On the Construction of Polynomial $\beta$-algebras over a Field

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Abstract. In this paper we construct quadratic $\beta$-algebras on a field, and we discuss both linear-quadratic $\beta$-algebras and quadratic-linear $\beta$-algebras in a field. Moreover, we discuss some relations of binary operations in $\beta$-algebras.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras([3, 4]). We refer useful textbooks for $BCK/BCI$-algebra to [2, 7, 10]. J. Neggers and H. S. Kim([8]) introduced another class related to some of the previous ones, viz., $B$-algebras and studied some of its properties. They also introduced the notion of $\beta$-algebra([9]) where two operations are coupled in such a way as to reflect the natural coupling which exists between the usual group operation and its associated $B$-algebra which is naturally defined by it. P. J. Allen et al.([1]) gave another proof of the close relationship of $B$-algebras with groups using the observation that the zero adjoint mapping is surjective. H. S. Kim and H. G. Park([5]) showed that if $X$ is a 0-commutative $B$-algebra, then $(x * a) * (y * b) = (b * a) * (y * x)$. Using this property they showed that the class of $p$-semisimple $BCI$-algebras is equivalent to the class of 0-commutative $B$-algebras. Y. H. Kim and K. S. So ([6]) investigated some properties of $\beta$-algebras and further relations with $B$-algebras. Especially, they showed that if $(X, -, +, 0)$ is a $B$-algebra, then $(X, +)$ is a semigroup with identity 0. They discussed some constructions of linear $\beta$-algebras in a field $K$.

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In this paper we construct quadratic $\beta$-algebras on a field, and we discuss both linear-quadratic $\beta$-algebras and quadratic-linear $\beta$-algebras in a field. Moreover, we discuss some relations of binary operations in $\beta$-algebras.

2. Preliminaries

A $\beta$-algebra([9]) is a non-empty set $X$ with a constant 0 and two binary operations “$+$” and “$-$” satisfying the following axioms: for any $x, y, z \in X$,

(I) $x - 0 = x$,

(II) $(0 - x) + x = 0$,

(III) $(x - y) - z = x - (z + y)$.

Example 2.1([9]) Let $X := \{0, 1, 2, 3\}$ be a set with the following tables:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
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<tr>
<td>2</td>
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<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Then $(X, +, -)$ is a $\beta$-algebra.

Given a $\beta$-algebra $X$, we denote $x^* := 0 - x$ for any $x \in X$.

Theorem 2.2([6]) Let $(K, +, \cdot, e)$ be a field (sufficiently large) and let $x, y \in K$. Then $(K, \oplus, \ominus, e)$ is a $\beta$-algebra, where $x \oplus y = x + y - e$ and $x \ominus y = x - y + e$ for any $x, y \in K$.

We call such a $\beta$-algebra described in Theorem 2.2 a linear $\beta$-algebra.

If we let $\varphi : K \to K$ be a map defined by $\varphi(x) = e + bx$ for some $b \in K$. Then we have

$$\varphi(x + y) = e + b(x + y)$$
$$\quad = (e + bx) + (e + by) - e$$
$$\quad = \varphi(x) \ominus \varphi(y)$$

and

$$\varphi(x - y) = e + b(x - y)$$
$$\quad = (e + bx) - (e + by) + e$$
$$\quad = \varphi(x) \ominus \varphi(y),$$

so that $\varphi(0) = e$ implies $\varphi : (K, -, +, 0) \to (K, \ominus, \oplus, e)$ is a homomorphism of $\beta$-algebras, where “$-$” is the usual subtraction in the field $K$. If $b \neq 0$, then
ψ : (K, ⊕, ⊖, e) → (K, −, +, 0) defined by ψ(x) := (x − e)/b is a homomorphism of β-algebras and the inverse mapping of the mapping φ, so that (K, ⊕, ⊖, e) and (K, −, +, 0) are isomorphic as β-algebras, i.e., there is only one isomorphism type in this case. We summarize:

**Proposition 2.3. ([6])** The β-algebra (K, ⊕, ⊖, e) discussed in Theorem 2.2 is unique up to isomorphism.

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra (X; ∗, 0) of type (2, 0) is called a BCI-algebra ([3, 4]) if it satisfies the following conditions: for all x, y, z ∈ X,

(i) (((x ∗ y) ∗ (x ∗ z)) ∗ (z ∗ y) = 0),

(ii) ((x ∗ (x ∗ y)) ∗ y = 0),

(iii) (x ∗ x = 0),

(iv) (x ∗ y = 0, y ∗ x = 0 ⇒ x = y).

If a BCI-algebra X satisfies the following identity: for all x ∈ X,

(v) (0 ∗ x = 0),

then X is called a BCK-algebra ([3, 4]).

3. Constructions of Polynomial β-algebras

Let (K, +, ·, e) be a field (sufficiently large) and let x, y ∈ K. First, we consider the case of quadratic-linear β-algebras, i.e., x ⊕ y is the polynomial of x, y with degree 2, and x ⊖ y is the polynomial of x, y with degree 1. Define two binary operations “⊕, ⊖” on K as follows:

\[ x ⊕ y := A + Bx + Cy + Dx^2 + Exy + Fy^2, \]

\[ x ⊖ y := α + βx + γy \]

where α, β, γ, A, B, C, D, E, F ∈ K (fixed). Assume that (K, ⊕, ⊖, 0) is a β-algebra. It is necessary to find proper solutions for two equations. Since x = x ⊕ e = α + βx + γe, we obtain (β − 1)x + (α + γe) = 0, and hence β = 1 and α = −γe. It follows that

\[ \text{(3.1)} \quad x ⊕ y = x + γ(y − e) \]

From (3.1) we obtain e ⊕ x = e + γ(x − e) = (1 − γ)e + γx. Since (e ⊕ x) ⊕ x = e,
we obtain

\[
e = (e \odot x) \oplus x
\]

\[
= [(1 - \gamma)e + \gamma x] \oplus x
\]

\[
= A + B[(1 - \gamma)e + \gamma x] + Cx + D[(1 - \gamma)e + \gamma x]^2
\]

\[
+ E[(1 - \gamma)e + \gamma x]x + Fx^2
\]

\[
= [A + B(1 - \gamma)e + D(1 - \gamma)^2e^2] + [B\gamma + C + 2D\gamma(1 - \gamma)e
\]

\[
+ E(1 - \gamma)e]x + [D\gamma^2 + E\gamma + F]x^2
\]

If we assume that \(|K| \geq 3\), then we obtain

\[
D\gamma^2 + E\gamma + F = 0
\]

(3.2)

\[
B\gamma + C + 2D\gamma(1 - \gamma)e + E(1 - \gamma)e = 0
\]

\[
A + B(1 - \gamma)e + D(1 - \gamma)^2e^2 = 0
\]

Case(i). \(\gamma \neq 0\). By formula (3.1) we obtain

\[
(x \odot y) \odot z = x \odot y + \gamma(z - e)
\]

\[
= x + \gamma(y - e) + \gamma(z - e)
\]

\[
= x + \gamma(y + z - 2e)
\]

and

\[
x \oplus (z \odot y) = x + \gamma(z \oplus y - e)
\]

By (III), we obtain \(x + \gamma(y + z - 2e) = x + \gamma(z \oplus y - e)\). Since \(\gamma \neq 0\), we have \(z \oplus y - e = z + y - 2e\). This shows that \(x \oplus y = x + y - e\) for all \(x, y \in K\). In this case, we obtain the linear case as described in Theorem 2.2.

Case (ii). \(\gamma = 0\). By (3.1), we obtain \(x \odot y = x\). Since \(|K| \geq 3\), the formula (3.2) can be represented as

\[
F = 0, C + Ee = 0, A + Be + De^2 = e
\]

It follows that \(F = 0, C = -Ee, A = e - Be - De^2\). This shows that

\[
x \oplus y = (e - Be - De^2) + Bx + (-Ee)y + Dx^2 + Exy
\]

\[
= e + B(x - e) + D(x^2 - e^2) + E(x - e)y
\]

\[
= e + [B + D(x + e) + Ey](x - e)
\]

which means that \(x \oplus y\) is of the form:

\[
x \oplus y = e + (a + bx + cy)(x - e).
\]

It is a quadratic form (not a linear form) and so \((K, \oplus, \odot, e)\) is a quadratic-linear \(\beta\)-algebra where \(x \oplus y = e + (a + bx + cy)(x - e)\) and \(x \odot y = x\). We summarize:
Theorem 3.1. Let \((K, +, \cdot, e)\) be a field (sufficiently large). If we define two binary operations “⊕, ⊖” on \(K\) by
\[
x \oplus y := e + (a + bx + cy)(x - e) \\
x \ominus y := x
\]
for all \(x, y \in K\), then \((K, \oplus, \ominus, e)\) is a quadratic-linear \(β\)-algebra.

Corollary 3.2. Let \((K, +, \cdot, 0)\) be a field (sufficiently large) and let \(a, b, c \in K\). If we define two binary operations “⊕, ⊖” on \(K\) by
\[
x \oplus y := ax + bx^2 + cxy \\
x \ominus y := x
\]
for all \(x, y \in K\), then \((K, \oplus, \ominus, 0)\) is a quadratic-linear \(β\)-algebra.

Proof. It follows immediately from Theorem 3.1 by letting \(e := 0\). □

Example 3.3. Let \(R\) be the set of all real numbers. If we define
\[
x \oplus y := x - x^2 - 2xy \\
x \ominus y := x
\]
for all \(x, y \in R\), then \((R, \oplus, \ominus, 0)\) is a quadratic-linear \(β\)-algebra.

Let \((K, +, \cdot, e)\) be a field (sufficiently large) and let \(x, y \in K\). Next, we consider the case of linear-quadratic \(β\)-algebras, i.e., \(x \oplus y\) is the polynomial of \(x, y\) with degree 1, and \(x \ominus y\) is the polynomial of \(x, y\) with degree 2.

Theorem 3.4. There is no linear-quadratic \(β\)-algebras over a field \((K, +, - , e)\).

Proof. Let \((K, +, \cdot, e)\) be a field (sufficiently large). Define two binary operations “⊕, ⊖” on \(K\) as follows:
\[
x \oplus y := A + Bx + Cy, \\
x \ominus y := α + βx + γy + δx^2 + εxy + ξy^2
\]
where \(A, B, C, α, β, γ, δ, ε, ξ ∈ K\) (fixed). Assume that \((K, \oplus, \ominus, e)\) is a \(β\)-algebra and \(|K| ≥ 3\). Then
\[
x = x \ominus e = α + βx + γe + δx^2 + εex + ξe^2 = [α + γe + ξe^2] + [β + εe]x + δx^2
\]
It follows that
\[
α + γe + ξe^2 = 0 \\
β + εe = 1 \\
δ = 0
\]
(3.3)

Case (i). \(e = 0\). By formula (3.3) we obtain \(α = 0, β = 1, δ = 0\). It follows that
\[
x \ominus y = x + γy + εxy + ξy^2
\]
(3.4)
It follows that $0 \ominus x = 0 + \gamma x + \varepsilon 0x + \xi x^2 = \gamma x + \xi x^2$ and hence

$$
0 = (0 \ominus x) \ominus x \\
= (\gamma x + \xi x^2) \ominus x \\
= A + B(\gamma x + \xi x^2) + Cx \\
= A + (B\gamma + C)x + B\xi x^2
$$

for all $x \in K$. This shows that $A = 0, B\gamma + C = 0, B\xi = 0$.

**Subcase (i).** $B = 0$. Since $C = -B\gamma$, we obtain $C = 0$ and hence $x \oplus y = 0$ for all $x, y \in K$. This shows that $(0 \ominus x) \ominus x = 0$. Using formula (3.4) we obtain

$$(x \ominus y) \ominus z = (x + \gamma y + \varepsilon xy + \xi y^2) \ominus z \\
= (x + \gamma y + \varepsilon xy + \xi y^2) + \gamma z + \\
\varepsilon(x + \gamma y + \varepsilon xy + \xi y^2)z + \xi z^2 \\
= x + \gamma y + \gamma z + \varepsilon xy + \xi(y^2 + z^2) + \\
\varepsilon(x + \gamma y + \varepsilon xy + \xi y^2)z
$$

and

$$
x \ominus (z \oplus y) = x \ominus 0 = x
$$

By (III), we obtain $\gamma = 0, \varepsilon = 0, \xi = 0$, proving that $x \ominus y = x$. Hence $x \oplus y = x$ and $x \ominus y = x$ show that $(K, \ominus, \ominus, 0)$ is not a linear-quadratic $\beta$-algebra.

**Subcase (i-1).** $B = 0$. Since $C = -B\gamma$, we obtain $C = 0$ and hence $x \ominus y = 0$ for all $x, y \in K$. This shows that $(0 \ominus x) \ominus x = 0$. Using formula (3.4) we obtain

$$(x \ominus y) \ominus z = (x + \gamma y + \varepsilon xy + \xi y^2) \ominus z \\
= (x + \gamma y + \varepsilon xy + \xi y^2) + \gamma z + \\
\varepsilon(x + \gamma y + \varepsilon xy + \xi y^2)z + \xi z^2 \\
= x + \gamma y + \gamma z + \varepsilon xy + \xi(y^2 + z^2) + \\
\varepsilon(x + \gamma y + \varepsilon xy + \xi y^2)z
$$

and

$$(x \ominus y) \ominus z = (x + \gamma y + \varepsilon xy) \ominus z \\
= x + \gamma y + \varepsilon xy + \gamma z + \varepsilon(x + \gamma y + \varepsilon xy)z
$$

By (III), we obtain $\varepsilon = 0$ and $\gamma(1 + \gamma)B = 0$. If $\gamma = 0$, then $x \ominus y = Bx$ and $x \ominus y = x$. If $B = 0$, then it is the same case as subcase (i-1). If $\gamma = -1$, then $x \ominus y = B(x + y)$ and $x \ominus y = x - y$. This shows that $(K, \ominus, \ominus, 0)$ is not a linear-quadratic $\beta$-algebra.

**Case (ii).** $e \neq 0$. It follows from (3.3) that $\alpha = -\gamma e - \xi e^2, \beta = 1 - \varepsilon e, \delta = 0$. Hence

$$(3.5) \\
x \ominus y = (\gamma e - \xi e^2) + (1 - \varepsilon e)x + \gamma y + \varepsilon xy + \xi y^2$$
Subcase (ii-1). \( \xi \neq 0 \). The formula (3.5) can be written as

\[
\text{(3.6)} \quad x \ominus y = x + (\gamma + \varepsilon x)(y - e) + \xi(y^2 - e^2)
\]

If \( y := e \) in (3.6), then \( x \ominus e = x + (\gamma + \varepsilon x)(e - e) + \xi(e^2 - e^2) = x \). If \( x := e \) and \( y := x \) in (3.6), then \( e \ominus x = x + (\gamma + \varepsilon e)(x - e) + \xi(x^2 - e^2) \). It follows that

\[
e = (e \ominus x) \ominus x
\]

\[
= A + B(e \ominus x) + Cx
\]

\[
= A + B[e + (\gamma + \varepsilon e)(x - e) + \xi(x^2 - e^2)] + Cx
\]

\[
= [A + Be - B(\gamma + \varepsilon e)e - B\xi e^2] + [B(\gamma + \varepsilon e) + C]x + B\xi x^2
\]

for all \( x \in K \), and hence we obtain

\[
B\xi = 0
\]

\[
B(\gamma + \varepsilon e) + C = 0
\]

\[
A + Be - B(\gamma + \varepsilon e)e - B\xi e^2 = 0
\]

Since \( \xi \neq 0 \), we have \( B = 0 \) and hence \( C = 0, A = 0 \), i.e., \( x \oplus y = 0 \) and \( x \ominus y = x + (\gamma + \varepsilon x)(y - e) + \xi(y^2 - e^2) \) does not form a \( \beta \)-algebra, since \( e \ominus (e \ominus e) = e - e(\gamma + \varepsilon e) - \xi e^2 \) and \( (e \ominus e) \ominus e = e \).

Subcase (ii-2). \( \xi = 0 \). The formula can be written as

\[
\text{(3.7)} \quad x \ominus y = x + (\gamma + \varepsilon x)(y - e)
\]

It follows that \( x \ominus e = x + (\gamma + \varepsilon x)(e - e) = x \) and \( e \ominus x = e + (\gamma + \varepsilon e)(x - e) \). By (II) we obtain the following.

\[
e = (e \ominus x) \ominus x
\]

\[
= [e + (\gamma + \varepsilon e)(x - e)] \ominus x
\]

\[
= A + B[e + (\gamma + \varepsilon e)(x - e)] + Cx
\]

\[
= [A + Be - B(\gamma + \varepsilon e)e] + [B(\gamma + \varepsilon e) + C]x
\]

for all \( x \in K \), and hence we obtain

\[
A + Be - B(\gamma + \varepsilon e)e = e
\]

\[
B(\gamma + \varepsilon e) + C = 0
\]

Hence \( A = e + Be[\gamma + \varepsilon e - 1] \) and \( C = -B(\gamma + \varepsilon e) \). It follows that

\[
\text{(3.8)} \quad x \oplus y = [e + Be(\gamma + \varepsilon e - 1)] + Bx - B(\gamma + \varepsilon e)y
\]

Using formulas (3.7) and (3.8), we obtain

\[
x \ominus (z \oplus y) = x + (\gamma + \varepsilon x)(z \ominus y - e)
\]

\[
= x + (\gamma + \varepsilon x)[e + Be(\gamma + \varepsilon e - 1)] + Bz - B(\gamma + \varepsilon e)y - e]
\]

\[
= x + (\gamma + \varepsilon x)[Be(\gamma + \varepsilon e - 1)] + Bz - B(\gamma + \varepsilon e)y]
\]

\[
= x + B(\gamma + \varepsilon x)[e(\gamma + \varepsilon e - 1) + z - (\gamma + \varepsilon e)y]
\]
Proposition 4.2. Let

\[ (x \circ y) \circ z = x \circ y + (\gamma + \varepsilon(x \circ y))(z - e) \]

\[ = x + (\gamma + \varepsilon x)(y - e) + [\gamma + \varepsilon x(y + (\gamma + \varepsilon x)(y - e))](z - e) \]

\[ = x + (\gamma + \varepsilon x)((y - e) + (1 + \varepsilon(y - e))(z - e)) \]

By (III), we have

\[ B(\gamma + \varepsilon x)[\gamma + \varepsilon y - 1] + z - (\gamma + \varepsilon x)y \]

\[ = (\gamma + \varepsilon x)(y - e) + (1 + \varepsilon(y - e))(z - e) \]

Subcase (ii-2-a). \( \gamma = \varepsilon = 0 \). We obtain \( \beta = 1, \alpha = 0 \) and hence \( x \circ y = x, \)
\( x \circ y = (e - Be) + Bx, \) a linear \( \beta \)-algebra.
Subcase (ii-2-b). \( \gamma + \varepsilon x \neq 0 \) for all \( x \in K \). We obtain the following formula.

\[ B[e(\gamma + \varepsilon x - 1) + z - (\gamma + \varepsilon x)y] \]

\[ = (y - e) + (1 + \varepsilon(y - e))(z - e) \]

for all \( y, z \in K \). This shows that \( \varepsilon = 0, \gamma = -1, \delta = 0, \beta = 1, \alpha = e \) and \( A = -e \)
and \( B = C = 1 \). Hence \( x \circ y = x + y - e \) and \( x \circ y = x - y + e \), i.e., \( (K, \circ, \circ, \circ) \)
is a linear \( \beta \)-algebra which is not a quadratic-linear \( \beta \)-algebra. Hence there is no linear-quadratic \( \beta \)-algebra which is not a linear \( \beta \)-algebra.

Problem. Construct a complete quadratic \( \beta \)-algebra over a field, i.e., \( x \circ y \) and \( x \circ y \) are both polynomials of \( x \) and \( y \) of degree 2.

4. Some Relations of Binary Operations in \( \beta \)-algebras

In this section, we discuss some relations of binary operations in \( \beta \)-algebras. For example, given a groupoid \((X, +)\), we want to know the structure of \((X, -)\) if \((X, +, -, 0)\) is a \( \beta \)-algebra. Given a non-empty set \( X \), a groupoid \((X, *)\) is said to be a left-zero semigroup if \( x * y = x \) for all \( x, y \in X \). Similarly, a groupoid \((X, *)\) is said to be a right-zero semigroup if \( x * y = y \) for all \( x, y \in X \).

Proposition 4.1. If \((X, \oplus)\) is a left-zero semigroup and if \((X, \oplus, \ominus, 0)\) is a \( \beta \)-algebra, then \((X, \circ)\) is also a left-zero semigroup.

Proof. Assume that \((X, \oplus, \ominus, 0)\) is a \( \beta \)-algebra. Then \( x \ominus 0 = x \) for all \( x \in X \), and \( (x \ominus y) \circ z = x \ominus (z \ominus y) = x \ominus z \), i.e., \( (x \ominus y) \circ z = x \ominus z \) for all \( x, y, z \in X \). It follows that \( x \ominus y = (x \ominus y) \ominus 0 = x \ominus 0 = x \), i.e., \( x \ominus y = x \), for all \( x, y \in X \), proving that \((X, \circ)\) is a left-zero semigroup.

Proposition 4.2. Let \((X, \ominus, 0)\) be an algebra with \( 0 \ominus x = 0 \) for all \( x \in X \). If \((X, \ominus)\) is a left-zero semigroup, then \((X, \oplus, \ominus, 0)\) is a \( \beta \)-algebra.

Proof. Since \((X, \ominus)\) is a left-zero semigroup, the conditions (I) and (III) hold. It follows from \( 0 \ominus x = 0 \) for all \( x \in X \) that \( 0 = 0 \ominus x = (0 \ominus x) \ominus x \), proving the proposition.
Corollary 4.3. Let \((X, *, 0)\) be a BCK-algebra. If \((X, \ominus)\) is a left-zero semigroup, then \((X, *, \ominus, 0)\) is a \(\beta\)-algebra.

Proof. Straightforward. \(\square\)

Proposition 4.4. Let \((X, \ominus, 0)\) be an algebra with \(0 \oplus x = 0\) for all \(x \in X\). If \((X, \ominus, 0)\) is a \(\beta\)-algebra, then \((X, \ominus)\) is a left-zero semigroup.

Proof. Let \((X, \ominus, 0)\) be a \(\beta\)-algebra. If we let \(z := 0\) in (III), then \(x \ominus y = (x \ominus y) \ominus 0 = x \ominus (0 \ominus y) = x \ominus 0 = x\) for all \(x, y \in X\), proving the proposition. \(\square\)

Proposition 4.5. Let \((X, \oplus)\) be a right-zero semigroup. If \((X, \ominus, 0)\) is a \(\beta\)-algebra, then \(X = \{0\}\).

Proof. Assume \((X, \ominus, 0)\) is a \(\beta\)-algebra. Since \((X, \ominus)\) is a right-zero semigroup, by (II), we have \(0 = (0 \ominus x) \ominus x = x \ominus x\) for all \(x \in X\), proving that \(X = \{0\}\). \(\square\)

Proposition 4.6. Let \((X, \ominus)\) be a right-zero semigroup. If \((X, \ominus, 0)\) is a \(\beta\)-algebra, then \(X = \{0\}\).

Proof. Assume \((X, \ominus, 0)\) is a \(\beta\)-algebra. Since \((X, \ominus)\) is a right-zero semigroup, by (II), we have \(0 = (0 \ominus x) \ominus x = x \ominus x\) for all \(x \in X\). By (III), we have \(z = (x \ominus y) \ominus z = x \ominus (z \ominus y) = z \ominus y\) for all \(y, z \in X\). It follows that \(x = x \ominus x = 0\) for all \(x \in X\), proving the proposition. \(\square\)

Theorem 4.7. Let \((K, +, -)\) be a field with \(|K| \geq 4\) and let \(e \in K\). Define a binary operation \(\ominus\) on \(K\) by \(x \ominus y := p(x, y)\), i.e., a quadratic polynomial of \(x\) and \(y\) in \(K\). If \(e \ominus x = e\) for all \(x \in K\), then \(x \ominus y = e + (A + Bx + Cy)(x - e)\) for all \(x, y \in K\) where \(A, B, C \in K\).

Proof. Assume \(p(x, y) := A + Bx + Cy\) for all \(x, y \in K\) where \(A, B, C \in K\). Since \(e \ominus x = e\), we have \(e = e \ominus y = A + Be + Cy\) for all \(y \in K\). It shows that \(A = e(1 - B), C = 0\). Hence \(x \ominus y = e(1 - B) + Bx = e + B(x - e)\). Assume \(p(x, y) := A + Bx + Cy + Dx^2 + Exy + Fy^2\). Then

\[
e = e \ominus y = A + Be + Cy + De^2 + Ey + Fy^2 = (A + Be + De^2) + (C + E)e + Fy^2
\]

It follows that \(F = 0, C + Ee = 0, A + Be + De^2 = e\). Hence

\[
p(x, y) = (e - Be - De^2) + Bx - Eey + Dx^2 + Exy = e + [B + D(x + e) + E][x - e] = e + (B + De + Dx + Ey)(x - e),
\]

i.e., \(p(x, y)\) is of the form \(p(x, y) = e + (A + Bx + Cy)(x - e)\). This shows that \(x \ominus y = e + q(x, y)(x - e)\) where \(q(x, y)\) is a linear polynomial of degree \(\leq 1\). \(\square\)

Using Theorem 4.7 and Proposition 4.2, we obtain the following.
Corollary 4.8. Let $(K, +, -, 0)$ be a field with $|K| \geq 4$ and let $e \in K$. Define a binary operation “$\oplus$” on $K$ by
\[ x \oplus y := e + q(x, y)(x - e) \]
where $q(x, y)$ is any polynomial of $x$ and $y$ in $K$. If $x \ominus y := x$ for all $x, y \in K$, then $(K, \oplus, \ominus, e)$ is a $\beta$-algebra.

References