beta_2+\beta_3)}$,

$F_a(x_1,x_2) = \frac{\beta_1+\beta_2+\beta_3}{\beta_2+\beta_3}(1+\lambda x_1^{-\delta})^{\beta_1}(1+\lambda x_2^{-\delta})^{\beta_2}(1+\lambda x_3^{-\delta})^{\beta_3} - \frac{\beta_3}{\beta_1+\beta_2}(1+\lambda x_3^{-\delta})^{-(\beta_1+\beta_2+\beta_3)}$.

Here $F_s(.,.)$ and $F_a(.,.)$ are the singular and the absolutely continuous part respectively.

Proof

Suppose A is the following event

$A = \{U_1 < U_3\} \cap \{U_2 < U_3\}$,

Then $P(A) = \frac{\beta_3}{\beta_1+\beta_2+\beta_3}$ and $P(A') = \frac{\beta_2+\beta_3}{\beta_1+\beta_2+\beta_3}$.

Therefore,

$$F_{X_1,X_2}(x_1,x_2) = P(X_1 \le x_1, X_2 \le x_2 \mid A)P(A) + P(X_1 \le x_1, X_2 \le x_2 \mid A')P(A').$$

Moreover for x_3 as defined before,

$$P(X_1 \le x_1, X_2 \le x_2 \mid A) = [P(A)]^{-1} P(U_1 \le U_3, U_2 \le U_3, U_3 \le x_3)$$
$$= (1+\lambda x_3^{-\delta})^{-(\beta_1+\beta_2+\beta_3)}$$

and $P(X_1 \le x_1, X_2 \le x_2 \mid A')$ can be obtained by subtraction as

$$P(X_1 \le x_1, X_2 \le x_2 \mid A') = \frac{\beta_1+\beta_2+\beta_3}{\beta_2+\beta_3}(1+\lambda x_1^{-\delta})^{\beta_1}(1+\lambda x_2^{-\delta})^{\beta_2}(1+\lambda x_3^{-\delta})^{\beta_3}$$
$$-\frac{\beta_3}{\beta_1+\beta_2}(1+\lambda x_3^{-\delta})^{-(\beta_1+\beta_2+\beta_3)}.$$

Clearly, $(1+\lambda x_3^{-\delta})^{-(\beta_1+\beta_2+\beta_3)}$ is the singular part as its mixed second partial derivative is zero when $x_1 \ne x_2$, and $P(X_1 \le x_1, X_2 \le x_2 \mid A')$ is the absolute continuous part as its mixed partial derivative is a density function.

Using Proposition 2.3, we immediately obtain the joint pdf of X_1 and X_2 also in the following form for $x_3 = \min\{x_1, x_2\}$;

$$f_{X_1,X_2}(x_1,x_2) = \frac{\beta_2+\beta_3}{\beta_1+\beta_2+\beta_3} f_a(x_1,x_2) + \frac{\beta_3}{\beta_1+\beta_2+\beta_3} f_s(x_3) \tag{4}$$

where

$$f_a(x_1,x_2) = \frac{\beta_1+\beta_2+\beta_3}{\beta_2+\beta_3} \times \begin{cases} f_D(x_1;\beta_1+\beta_3)\cdot f_D(x_2;\beta_2) & \text{if} \quad x_1 < x_2 \\ f_D(x_1;\beta_1)\cdot f_D(x_2;\beta_2+\beta_3) & \text{if} \quad x_1 > x_2 \end{cases}$$

and

$$f_s(x_3) = f_D(x_3;\beta_1+\beta_2+\beta_3).$$

Clearly, here $f_a(x_1,x_2)$ and $f_s(x_3)$ are the absolutely continuous and singular part respectively.

The absolutely continuous part of the BVD density may be unimodal depending on the values of λ, δ β_1, β_2 and β_3 that is $f_a(x_1,x_2)$ is unimodal and the respective modes are

$$\left\{[\tfrac{(\beta_1+\beta_3)\lambda\,\delta-1}{\delta+1}]^{\frac{1}{\delta}}, [\tfrac{\beta_2\lambda\,\delta-1}{\delta+1}]^{\frac{1}{\delta}}\right\} \text{ and } \left\{[\tfrac{\beta_1\lambda\,\delta-1}{\delta+1}]^{\frac{1}{\delta}}, [\tfrac{(\beta_2+\beta_3)\lambda\,\delta-1}{\delta+1}]^{\frac{1}{\delta}}\right\}.$$

The median for the BVD distribution is obtained as

$$\left[\frac{\lambda}{2^{1/\beta_1+\beta_2+\beta_3}-1}\right]^{\frac{1}{\delta}}.$$

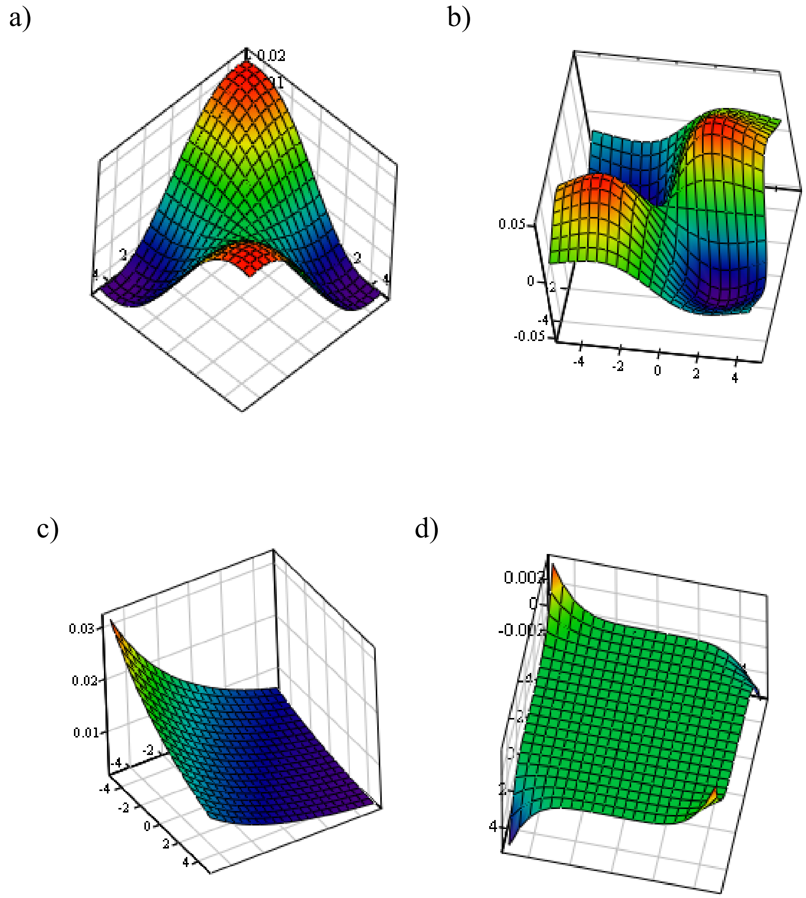

Figure 1. Surface plots of the absolutely continuous part of the joint pdf of the BVD distribution for different values of ($\beta_1, \beta_2, \beta_3, \lambda, \delta$): (a) (5, 3, 2, 0.005, 2), (b) (1, 1, 1, 0.05,2), (c) (1, 1, 1, 0.05, 1) and (d) (4, 5, 3, 0.0005, 6).

3. DIFFERENT PROPORTIES

In this Section we provide different basic properties of the BVD model. First we provide the marginal and conditional distributions of BVD model.

Proposition 3.1 If $(X_1, X_2) \sim BVD(\beta_1, \beta_2, \beta_3, \lambda, \delta)$. Then,

1. $X_1 \sim D(\lambda, \delta, \beta_1 + \beta_3)$ and $X_2 \sim D(\lambda, \delta, \beta_2 + \beta_3)$
2. $max(X_1, X_2) \sim D(\lambda, \delta, \beta_1 + \beta_2 + \beta_3)$
3. The conditional pdf of X_i , given $X_j = x_j$, denoted by $f_{i/j}\left(x_i/\, x_j\right)$ $(i \neq j = 1,2)$, is given by

$$f_{i/j}(x_1,x_2)=\begin{cases} f_{i/j}^{(1)}(x_i/x_j) & if \quad x_i<x_j \\ f_{i/j}^{(2)}(x_i/x_j) & if \quad x_j<x_i \\ f_{i/j}^{(3)}(x_i/x_j) & if \quad x_i=x_j, \end{cases} \tag{5}$$

where

$$f_{i/j}^{(1)}(x_i/x_j)=\frac{\lambda\delta(\beta_1+\beta_3)\beta_2}{(\beta_2+\beta_3)}x_i^{-\delta-1}(1+\lambda x_i^{-\delta})^{-(\beta_1+\beta_3)-1}(1+\lambda x_j^{-\delta})^{\beta_3}$$

$$f_{i/j}^{(2)}(x_i/x_j)=\lambda\delta\ \beta_1 x_i^{-\delta-1}(1+\lambda x_j^{-\delta})^{-\beta_1-1},$$

$$f_{i/j}^{(3)}(x_i/x_j)=\frac{\beta_3\ x_i^{-\delta-1}}{(\beta_2+\beta_3)x_j^{-\delta-1}}(1+\lambda x_i^{-\delta})^{-(\beta_1+\beta_2+\beta_3)-1}(1+\lambda x_j^{-\delta})^{\beta_2+\beta_3+1}.$$

Proof: They can be obtained by routine calculation.
The bivariate survival function of (X_1,X_2) is given as

$$S_{BVD}(x_1,x_2)=\begin{cases} F_D(x_1;\beta_1+\beta_3)[F_D(x_2;\beta_2)-1]+[1-F_D(x_2;\beta_2+\beta_3)], x_1<x_2 \\ F_D(x_2;\beta_2+\beta_3)[F_D(x_1;\beta_1)-1]+[1-F_D(x_1;\beta_1+\beta_3)], x_2<x_1 \\ 1-F_D(x;\beta_1+\beta_2+\beta_3),\ \ x_1=x_2=x. \end{cases}$$

The bivariate reversed hazard function for (X_1,X_2) can be written as

$$r_{BVD}(x_1,x_2)=\begin{cases} (\lambda\,\delta)^2(\beta_1+\beta_3)\beta_2\ \ x_1^{-\delta-1}\,x_2^{-\delta-1}\left(1+\lambda\,x_1^{-\delta}\right)\left(1+\lambda\,x_2^{-\delta}\right), x_1<x_2 \\ (\lambda\,\delta)^2(\beta_2+\beta_3)\,\beta_1\,x_1^{-\delta-1}\,x_2^{-\delta-1}\left(1+\lambda\,x_1^{-\delta}\right)\left(1+\lambda\,x_2^{-\delta}\right), x_1>x_2 \\ \beta_3\lambda\,\delta\,x^{-\delta-1}\left(1+\lambda\,x^{-\delta}\right), x_1=x_2=x. \end{cases}$$

Algorithm to generate from BVD distribution

Step 1. Generate U_1, U_2 and U_3 from $U(0,1)$,

Step 2. Compute

$$Z_1=\left[\frac{U_1^{-\frac{1}{\beta_1}}-1}{\lambda}\right]^{\frac{-1}{\delta}},\ Z_2=\left[\frac{U_2^{-\frac{1}{\beta_2}}-1}{\lambda}\right]^{\frac{-1}{\delta}}\ \text{and}\ Z_3=\left[\frac{U_3^{-\frac{1}{\beta_3}}-1}{\lambda}\right]^{\frac{-1}{\delta}},$$

Step3. Obtain

$$X_1=\max(Z_1,Z_3)\ \text{ and }\ X_2=\max(Z_2,Z_3).$$

Since the marginal distributions of the bivariate vector (X_1,X_2) are Dagum distributions, the moments of X_1 and X_2 can be obtained directly from their marginals

$$E(X_i^{\ r})=(\beta_i+\beta_3)\lambda^{\frac{r}{\delta}}\ B\left[(\beta_i+\beta_3)+\frac{r}{\delta},1-\frac{r}{\delta}\right], i=1,2.$$

where $\boldsymbol{B}(.,.)$ is the beta function. Now, we present the product moments.

Proposition 3.2. If $(X_1, X_2) \sim BVD(\beta_1, \beta_2, \beta_3, \lambda, \delta)$. Then, The r^{th} and s^{th} joint moments of the X_1 and X_2, denoted by $\mu'_{r,s}$ is given by

$$\acute{\mu}_{r,s} = E(X_1{}^r X_2{}^s) = \beta_2(\beta_1+\beta_3)\frac{\lambda^{\frac{r+s}{\delta}-2}}{1-\frac{r}{\delta}}\, B\left[2-\frac{r+s}{\delta}, \beta_2+\frac{r+s}{\delta}-1\right]$$
$$\cdot\ {}_2F_1\left(2-\frac{r+s}{\delta}, 1-\frac{r}{\delta}, 1-(\beta_1+\beta_3)+\frac{r}{\delta}; -\frac{r}{\delta}, \beta_2+1; 1\right)$$
$$+\beta_1(\beta_2+\beta_3)\frac{\lambda^{\frac{r+s}{\delta}-2}}{1-\frac{r}{\delta}}\, B\left[2-\frac{r+s}{\delta}, \beta_1+\frac{r+s}{\delta}-1\right]$$
$$\cdot\ {}_2F_1\left(2-\frac{r+s}{\delta},\ 1-\frac{s}{\delta}, 1-(\beta_2+\beta_3)+\frac{s}{\delta}; -\frac{s}{\delta},\ \beta_1+1\,;1\right)$$
$$+(\beta_3)\lambda^{\frac{r+s}{\delta}}\ B\left[(\beta_1+\beta_2+\beta_3)+\frac{r+s}{\delta}, 1-\frac{r+s}{\delta}\right]. \tag{6}$$

Where $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du$ is the beta function,

${}_pF_q(b_1,\ldots,b_P;c_1,\ldots,c_q;u) = \sum_{i=0}^{\infty} \frac{(b_1)_i\ldots(b_P)_i}{(c_1)_i\ldots(c_q)_i}\frac{u^i}{i!}$ is a hypergeometric function,

$(b)_i = b(b+1)\ldots(b+i-1) = \frac{\Gamma(b+i)}{\Gamma(b)}(b \neq 0, i = 1,2,\ldots)$. and p,q are nonnegative integers.

Proof

Starting with $E(X_1^r, X_2^s) = \int_0^\infty\int_0^\infty x_1^r x_2^s f(x_1,x_2)dx_1dx_2$ $\quad r,s = 1,2,3,\ldots$ and substituting for $f(x_1,x_2)$ from (2.2).Then, using the fact that.

$$B_x(\alpha,\beta) = \int_0^x u^{\alpha-1}(1-u)^{\beta-1}du = \frac{x^\alpha}{\alpha}\ {}_2F_1(\alpha, 1-\beta; \alpha+1; x),$$

where $B_x(\alpha,\beta)$ is an incomplete beta function and the identity

$$\int u^{\alpha-1}(1-u)^{\beta-1}\ {}_2F_1(c,d;\rho;u)du = B(\alpha,\beta)\ {}_3F_2(\alpha,c,d;\rho,\alpha+\beta;1)$$

for $\alpha,\beta > 0$ and $d+\beta-\alpha-c > 0$, We can derive the expression for $E(X_1^r, X_2^s)$.

4. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we consider the problem of computing the maximum likelihood estimators (MLEs) of the unknown parameters of the BVD distribution. Suppose

$\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ is a random sample from $BVD(\beta_1, \beta_2, \beta_3, \lambda, \delta)$ distribution. Consider the following notation

$$I_1 = \{i; x_{1i} < x_{2i}\}, \quad I_2 = \{i; x_{1i} > x_{2i}\}, \quad I_3 = \{x_{1i} = x_{2i} = x_i\}, \quad I = I_1 \cup I_2 \cup I_3,$$

$$|I_1| = n_1, \quad |I_2| = n_2, \quad |I_3| = n_3, \quad \text{and} \quad n_1 + n_2 + n_3 = n.$$

The log-likelihood function of the sample of size n is given by

$$\ln L(\underline{\theta}) = \sum_{i \in I_1} \ln f_1(x_{1i}, x_{2i}) + \sum_{i \in I_2} \ln f_2(x_{1i}, x_{2i}) + \sum_{i \in I_3} \ln f_3(x_i) \tag{7}$$

$$\begin{aligned}
\ln L(\underline{\theta}) = {} & n_1 \ln(\beta_1 + \beta_3) + n_1 \ln \beta_2 + n_2 \ln(\beta_2 + \beta_3) + n_2 \ln \beta_1 + n_3 \ln \beta_3 + (2n_1 + 2n_2 + n_3) \ln(\lambda) \\
& + (2n_1 + 2n_2 + n_3) \ln(\delta) - (\delta + 1)[\sum_{i=1}^{n_1} \ln x_{1i} + \sum_{i=1}^{n_1} \ln x_{2i} + \sum_{i=1}^{n_2} (\ln x_{1i} + \ln x_{2i}) + \sum_{i=1}^{n_3} \ln x_i] \\
& - (\beta_1 + \beta_3) \sum_{i=1}^{n_1} \ln(1 + \lambda x_{1i}^{-\delta}) - (\beta_2 + 1) \sum_{i=1}^{n_1} \ln(1 + \lambda x_{2i}^{-\delta}) - (\beta_2 + \beta_3) \sum_{i=1}^{n_2} \ln(1 + \lambda x_{2i}^{-\delta}) \\
& - (\beta_1 + 1) \sum_{i=1}^{n_2} \ln(1 + \lambda x_{1i}^{-\delta}) - (\beta_1 + \beta_2 + \beta_3) \sum_{i=1}^{n_3} \ln(1 + \lambda x_i^{-\delta}).
\end{aligned} \tag{8}$$

where $\underline{\theta} = (\beta_1, \beta_2, \beta_3, \lambda, \delta)$.

On differentiating (4.2) with respect to $\beta_1, \beta_2, \beta_3, \lambda$ and δ in turn and equating to zero, we obtain the following likelihood equations

$$\frac{n_1}{\hat{\beta}_1 + \hat{\beta}_3} + \frac{n_2}{\hat{\beta}_1} - \sum_{i=1}^{n_1} \ln(1 + \lambda x_{1i}^{-\delta}) - \sum_{i=1}^{n_2} \ln(1 + \lambda x_{1i}^{-\delta}) - \sum_{i=1}^{n_3} \ln(1 + \lambda x_i^{-\delta}) = 0,$$

$$\frac{n_2}{\hat{\beta}_2 + \hat{\beta}_3} + \frac{n_1}{\hat{\beta}_2} - \sum_{i=1}^{n_1} \ln(1 + \lambda x_{1i}^{-\delta}) - \sum_{i=1}^{n_2} \ln(1 + \lambda x_{2i}^{-\delta}) - \sum_{i=1}^{n_3} \ln(1 + \lambda x_i^{-\delta}) = 0,$$

$$\frac{n_1}{\hat{\beta}_1 + \hat{\beta}_3} + \frac{n_2}{\hat{\beta}_2 + \hat{\beta}_3} + \frac{n_3}{\hat{\beta}_3} - \sum_{i=1}^{n_1} \ln(1 + \lambda x_{1i}^{-\delta}) - \sum_{i=1}^{n_2} \ln(1 + \lambda x_{2i}^{-\delta}) - \sum_{i=1}^{n_3} \ln(1 + \lambda x_i^{-\delta}) = 0,$$

$$\begin{aligned}
& \frac{2n_1 + 2n_2 + n_3}{\hat{\lambda}} - (\hat{\beta}_1 + \hat{\beta}_3 + 1) \sum_{i=1}^{n_1} \frac{x_{1i}^{-\delta}}{1 + \lambda x_{1i}^{-\delta}} - (\hat{\beta}_2 + 1) \sum_{i=1}^{n_1} \frac{x_{2i}^{-\delta}}{1 + \lambda x_{2i}^{-\delta}} - (\hat{\beta}_1 + 1) \sum_{i=1}^{n_2} \frac{x_{1i}^{-\delta}}{1 + \lambda x_{1i}^{-\delta}} \\
& - (\hat{\beta}_2 + \hat{\beta}_3 + 1) \sum_{i=1}^{n_2} \frac{x_{2i}^{-\delta}}{1 + \lambda x_{2i}^{-\delta}} - (\hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_3 + 1) \sum_{i=1}^{n_3} \frac{x_i^{-\delta}}{1 + \lambda x_i^{-\delta}} = 0,
\end{aligned}$$

and

$$\frac{2n_1+2n_2+n_3}{\hat{\delta}}+(\hat{\beta}_1+\hat{\beta}_3+1)\sum_{i=1}^{n_1}\frac{\lambda x_{1i}^{-\delta}\ln x_{1i}}{1+\lambda x_{1i}^{-\delta}}+(\hat{\beta}_2+1)\sum_{i=1}^{n_1}\frac{\lambda x_{2i}^{-\delta}\ln x_{2i}}{1+\lambda x_{2i}^{-\delta}}$$
$$(\hat{\beta}_1+1)\sum_{i=1}^{n_2}\frac{\lambda x_{1i}^{-\delta}\ln x_{1i}}{1+\lambda x_{1i}^{-\delta}}+(\hat{\beta}_2+\hat{\beta}_3+1)\sum_{i=1}^{n_2}\frac{\lambda x_{2i}^{-\delta}\ln x_{2i}}{1+\lambda x_{2i}^{-\delta}}+(\hat{\beta}_1+\hat{\beta}_2+\hat{\beta}_3+1)\sum_{i=1}^{n_3}\frac{\lambda x_{i}^{-\delta}\ln x_{i}}{1+\lambda x_{i}^{-\delta}}$$
$$-[\sum_{i=1}^{n_1}\ln x_{1i}+\sum_{i=1}^{n_1}\ln x_{2i}+\sum_{i=1}^{n_2}(\ln x_{1i}+\ln x_{2i})+\sum_{i=1}^{n_3}\ln x_i]=0. \tag{9}$$

These five equations have not explicit form; therefore, their solutions are numerically obtained using Newton-Raphson method as will be seen in Section 5. They are solved simultaneously to obtain $\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda}$ and $\hat{\delta}$.

The asymptotic variance-covariance matrix of $\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda}$ and $\hat{\delta}$, is obtained as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}^{-1}$$

where

$$a_{11}=-\left.\frac{\partial^2 \ln L}{\partial\beta_1^2}\right|_{\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda},\hat{\delta}}=\frac{n_1}{(\hat{\beta}_1+\hat{\beta}_3)^2}+\frac{n_2}{\hat{\beta}_1^2},$$

$$a_{12}=-\left.\frac{\partial^2 \ln L}{\partial\beta_1\,\partial\beta_2}\right|_{\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda},\hat{\delta}}=0,$$

$$a_{13}=-\left.\frac{\partial^2 \ln L}{\partial\beta_1\,\partial\beta_3}\right|_{\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda},\hat{\delta}}=\frac{n_1}{(\hat{\beta}_1+\hat{\beta}_3)^2}$$

$$a_{14}=-\left.\frac{\partial^2 \ln L}{\partial\beta_1\,\partial\lambda}\right|_{\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda},\hat{\delta}}=\sum_{i=1}^{n_1}\frac{x_{1i}^{-\hat{\delta}}}{1+\hat{\lambda}x_{1i}^{-\hat{\delta}}}+\sum_{i=1}^{n_2}\frac{x_{1i}^{-\hat{\delta}}}{1+\hat{\lambda}x_{1i}^{-\hat{\delta}}}+\sum_{i=1}^{n_3}\frac{x_{i}^{-\hat{\delta}}}{1+\hat{\lambda}x_{i}^{-\hat{\delta}}},$$

$$a_{15}=-\left.\frac{\partial^2 \ln L}{\partial\beta_1\,\partial\delta}\right|_{\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda},\hat{\delta}}=\sum_{i=1}^{n_1}\frac{\hat{\lambda}\,x_{1i}^{-\hat{\delta}}\ln x_{1i}}{1+\hat{\lambda}x_{1i}^{-\hat{\delta}}}+\sum_{i=1}^{n_2}\frac{\hat{\lambda}\,x_{1i}^{-\hat{\delta}}\ln x_{1i}}{1+\hat{\lambda}x_{1i}^{-\hat{\delta}}}+\sum_{i=1}^{n_3}\frac{\hat{\lambda}\,x_{i}^{-\hat{\delta}}\ln x_{i}}{1+\hat{\lambda}x_{i}^{-\hat{\delta}}},$$

$$a_{22}=-\left.\frac{\partial^2 \ln L}{\partial\beta_2^2}\right|_{\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda},\hat{\delta}}=\frac{n_2}{(\hat{\beta}_2+\hat{\beta}_3)^2}+\frac{n_1}{\hat{\beta}_2^2},$$

$$a_{23} = -\left.\frac{\partial^2 \ln L}{\partial\beta_2\,\partial\beta_3}\right|_{\hat\beta_1,\hat\beta_2,\hat\beta_3,\hat\lambda,\hat\delta} = \frac{n_2}{(\hat\beta_2+\hat\beta_3)^2},$$

$$a_{24} = -\left.\frac{\partial^2 \ln L}{\partial\beta_2\,\partial\lambda}\right|_{\hat\beta_1,\hat\beta_2,\hat\beta_3,\hat\lambda,\hat\delta} = \sum_{i=1}^{n_1}\frac{x_{1i}^{-\hat\delta}}{1+\hat\lambda x_{1i}^{-\hat\delta}} + \sum_{i=1}^{n_2}\frac{x_{2i}^{-\hat\delta}}{1+\hat\lambda x_{2i}^{-\hat\delta}} + \sum_{i=1}^{n_3}\frac{x_{i}^{-\hat\delta}}{1+\hat\lambda x_{i}^{-\hat\delta}},$$

$$a_{25} = -\left.\frac{\partial^2 \ln L}{\partial\beta_2\,\partial\delta}\right|_{\hat\beta_1,\hat\beta_2,\hat\beta_3,\hat\lambda,\hat\delta} = \sum_{i=1}^{n_1}\frac{\hat\lambda\, x_{1i}^{-\hat\delta}\ln x_{1i}}{1+\hat\lambda x_{1i}^{-\hat\delta}} + \sum_{i=1}^{n_2}\frac{\hat\lambda\, x_{2i}^{-\hat\delta}\ln x_{2i}}{1+\hat\lambda x_{2i}^{-\hat\delta}} + \sum_{i=1}^{n_3}\frac{\hat\lambda\, x_{i}^{-\hat\delta}\ln x_{i}}{1+\hat\lambda x_{i}^{-\hat\delta}},$$

$$a_{33} = -\left.\frac{\partial^2 \ln L}{\partial\beta_3^2}\right|_{\hat\beta_1,\hat\beta_2,\hat\beta_3,\hat\lambda,\hat\delta} = \frac{n_1}{(\hat\beta_1+\hat\beta_3)^2} + \frac{n_2}{(\hat\beta_2+\hat\beta_3)^2} + \frac{n_3}{\hat\beta_3^2},$$

$$a_{34} = -\left.\frac{\partial^2 \ln L}{\partial\beta_3\,\partial\lambda}\right|_{\hat\beta_1,\hat\beta_2,\hat\beta_3,\hat\lambda,\hat\delta} = \sum_{i=1}^{n_1}\frac{x_{1i}^{-\hat\delta}}{1+\hat\lambda x_{1i}^{-\hat\delta}} + \sum_{i=1}^{n_2}\frac{x_{2i}^{-\hat\delta}}{1+\hat\lambda x_{2i}^{-\hat\delta}} + \sum_{i=1}^{n_3}\frac{x_{i}^{-\hat\delta}}{1+\hat\lambda x_{i}^{-\hat\delta}},$$

$$a_{35} = -\left.\frac{\partial^2 \ln L}{\partial\beta_3\,\partial\delta}\right|_{\hat\beta_1,\hat\beta_2,\hat\beta_3,\hat\lambda,\hat\delta} = \sum_{i=1}^{n_1}\frac{\hat\lambda\, x_{1i}^{-\hat\delta}\ln x_{1i}}{1+\hat\lambda x_{1i}^{-\hat\delta}} + \sum_{i=1}^{n_2}\frac{\hat\lambda\, x_{2i}^{-\hat\delta}\ln x_{2i}}{1+\hat\lambda x_{2i}^{-\hat\delta}} + \sum_{i=1}^{n_3}\frac{\hat\lambda\, x_{i}^{-\hat\delta}\ln x_{i}}{1+\hat\lambda x_{i}^{-\hat\delta}},$$

$$a_{44} = -\left.\frac{\partial^2 \ln L}{\partial\lambda^2}\right|_{\hat\beta_1,\hat\beta_2,\hat\beta_3,\hat\lambda,\hat\delta} = \frac{2n_1+2n_2+n_3}{\hat\lambda^2} + (\hat\beta_1+\hat\beta_3+1)\sum_{i=1}^{n_1}\frac{x_{1i}^{-2\hat\delta}}{(1+\hat\lambda x_{1i}^{-\hat\delta})^2} + (\hat\beta_2+1)\sum_{i=1}^{n_1}\frac{x_{2i}^{-2\hat\delta}}{(1+\hat\lambda x_{2i}^{-\hat\delta})^2}$$
$$+(\hat\beta_1+1)\sum_{i=1}^{n_2}\frac{x_{1i}^{-2\hat\delta}}{(1+\hat\lambda x_{1i}^{-\hat\delta})^2} + (\hat\beta_2+\hat\beta_3+1)\sum_{i=1}^{n_2}\frac{x_{2i}^{-2\hat\delta}}{(1+\hat\lambda x_{2i}^{-\hat\delta})^2} + (\hat\beta_1+\hat\beta_2+\hat\beta_3+1)\sum_{i=1}^{n_3}\frac{x_{i}^{-2\hat\delta}}{(1+\hat\lambda x_{i}^{-\hat\delta})^2},$$

$$a_{45} = -\left.\frac{\partial^2 \ln L}{\partial\lambda\partial\delta}\right|_{\hat\beta_1,\hat\beta_2,\hat\beta_3,\hat\lambda,\hat\delta} = (\hat\beta_1+\hat\beta_3+1)\sum_{i=1}^{n_1}\frac{x_{1i}^{-\hat\delta}\ln x_{1i}}{(1+\hat\lambda x_{1i}^{-\hat\delta})^2} + (\hat\beta_2+1)\sum_{i=1}^{n_1}\frac{x_{2i}^{-\hat\delta}\ln x_{2i}}{(1+\hat\lambda x_{2i}^{-\hat\delta})^2}$$
$$+(\hat\beta_1+1)\sum_{i=1}^{n_2}\frac{x_{1i}^{-\hat\delta}\ln x_{1i}}{(1+\hat\lambda x_{1i}^{-\hat\delta})^2} + (\hat\beta_2+\hat\beta_3+1)\sum_{i=1}^{n_2}\frac{x_{2i}^{-\hat\delta}\ln x_{2i}}{(1+\hat\lambda x_{2i}^{-\hat\delta})^2} + (\hat\beta_1+\hat\beta_2+\hat\beta_3+1)\sum_{i=1}^{n_3}\frac{x_{i}^{-\hat\delta}\ln x_{i}}{(1+\hat\lambda x_{i}^{-\hat\delta})^2},$$

$$a_{55} = -\left.\frac{\partial^2 \ln L}{\partial\delta^2}\right|_{\hat\beta_1,\hat\beta_2,\hat\beta_3,\hat\lambda,\hat\delta} = \frac{2n_1+2n_2+n_3}{\hat\delta^2} + (\hat\beta_1+\hat\beta_3+1)\sum_{i=1}^{n_1}\frac{\lambda x_{1i}^{-\hat\delta}(\ln x_{1i})^2}{(1+\hat\lambda x_{1i}^{-\hat\delta})^2} + (\hat\beta_2+1)\sum_{i=1}^{n_1}\frac{\lambda x_{2i}^{-\hat\delta}(\ln x_{2i})^2}{(1+\hat\lambda x_{2i}^{-\hat\delta})^2}$$
$$+(\hat\beta_1+1)\sum_{i=1}^{n_2}\frac{\lambda x_{1i}^{-\hat\delta}(\ln x_{1i})^2}{(1+\hat\lambda x_{1i}^{-\hat\delta})^2} + (\hat\beta_2+\hat\beta_3+1)\sum_{i=1}^{n_2}\frac{\lambda x_{2i}^{-\hat\delta}(\ln x_{2i})^2}{(1+\hat\lambda x_{2i}^{-\hat\delta})^2} + (\hat\beta_1+\hat\beta_2+\hat\beta_3+1)\sum_{i=1}^{n_3}\frac{\lambda x_{i}^{-\hat\delta}(\ln x_{i})^2}{(1+\hat\lambda x_{i}^{-\hat\delta})^2}.$$

Now we state the asymptotic normality results to obtain the asymptotic confidence intervals of $\beta_1, \beta_2, \beta_3, \lambda$ and δ. It can be stated as follows

$$\sqrt{n}\,[(\hat\lambda-\lambda),(\hat\delta-\delta),(\hat\beta_1-\beta_1),(\hat\beta_2-\beta_2),(\hat\beta_3-\beta_3)] \rightarrow N_5(0\,,\,I(\underline{\theta})^{-1}) \text{ as } n\rightarrow\infty \tag{10}$$

Where $I^{-1}(\underline{\theta})$ is the variance-covariance matrix, $\hat{\underline{\theta}}=(\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\lambda},\hat{\delta})$. and $\underline{\theta}=(\beta_1,\beta_2,\beta_3,\lambda,\delta)$. Since $\underline{\theta}$ is unknown in (4.4), $I^{-1}(\underline{\theta})$ is estimated by $I^{-1}(\hat{\underline{\theta}})$; the asymptotic variance-covariance matrix that defined above and this can be used to obtain the asymptotic confidence intervals of $\beta_1,\beta_2,\beta_3,\lambda$ and δ.

5. SIMULATION RESULTS

In this section we present, a simulation experiment in which we evaluated the estimation of the model parameters of the BVD distribution by considering the direct maximization of the log-likelihood. The simulations were performed using the Mathcad program, the number of replications $R = 1000$ and the tolerance level was 0.001.

The evaluation of the point estimation was performed based on the following quantities for each sample size: the Average Estimates (AE) and the Mean Squared Error, (MSE) are estimated from R replications and the coverage rate of the 95% confidence interval for $\beta_1,\beta_2,\beta_3,\lambda$ and δ. We set the sample size at $n = 20{,}40$ and 60, and considered some values for the parameters $\beta_1,\beta_2,\beta_3,\lambda$ and δ.

It can be seen from Table 1 that the estimates are quite stable and, more importantly, are close to the true values for the sample sizes considered and that the MSE decreases as the sample size increases, as expected. We also notice that the coverage probabilities of the asymptotic confidence interval are close to the nominal level. These results indicate that the proposed model and the asymptotic approximation work well under the situation where no censoring occurs.

Table 1. The average estimates (AE), the mean squared errors (MSE), and the Coverage percentages (CI) of $\beta_1,\beta_2,\beta_3,\lambda$ and δ for BVD model

n	parameters	AE	MSE	95% CI Coverage
20	β_1 (1.8)	1.7864	0.00055	0.97
	β_2 (1.5)	1.5088	0.00057	0.94
	β_3 (1.5)	1.6955	0.02653	0.91
	λ (1.8)	2.0146	0.043	0.95
	δ (1.8)	2.0241	0.044	0.92
40	β_1 (1.6)	1.5544	0.00045	0.92
	β_2 (2)	2.0061	0.00054	0.91
	β_3 (2)	2.1607	0.0179	0.97
	λ (1.6)	1.5320	0.002	0.95
	δ (1.6)	1.40326	0.03	0.89

60	β_1 (1.4)	1.3399	0.00056	0.94
	β_2 (1.7)	1.6582	0.00056	0.95
	β_3 (1.7)	1.5801	0.010091	0.98
	λ (1.1)	0.9879	0.004	0.96
	δ (1.1)	1.2157	0.026	0.89

6. ABSOLUTELY CONTINUOUS BVD DISTRIBUTION

Block and Basu (1974) obtained an absolutely continuous bivariate exponential distribution, which is known as the Block and Basu exponential (BBBE) distribution. This distribution arises from the Marshal and Olkin bivariate exponential (MOBE) distribution by removing the singular part and retaining only the absolutely continuous part. Although MOBE is a singular bivariate exponential distribution, BBBE distribution enjoys all the properties of an absolutely continuous distribution. Because of this reason, BBBE is a very popular bivariate distribution. It has been used extensively for data analysis purposes, even though it is known that the marginals of BBBE are not exponential unlike MOBE model.

Based on the study of Block and Basu (1974), we introduce an absolutely continuous bivariate Dagum (BVD_{ac}) distribution. Some results in this Section are similar to those derived in the previous Sections and therefore the proofs are omitted.

A random vector (Y_1, Y_2) follows a BVD_{ac} distribution if its pdf of is given by

$$f_{Y_1,Y_2}(y_1,y_2)=\begin{cases} c\, f_1(y_1,y_2) & if \quad y_1<y_2 \\ c\, f_2(y_1,y_2) & if \quad y_1>y_2 \end{cases}$$

$$=c.\begin{cases} f_D(y_1;\beta_1+\beta_3)\cdot f_D(y_2;\beta_2) & if \quad y_1<y_2 \\ f_D(y_1;\beta_1)\cdot f_D(y_2;\beta_2+\beta_3) & if \quad y_1>y_2 \end{cases}, \tag{11}$$

Here c is the normalizing constant and $= \dfrac{\beta_1+\beta_2+\beta_3}{\beta_2+\beta_3}$. We denote $(Y_1, Y_2) \sim BVD_{ac}(\beta_1, \beta_2, \beta_3, \lambda, \delta)$ if (X_1, X_2) has a BVD distribution, then (X_1, X_2) given $X_1 \neq X_2$ has a BVD_{ac} distribution. Thus , a simple method to generate the BVD_{ac} distribution is as follows

Step1: Generate $(X_1, X_2) \sim BVD(\beta_1, \beta_2, \beta_3, \lambda, \delta)$;

Step2: If $X_1 = X_2$, go back to step1 ; otherwise take (X_1, X_2).

In what follows the joint cdf corresponding to Equation (11), the marginal distributions and the product moments of the BVD_{ac} are presented.

Proposition 6.1. Let $(Y_1, Y_2) \sim BVD_{ac}(\beta_1, \beta_2, \beta_3, \lambda, \delta)$. The associated failure function is

$$F_{Y_1,Y_2}(y_1,y_2) = \frac{\beta_1+\beta_2+\beta_3}{\beta_1+\beta_2} F_D(y_1;\lambda,\delta,\beta_1)\, F_D(y_2;\lambda,\delta,\beta_2) F_D(y;\lambda,\delta,\beta_3) - \frac{\beta_3}{\beta_1+\beta_2} F_D(y;\lambda,\delta,\beta_{123});$$

Where $y = \min(y_1, y_2)$. Furthermore, the marginal failure functions are given by

$$F_{Y_1}(y_1) = \frac{\beta_1+\beta_2+\beta_3}{\beta_1+\beta_2} F_D(y_1;\lambda,\delta,\beta_1+\beta_3) - \frac{\beta_3}{\beta_1+\beta_2} F_D(y_1;\lambda,\delta,\beta_{123})$$

$$F_{Y_2}(y_2) = \frac{\beta_1+\beta_2+\beta_3}{\beta_1+\beta_2} F_D(y_2;\lambda,\delta,\beta_2+\beta_3) - \frac{\beta_3}{\beta_1+\beta_2} F_D(y_2;\lambda,\delta,\beta_{123})$$

Proof

The joint cdf given in (2.3) can be written as follows

$$F_{X_1,X_2}(x_1,x_2) = \frac{\beta_2+\beta_3}{\beta_1+\beta_2+\beta_3} F_a(x_1,x_2) + \frac{\beta_3}{\beta_1+\beta_2+\beta_3} F_s(x),$$

Here $F_s(.,.)$ and $F_a(.,.)$ are the singular and the absolutely continuous part respectively. For $x = \min\{x_1, x_2\}$,

$$F_s(x) = F_D(x,\lambda,\delta,\beta_1+\beta_2+\beta_3),$$

and $F_a(x_1,x_2) = \frac{\beta_1+\beta_2+\beta_3}{\beta_1+\beta_2} F_D(x_1;\lambda,\delta,\beta_1)\, F_D(x_2;\lambda,\delta,\beta_2) F_D(x;\lambda,\delta,\beta_3)$

$$- \frac{\beta_3}{\beta_1+\beta_2} F_D(x;\lambda,\delta,\beta_{123})$$

Once $F_{X_1,X_2}(x_1,x_2) = F_a(x_1,x_2)$, the result holds. The marginal cdfs are obtained trivially.

Proposition 6.2. The marginal pdfs associated with the cdf function given in Proposition 6.1 are as follows

$$f_{Y_1}(y_1) = c f_D(y_1;\lambda,\delta,\beta_1+\beta_3) - c\frac{\beta_3}{\beta_1+\beta_2+\beta_3} f_D(y_1;\lambda,\delta,\beta_1+\beta_2+\beta_3), \quad y_1 > 0$$

and

$$f_{Y_2}(y_2) = c f_D(y_2;\lambda,\delta,\beta_2+\beta_3) - c\frac{\beta_3}{\beta_1+\beta_2+\beta_3} f_D(y_2;\lambda,\delta,\beta_1+\beta_2+\beta_3),$$

$y_2 > 0.$

Proof By computing $f_{Y_1}(y_1) = \frac{dF_{Y_1}(y_1)}{dy_1}$ and $f_{Y_2}(y_2) = \frac{dF_{Y_2}(y_2)}{dy_2}$, the results follows.

Unlike those of the BVD distribution, the marginals of the BVD_{ac} distribution are not Dagum distributions. If $\beta_3 \to 0^+$, then Y_1 and Y_2 follow Dagum distributions and in this case, Y_1 and Y_2 become independent .

Proposition 6.3. The product moments of $(Y_1, Y_2) \sim BVD_{ac}(\beta_1, \beta_1, \beta_1, \lambda, \delta)$. are given by

$$E(Y_1{}^r Y_2{}^s) = c\,\beta_2(\beta_1+\beta_3)\frac{\lambda^{\frac{r+s}{\delta}-2}}{1-\frac{r}{\delta}}\;B\left[2-\frac{r+s}{\delta},\beta_2+\frac{r+s}{\delta}-1\right]$$
$$.\quad {}_2F_1\left(2-\frac{r+s}{\delta},1-\frac{r}{\delta},1-(\beta_1+\beta_3)+\frac{r}{\delta};-\frac{r}{\delta},\beta_2+1;1\right)$$
$$+c\,\beta_1(\beta_2+\beta_3)\frac{\lambda^{\frac{r+s}{\delta}-2}}{1-\frac{r}{\delta}}\;B\left[2-\frac{r+s}{\delta},\beta_1+\frac{r+s}{\delta}-1\right]$$
$$.\quad {}_2F_1\left(2-\frac{r+s}{\delta},\;1-\frac{s}{\delta},1-(\beta_2+\beta_3)+\frac{s}{\delta};-\frac{s}{\delta},\;\beta_1+1\,;1\right).$$

The marginal moments of Y_1 and Y_2 are as follows

$$E(Y_i{}^r) = c\left[(\beta_i+\beta_3)\lambda^{\frac{r}{\delta}}\;B\left[(\beta_i+\beta_3)+\frac{r}{\delta},1-\frac{r}{\delta}\right]\right.$$
$$\left.-\beta_3\lambda^{\frac{r}{\delta}}\;B\left[(\beta_1+\beta_2+\beta_3)+\frac{r}{\delta},1-\frac{r}{\delta}\right]\right],\qquad i=1,2.$$

Proposition 6.4. Let $(Y_1,Y_2)\sim BVD_{ac}(\beta_1,\beta_2,\beta_3,\lambda,\delta)$. Then the

i. Stress- Strength parameter has the following form;

$$R = P(Y_1 < Y_2) = \frac{\beta_1}{\beta_1+\beta_2},$$

ii. $max(Y_1,Y_2)\sim\; D(\beta_1+\beta_2+\beta_3)$.

7. DATA ANALYSIS

For illustrative purposes we have analyzed one data set to see how the proposed model works in practice. The data set has been obtained from Meintanis (2007). The data represent the football (soccer) data where at least one goal scored by the home team and at least one goal scored directly from a penalty kick, foul kick or any other direct kick (all of them together will be called as kick goal) by any team have been considered. Here X_1 represents the time in minutes of the first kick goal scored by any team and X_2 represents the first goal of any type scored by the home team. In this case all possibilities are open, for example $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$. All the data points have been divided by 100 so that the shape and scale parameters are of the same order. This is not going to make any difference in any statistical inference. Meintanis (2007) used the Marshal-Olkin distribution to analyze the data. We would like to analyze the data using BVD model. Before going to analyze the data using BVD model, we fit the Dagum distribution to X_1 and X_2 separately. The MLEs of the parameters β,λ and δ of the respective Dagum distribution for X_1 , X_2 and $\max(X_1,X_2)$ are (1.2, 1.80, 2.06), (1.089, 2.50, 2.56) and (1.58, 1.95, 1.098) respectively. The Kolmogorov-Smirnov distances between the fitted distribution and the empirical distribution function and the corresponding p values (in brackets) for X_1 , X_2 and $\max(X_1,X_2)$ are 0.19 (0.46) , 0.141(0.53) and 0.20(0.34) respectively. Based on the p values Dagum distribution cannot be rejected for the marginals and for the maximum also. Now from the Kolmogorov-Smirnov distances, it is

clear that although Meintanis (2007) suggested using bivariate exponential (BVE) distribution, BVD model is preferable in this case.
Now we try to test whether BVD model or BVE model provides better fit to the above data set. The Akaike information criterion (AIC), Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) are used to compare the candidate distributions. In case of BVD, based on the above estimates the log-likelihood value is -30.459 and in case of BVE model, using the estimates obtained by Meintanis (2007), the log-likelihood value becomes -44.56. The corresponding AIC, BIC, CAIC and HQIC values are (70.918, 78.973, 72.853and 73.758) and (96.12, 95.46, 96.11 and 97.62) respectively. Therefore, all of the criteria suggest that BVD provides a better fit than the BVE model.

8. CONCLUSIONS

In this paper, the BVD distribution is introduced for the first time. Marginals of this bivariate distribution are also Dagum distributions. It is observed that the BVD distribution is a singular distribution and it has an absolute continuous part and a singular part. Since the joint distribution function and the joint density function are in closed forms, therefore this distribution can be used in practice for non-negative and positively correlated random variables. Several properties of the BVD such as conditional distributions, product moments, joint survival function and joint reversed hazard function have been discussed.
This model has five unknown parameters we obtained the maximum likelihood estimates for the five unknown parameters and their approximate variance- covariance matrix. We perform some simulations to see the performances of the MLEs and the simulation results indicate that the BVD distribution and the asymptotic approximation work well under the situation when no censoring occur. One data set has been re-analyzed and it is observed that the bivariate Dagum distribution provides a better fit than the bivariate exponential distribution. Along the same line as Block and Basu (1974) bivariate exponential model, an absolute continuous version of the BVD also obtained in this paper and several of its properties such as the joint failure function, marginal densities, product moments, marginal moments, median and mode are presented. Distribution of sum, product and ratio of two absolutely continuous BVD random variables will be consider and it will be reported elsewhere.

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