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ON SUBDIRECT PRODUCT OF PRIME MODULES

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Abstract. In the various module generalizations of the concepts of prime (semiprime) for a ring, the question "when are semiprime modules subdirect product of primes?" is a serious question in this context and it is considered by earlier authors in the literature. We continue study on the above question by showing that: If R is Morita equivalent to a right pre-duo ring (e.g., if R is commutative) then weakly compressible R-modules are precisely subdirect products of prime R-modules if and only if $\dim(R) = 0$ and $R/\mathrm{N}(R)$ is a semi-Artinian ring if and only if every classical semiprime module is semiprime. In this case, the class of weakly compressible R-modules is an enveloping for Mod-R. Some related conditions are also investigated.

1. Introduction

Throughout this paper rings will have a nonzero identity, modules will be right and unitary. In the literature, there are several module generalizations of a semiprime (prime) ring, see [15, Sections 13 and 14] for an excellent reference on the subject. These generalizations introduce various concepts of semiprime (prime) modules and many important theories on semiprime (prime) rings are generalized to modules by them, see; [3], [7], [8], [10] and [17]. The natural question "when are semiprime modules subdirect product of primes?" is then appeared related to these generalizations. In the following, we first recall some definitions of the literature and explain some where the above question was studied. Then we illustrate the main results about the above question. Following [10], a module M_R is called *$\star$-prime* if $M \in \mathrm{Cog}(N)$ for any $0 \neq N \leq M_R$. These modules were originally studied in [4]. It is easy to see that $\star$-prime modules M_R are prime (i.e., $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$ for any $0 \neq N \leq M_R$). In [3], *classical prime* modules M_R (i.e., for any $0 \neq N \leq M_R$, $\mathrm{ann}_R(N)$ is a prime ideal of R) was studied. A widely used generalization of semiprime rings is *weakly compressible* modules M_R that was defined in [2] by $\mathrm{Hom}_R(M, N)N \neq 0$ for all $0 \neq N \leq M_R$. They are *semiprime* in the sense of [10] (i.e., $M \in \mathrm{Cog}(N)$

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for every $N \leq_{ess} M_R$). As we will see in Proposition 2.1, semiprime R-modules have the property " $\mathrm{ann}_R(N)$ is a semiprime ideal of R for all $0 \neq N \leq M_R$". We call such modules M_R *classical semiprime*. Consider the following classes of R-modules:
$\mathcal{P}_\star = \{\star$-prime R-modules$\}$, $\mathcal{P} = \{$prime R-modules$\}$, $\mathcal{CP} = \{$classical prime R-modules$\}$, $\mathcal{W} = \{$weakly compressible R-modules$\}$, $\mathcal{S} = \{$semiprime R-modules$\}$, $\mathcal{CS} = \{$classical semiprime R-modules$\}$. We have the following diagram and in Examples 2.4, we will show that all of the implications in the diagram are in general not reversible.

$$\begin{array}{ccccc} \mathcal{P}_\star & \subsetneq & \mathcal{P} & \subsetneq & \mathcal{CP} \\ \cap & & & & \cap \\ \mathcal{W} & \subsetneq & \mathcal{S} & \subsetneq & \mathcal{CS} \end{array}$$

Let $\mathcal{C}$, $\mathcal{D}$ be two classes of R-modules and $\mathcal{C}$ is closed under taking submodules. If we denote $\Omega_{\mathcal{C}} = \{M_R \mid \mathrm{Rej}(M, \mathcal{C}) = 0\}$ where $\mathrm{Rej}(M, \mathcal{C}) = \bigcap\{\mathrm{Ker} f \mid f : M_R \to C_R$ for some $C \in \mathcal{C}\}$, then $\mathcal{D} \subseteq \Omega_{\mathcal{C}}$ means every element in $\mathcal{D}$ is a subdirect product of some elements in $\mathcal{C}$. In [7], for a commutative ring R, it is studied, when $\mathcal{CS} = \Omega_{\mathcal{P}}$. In [3, Thoerem 3.12], it is investigated when $\mathcal{CS} = \Omega_{\mathcal{CP}}$ for certain commutative rings. In [5, Theorems 2.6 and 3.3], the conditions $\mathcal{S} \subseteq \Omega_{\mathcal{P}_\star}$ and $\mathcal{W} = \Omega_{\mathcal{P}_\star}$ are investigated for certain duo rings. In [10, Corollary 5.4], it is shown that $\mathcal{W} \subseteq \Omega_{\mathcal{P}}$. The aim of this paper is study the equality "$\mathcal{W} = \Omega_{\mathcal{P}}$" for commutative rings (or more generally duo rings) as stated in the abstract of the paper. However, in view of the mentioned works, the study of relations between these generalizations is not out of place. We observe in (Theorems 2.10 and 2.12) that the study of conditions on a ring R, under which $\mathcal{W} = \Omega_{\mathcal{P}}$ leads to knowing where the equalities $\mathcal{CP} = \mathcal{P}$ and $\mathcal{S} = \mathcal{CS}$ may occur. This in turn describes how are the generalizations far away from each others. Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [1] and [11].

2. Main results

We first collect some properties of the classes stated in the introduction for latter uses, and show that they are invariant under Morita equivalences. Then we investigate when $\mathcal{W} = \Omega_{\mathcal{P}}$. A nonzero submodule N of a module M_R is called essential (and denoted by $N \leq_{ess} M_R$) if $N \cap K \neq 0$ for all $0 \neq K \leq M_R$. For R-modules X and Y, we write $X \hookrightarrow Y$ if there exists an injective R-homomorphism from X to Y. A module X_R is cogenerated by Y_R and denoted by $X_R \in \mathrm{Cog}(Y)$ if $\mathrm{Rej}(X, Y) = 0$. If A is a nonempty set and M is an R-module, then M^A means a direct product of copies of M_R.

Proposition 2.1. (a) *If M is a semiprime R-module and N is a fully invariant submodule of M_R, then N_R is semiprime.*

(b) *If M_R is semiprime* (*resp. prime*)*, then M_R is classical semiprime* (*resp. classical prime*)*. In particular, $ann_R(M)$ is a semiprime* (*resp. prime*) *ideal of R.*

(c) *Every weakly compressible module is semiprime.*

(d) *Let I be an ideal of R and "P" denote any one of the properties*: *"weakly compressible", "$\star$-prime", "prime", "semiprime", "classical prime", "classical semiprime". If M is an R-module such that $MI = 0$, then M_R satisfies the property "P" if and only if $M_{R/I}$ does so.*

(e) *Let M_R be a semiprime module. Then every simple submodule of M_R is a direct summand of M_R. In particular, if $Soc(M)$ is essential in M_R, then M_R is weakly compressible.*

(f) *The class of weakly compressible modules is closed under co-products and taking submodules.*

(g) *The class of semiprime modules is closed under products $\star$-prime and co-products.*

Proof. The part (d) has a routine arguments and the other parts are obtained by [5, Proposition 2.1]. □

Lemma 2.2. *Let R and S be Morita equivalent rings with category equivalence $\alpha : Mod\text{-}R \to Mod\text{-}S$ and $M \in Mod\text{-}R$. If $I = ann_R(M)$ and $B = ann_S(\alpha(M))$, then the rings R/I and S/B are Morita equivalent by the restriction α on $Mod\text{-}R/I$.*

Proof. Let $A = \mathrm{ann}_S(\alpha(R/I))$. Hence $R/I \overset{\alpha}{\approx} S/A$ by [1, Proposition 21.11]. On the other hand, $\alpha(M)$ is faithful as a module over S/A by [1, Proposition 21.6(4)]. It follows that $A = B$, as desired. □

Proposition 2.3. *The properties*: *"semiprime", "$\star$-prime", "prime", "classical prime", "classical semiprime" and "weakly compressible" are Morita invariant .*

Proof. Let R and S be Morita equivalent rings with equivalence category α : Mod-$R \to$ Mod-S and $M \in$ Mod-R.

Since category equivalences preserve (essential) monomorphisms and direct products [1, Proposition 21.6(3) and (5)], hence the properties of semiprime and $\star$-prime are Morita invariant.

For the prime case suppose that $\alpha(M)_S$ is prime, $0 \neq N \leq M_R$, $I = \mathrm{ann}_R(M)$ and $B = \mathrm{ann}_S(\alpha(M))$. Thus $\mathrm{ann}_S(\alpha(N)) = B$ and hence $\alpha(N)$ is a faithful S/B-module. Now since $R/I \approx S/B$ by Lemma 2.2, N must be a faithful R/I-module by [1, Proposition 21.6(4)]. Hence $\mathrm{ann}_R(N) = I$, proving that M_R is prime.

For classical (semi) prime cases note that every ring Morita equivalent to a prime (semiprime) ring is also a prime (semiprime) ring. Thus Lemma 2.2 shows that classical prime (classical semiprime) modules are Morita invariant.

Finally, if M_R is a weakly compressible module, then $N \not\hookrightarrow \mathrm{Rej}(M,N)$ for every nonzero R-submodule N. Because if with $0 \neq N_R$, then $\theta(N) \subseteq \mathrm{Rej}(M,N) = \mathrm{Rej}(M,\theta(N))$ that shows M_R is not weakly compressible. Now suppose that $\alpha(M)$ is a weakly compressible S-module and $0 \neq N \leq M_R$. If $N \subseteq \mathrm{Rej}(M,N)$, then $\alpha(N) \hookrightarrow \alpha(\mathrm{Rej}(M,N)) \hookrightarrow \mathrm{Rej}(\alpha(M),\alpha(N))$. This, by the above, contradicts weakly compressible condition on $\alpha(M)_S$. Hence $N \nsubseteq \mathrm{Rej}(M,N)$ and M_R is weakly compressible. □

A ring R is called *right semi-Artinian* if every nonzero R-module contains a simple submodule.

Examples 2.4. (a) Suppose that R is any commutative regular ring which is not semi-Artinian (for example $R = {\mathbb{Z}_2}^{\mathbb{N}}$). Then by [14, Theorem 3.2], there exists an R-module M such that $M \notin \mathcal{W}$. Since R is a regular ring, $\mathrm{Rad}(M) = 0$ and so $M \hookrightarrow L := \prod_{\lambda\in\Lambda} S_\lambda$ where Λ is a nonempty set and each S_λ is a simple R-module. By Lemma 2.1(g), $L \in \mathcal{S}$. Thus if $\mathcal{W} = \mathcal{S}$, then $M \in \mathcal{W}$, a contradiction. Therefore $\mathcal{W} \neq \mathcal{S}$.

(b) Let R be a commutative domain which is not field and Q be the quotient field of R. It is clear that $Q_R \in \mathcal{P}$. Since $\mathrm{Hom}_R(Q,R) = 0$, $Q_R \notin \mathcal{S}$. This example shows that $\mathcal{P}_\star \subsetneq \mathcal{P}$ and $\mathcal{S} \subsetneq \mathcal{CS}$.

(c) Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}_p$, where p is a prime number. Since $\mathrm{ann}_{\mathbb{Z}}(\mathbb{Q}) \neq \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}_p)$, $M \notin \mathcal{P}$. In fact, the annihilator of any nonzero submodule of $M_{\mathbb{Z}}$ is either 0 or $p\mathbb{Z}$. Hence $M \in \mathcal{CP}\backslash\mathcal{P}$ and so $\mathcal{CP} \neq \mathcal{P}$.

In order to study when $\mathcal{W} = \Omega_{\mathcal{P}}$, we first investigate the stronger cases: Mod-$R = \mathcal{P}_\star$ or $\mathcal{W}$. A ring R is said to be (*right quasi-duo*) *right duo* if (maximal) right ideals of R are ideal. We say that a ring R is *a right pre-duo ring* if every prime factor ring of R is right duo.

Proposition 2.5. *If R is a right duo ring, then $\mathcal{S} \subseteq \Omega_{\mathcal{P}_*}$.*

Proof. Let $M \in \mathcal{S}$. Note that if $m \in M$, then $mR \simeq R/\mathrm{ann}(mR)$ because R is right duo. Hence every cyclic submodule of M lies in $\Omega_{\mathcal{P}_*}$ by Proposition 2.1(b). On the other hand, since there exists an essential submodule N of M_R such that N is a direct sum of cyclic submodules of M, we deduce that M can be embedded in a product of its cyclic submodules. Therefore $M \in \Omega_{\mathcal{P}_*}$. The proof is complete. □

Lemma 2.6. *A ring R is right pre-duo $\Rightarrow$ R is a right quasi-duo ring $\Rightarrow$ R/J is a reduced ring.*

Proof. Since every maximal right ideal of R contains a prime ideal, we see that every right pre-duo ring is a right quasi-duo ring. For the second implication, let R be a right quasi-duo ring. Without loss of generality, we may suppose that $\mathrm{J}(R) = 0$. If $x^n = 0$ and $x^{n-1} \neq 0$ for some $x \in R$, then there exists a maximal right ideal P of R such that $x^{n-1} \notin P$. It follows that $x^{n-1}R + P = R$ and hence $xR = xP \subseteq P$, a contradiction. Thus R has no nonzero nilpotent elements. □

Following [11, Chapter 6], the *classical Krull dimension* of a ring R is originally defined to be the supremum of the lengths of all chains of prime ideals in R and is usually denoted by $\dim(R)$. A ring R is called *right V-ring* if every simple R-module is injective. A right quasi-duo ring R is a right V-ring if and only if R is a regular ring if and only if R is a semiprime ring with $\dim(R) = 0$ [16, Theorems 2.6 and 2.7].

Theorem 2.7. *Let R be a ring. The following statements hold.*

(a) *Every R-module is $\star$-prime if and only if R is a simple Artinian ring.*

(b) *Every R-module is prime if and only if R is a simple ring.*

(c) *Consider the following conditions.*

(i) *R is a right semi-Artinian right V-ring.*

(ii) *Mod-$R = \mathcal{W}$.*

(iii) *Mod-$R = \mathcal{S}$.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). *All conditions are equivalent if the ring R is Morita equivalent to a right quasi-duo ring.*

Proof. (a) For sufficiency note that by our assumption, there exists a unique simple R-module T (up to isomorphism) and every nonzero R-module M is isomorphic to a direct sum of copies of T. Hence $M \in \mathrm{Cog}(N)$ for all $0 \neq N \leq M_R$. Conversely, suppose that all R-modules are $\star$-prime. If M_1 and M_2 are nonzero R-modules, then $M_1 \in \mathrm{Cog}(M_2)$ because $M_1 \oplus M_2$ is a $\star$-prime R-module. This follows that R is a simple ring because $R \in \mathrm{Cog}(R/I)$ for any proper ideal I of R. Thus R is a right nonsingular ring. Now if J is any essential right ideal of R, the condition $R/J \in \mathrm{Cog}(R)$ implies $J = R$. Therefore R has no proper essential right ideal, proving that R is a simple Artinian ring.

(b) The sufficiency is clear. Conversely, suppose that every R-module is prime and I is any proper ideal of R. Since $R \oplus R/I$ is a prime R-module, $\mathrm{ann}_R(R/I) = \mathrm{ann}_R(R) = 0$. Hence $I = 0$, proving that R is a simple ring.

(c) (i) $\Rightarrow$ (ii). Since R is right semi-Artinian, we shall show that for every simple submodule $S \leq M_R$, there exists a homomorphism $f : M \to S$ such that $f(S) \neq 0$. The latter statement holds because R is a right V-ring.

(ii) $\Rightarrow$ (iii). By Proposition 2.1(c).

(iii) $\Rightarrow$ (ii). Let M be an R-module and N be any nonzero submodule of M_R. There exists a submodule $N^{'}$ of M_R such that $N \oplus N^{'}/N^{'}$ is an essential submodule of $M/N^{'}$ [1, Proposition 5.21]. Since $N \simeq N \oplus N^{'}/N^{'}$ and $M/N^{'}$ is a semiprime R-module by (iii), $M/N^{'} \in \mathrm{Cog}\ (N)$. This follows that there exists a homomorphism $f : M \to N$ such that $f(N) \neq 0$, proving that M_R is weakly compressible.

(ii) $\Rightarrow$ (i). Now suppose that R is Morita equivalent to a right quasi-duo ring. In view of Proposition 2.3, we may suppose that R is right quasi-duo. By (ii) and Proposition 2.1(e) we can show that $S = \mathrm{E}(S)$ for every simple R-module S. Therefore $\mathrm{J}(R) = 0$ and so R is a reduced ring by Lemma 2.6. Hence R is a right semi-Artinian and strongly regular ring by [14, Theorem 3.2]. Now, R is a V-ring by the above remarks and so (i) holds. $\square$

Remark 2.8. If R is any arbitrary ring, then the condition (ii) (or equivalently, the condition (iii)) of the above theorem implies that R is a V-ring because if $f : E(S) \to S$ with $f(S) \neq 0$ for a simple R-module S then $\ker f$ is a maximal submodule with $0 = S \cap \ker f$. Thus S_R is injective. We don't know whether R is also semi-Artinian. However, for many ring including duo rings, the answer is positive, see [14].

Lemma 2.9. *If $\{I_i\}_{i\in A}$ is a family of semiprime ideals in a ring R, then $\oplus_{i\in A}(R/I_i)^{\Lambda_i}$ is a weakly compressible R-module, where each Λ_i is a nonempty set.*

Proof. By Proposition 2.1(d) each $(R/I_i)^{\Lambda_i}$ is a weakly compressible R/I_i-module if and only if it is weakly compressible R-module. Hence by Proposition 2.1(f), we need to show that for every semiprime ring S and every nonempty set Λ, the S-module S^Λ is weakly compressible. The latter statement is well known, we give a proof for completeness. Let $0 \neq n = \{a_i\}_{i\in\Lambda} \in N \leq S^\Lambda$. Thus $a_t \neq 0$ for some $t \in \Lambda$. Since S is a semiprime ring, $a_tSa_t \neq 0$. Hence there exists $s \in S$ such that $a_tsa_t \neq 0$. Consider the nonzero element $x = ns \in N$. Let $f = \ell_x\pi_t$, where $\pi_t : S^\Lambda \to S$ is the canonical projection and $\ell_x : S \to xS$ is the left multiplication map by x. Thus $f : S^\Lambda \to N$ is a homomorphism such that $f(n) \neq 0$. The proof is complete. □

A ring R is called *right Goldie* if R has finite right uniform dimension and R satisfies the ACC on right annihilators. It is well known that a ring R is semiprime right Glodie if and only if R is semiprime, $\mathrm{Z}(R_R) = 0$ and R has finite right uniform dimension if and only if R has a semisimple Artinian classical right quotient ring [11, Theorem 2.3.6]. For any ring R, the intersection of all prime ideals of R is called *prime radical* and usually denoted by $\mathrm{N}(R)$.

Theorem 2.10. *Let R be a ring.*

(a) $\mathcal{CP} = \mathcal{P}$ *if and only if* $\dim(R) = 0$.

(b) *If R is a ring such that $R/N(R)$ is a right Goldie ring, then the following statements are equivalent.*

(i) $\mathcal{P}_\star = \mathcal{P}$.

(ii) $\mathcal{S} = \mathcal{CS}$.

(iii) $\mathcal{W} = \Omega_{\mathcal{P}}$.

(iv) *$R/N(R)$ is a semisimple Artinian ring.*

Proof. (a) For the sufficiency note that if $M_R \in \mathcal{CP}$, then $R/\mathrm{ann}_R(M)$ is a simple ring. Hence $M_R \in \mathcal{P}$. Conversely, suppose that $\mathcal{CP} = \mathcal{P}$ and P_1, P_2 are prime ideals of R such that $P_1 \subseteq P_2$. Let $M = R/P_1 \oplus R/P_2$ and $0 \neq N \leq M_R$. If $N \subseteq R/P_2$, then $\mathrm{ann}_R(N) = P_2$. Let $N \nsubseteq R/P_2$. Thus there exists an element $n = (a + P_1, b + P_2) \in N$ such that $a \notin P_1$. It follows that $\mathrm{ann}_R(nR) \subseteq P_1$ and so $\mathrm{ann}_R(N) \subseteq P_1$. On the other hand, $P_1 \subseteq \mathrm{ann}_R(N)$ because $MP_1 = 0$. Therefore $M_R \in \mathcal{CP}$ and so, by our assumption, $M_R \in \mathcal{P}$. It shows that $P_1 = P_2$, as desired.

(b) Let R be a ring with $\mathrm{N} = \mathrm{N}(R)$. Since all modules stated in conditions (i), (ii) or (iii), are annihilated by N, hence we may suppose that $\mathrm{N} = 0$ and R is a semiprime right Goldie ring.

Clearly (iv) implies (i), (ii) and (iii).

Suppose that one of the conditions (i), (ii) or (iii) holds. It is well known that R has only finitely many minimal prime ideals such that R/P is a right Goldie ring for each minimal prime ideal P of R [11, Propositions 3.2.2 and 3.2.5]. Thus it is enough to show that R/P is an Artinian ring for any minimal prime ideal P of R. Let now P be a minimal prime ideal of R. By Proposition 2.1(d), the conditions (i), (ii) and (iii) hold for the ring R/P. Thus we may suppose that R is a prime right Goldie ring with $Q =$ the classical right quotient ring of R [11, Theorem 2.3.6]. Hence for any $0 \neq A \leq Q_R$, we have $\mathrm{ann}_R(A) \subseteq \mathrm{ann}_R(A \cap R) = 0$ and so $Q_R \in \mathcal{P}$. Consequently, in any cases (i), (ii) or (iii), we have $Q_R \in \mathcal{S}$. Thus $Q \in \mathrm{Cog}(R)$. Hence by Lemma 2.9 and Proposition 2.1(f), Q_R is weakly compressible. It follows that $\mathrm{Hom}_R(Q, U) \neq 0$ for every uniform right ideal U of R. Therefore every uniform right ideal of R contains a nonzero divisible submodule. It is well known that in a semiprime right Goldie ring S, every nonsingular divisible S-module is injective, see instance [9, Theorem 3.3]. Hence every uniform right ideal of R contains a nonzero injective submodule. It follows that every uniform right ideal of R is a simple and injective R-module. Now, since the uniform dimension of R_R is finite, we deduce that $R = U_1 \oplus \cdots \oplus U_n$ $(n \geq 1)$ such that each U_i is a simple and injective R-module. Therefore R is an Artinian ring. The proof is complete. □

Let M be an R-module. The *Krull dimension* of M_R is defined by a transfinite induction [11, Chapter 6]. It is well known that every Noetherian module has a Krull dimension, and modules with Krull dimensions have finite uniform dimensions [11, Lemma 6.2.6].

Corollary 2.11. *Let R be a ring such that R_R has finite Krull dimension. The following statements are equivalent.*

(a) $\mathcal{P}_\star = \mathcal{P}$.

(b) $\mathcal{S} = \mathcal{CS}$.

(c) $\mathcal{W} = \Omega_{\mathcal{P}}$.

(d) $R/N(R)$ *is a semisimple Artinian ring.*

Proof. Note that semiprime rings with right Krull dimensions are known to be right Goldie [11, Proposition 6.3.5], and apply Theorem 2.10(b). □

Theorem 2.12. *Let R be a ring Morita equivalent to a right pre-duo ring. Then the following statements hold.*

(a) $\mathcal{P} = \mathcal{P}_\star$ *if and only if* $\dim(R) = 0$.

(b) *The following statements are equivalent.*

(i) $\mathcal{S} = \mathcal{CS}$.

(ii) $\mathcal{W} = \Omega_{\mathcal{P}}$.

(iii) $\dim(R) = 0$ *and* $R/N(R)$ *is a right semi-Artinian ring.*

Proof. By Lemma 2.2, if $R \overset{\alpha}{\approx} S$, then $R/\mathrm{N}(R) \approx S/\mathrm{N}(S)$. Therefore in view of Proposition 2.3, we may suppose that R is a right pre pre-duo ring. We first show that every prime factor ring of R is a right Goldie ring. Let P be a prime ideal of R. By our assumption on R, R/P is a domain and right duo ring. Hence R/P has finite right uniform dimension, by [12, Proposition 4.6]. Therefore R/P is a right Goldie ring [11, Theorem 2.3.6].

(a) For the sufficiency consider that every prime factor ring of R is a simple and right duo ring, hence is a division ring. Thus $\mathcal{P} = \mathcal{P}_\star$. Conversely, let P be any prime ideal of R. By the above, R/P is a right Goldie ring. Now an application of Theorem 2.10(b) for the ring R/P, shows that R/P is an Artinian ring and so P is a maximal ideal of R, as desired.

(b) The first note that R is a right quasi-duo ring by Lemma 2.6. Also if $\dim(R) = 0$, then by the remarks before Theorem 2.7, $R/\mathrm{N}(R)$ is a right V-ring.

(iii) $\Rightarrow$ (i) By Theorem 2.7(c).

(iii) $\Rightarrow$ (ii) By [10, Corollary 5.4] we have $\mathcal{W} \subseteq \Omega_{\mathcal{P}}$. On the other hand, every nonzero R-module M with $\mathrm{Rej}(M, \mathcal{P}) = 0$ is a module over $R/\mathrm{N}(R)$. The result is now obtained by Theorem 2.7(c).

(i) or (ii) $\Rightarrow$ (iii). By a similar argument, as it is seen in (a), we deduce that $\dim(R) = 0$. Thus by the first statements, $R/\mathrm{N}(R)$ is a right V-ring. Therefore every nonzero module over $R/\mathrm{N}(R)$ embeds in a product of simple R-modules. Hence Mod-$R/\mathrm{N}(R) = \Omega_{\mathcal{P}} \subseteq \mathcal{CS}$ [10, Proposition 5.5]. Therefore one of the conditions (i) or (ii) implies that Mod-$R/\mathrm{N}(R) \subseteq \mathcal{S}$. Hence $R/\mathrm{N}(R)$ is a right semi-Artinian ring by Theorem 2.7(c). □

A class of R-modules $\mathcal{C}$ is called *enveloping* if every R-module has an envelop in $\mathcal{C}$, see [6].

Corollary 2.13. *Let R be a right duo ring such that* $\dim(R) = 0$ *and $R/N(R)$ is a right semi-Artinian ring. Then the class $\mathcal{W}$ is enveloping.*

Proof. By Proposition 2.5 and Theorem 2.12, we have $\mathcal{W} = \Omega_{\mathcal{P}_\star}$. Hence the result is obtained by [5, Thereom 3.3]. □

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