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Spectrally Bounded Higher Derivations Mapping into the Jacobson Radical

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Abstract. We obtain commutativity-free characterizations of higher derivations on a unital Banach algebra that map into its Jacobson radical. We also investigate the spectral boundedness of generalized higher derivations.

1. Introduction

Throughout, let A be an algebra over the complex field. A *derivation* is an additive mapping $\delta : A \to A$ satisfying the Leibniz rule $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in A$.

Let A and B be Banach algebras. A linear mapping $\alpha : A \to B$ is called *spectrally bounded* if there is $M \geq 0$ such that $r(\alpha(x)) \leq Mr(x)$ for all $x \in A$. If $r(\alpha(x)) = r(x)$ for all $x \in A$, we say that α is a *spectral isometry*. If $r(x) = 0$, then x is called *quasinilpotent*. (Herein, $r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ denotes the *spectral radius* of $x \in A$).

One of principal results concerning derivations in Banach algebra theory is the classical Singer-Wermer theorem [6] which states that every continuous derivation on a commutative Banach algebra maps into the Jacobson radical $rad(A)$. In the same paper they conjectured that the assumption of continuity is not necessary and M. P. Thomas [7] proved the conjecture.

Now the problem concerning derivations on Banach algebras belongs to the noncommutative setting which states that a (possibly discontinuous) derivation δ on a (possibly noncommutative) Banach algebra A such that the commutator $\delta(x)x - x\delta(x)$ belongs to $rad(A)$ for all $x \in A$, maps A into its Jacobson radical. Equivalently, every derivation on A leaves primitive ideals of A invariant, which is called the *noncommutative Singer-Wermer conjecture*. But the question whether

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this is true, is still an open problem. There are various partial answers of the noncommutative Singer-Wermer conjecture, for example, Brešar and Mathieu proved that every spectrally bounded derivation on a unital Banach algebra maps into the radical [2, Theorem 2.5].

Let $\mathbb{N}$ be the set of the natural numbers. For $m \in \mathbb{N}$, a sequence $H = \{h_1, \cdots, h_m\}$ (resp., $H = \{h_1, \cdots, h_n, \cdots\}$) of linear mappings on an algebra A is called a *higher derivation* of rank m (resp. infinite rank) on A if the functional equation

$$h_n(xy) = \sum_{i=0}^{n} h_i(x)h_{n-i}(y).$$

holds for each $n = 1, 2, \cdots, m$ (resp. $n \in \mathbb{N}$) and for all $x, y \in A$, where h_0 is an identity mapping on A. We say that a higher derivation H of rank m (resp. infinite rank) on A maps into its Jacobson radical $rad(A)$, denoted by $H(A) \subseteq rad(A)$ if $h_n(A) \subseteq rad(A)$ for each $n = 1, 2, \cdots, m$ (resp. $n \in \mathbb{N}$). Note that a higher derivation of rank 1 is just a derivation.

A sequence $G = \{g_1, \cdots, g_m\}$ (resp. $G = \{g_1, \cdots, g_n, \cdots\}$) of linear mappings on a Banach algebra A is *spectrally bounded* on A if g_n is spectrally bounded on A for each $n = 1, 2, \cdots, m$ (resp. $n \in \mathbb{N}$).

R.J. Roy [5] and N.P. Jewell [3] showed that the continuity of derivations on semisimple Banach algebras can be extended to higher derivations. K.-W. Jun and Y.-W. Lee [4] generalized the Singer-Wermer theorem to higher derivations of any rank: *every higher derivation of any rank which is continuous on a commutative Banach algebra, maps into its Jacobson radical.*

In this note, using the arguments in [2], we extend Brešar and Mathieu's results [2] to higher derivations, that is, we obtain commutativity-free characterizations of higher derivations on a unital Banach algebra that map into its Jacobson radical. We also investigate the spectral boundedness of generalized higher derivations.

2. Main Results

Theorem 2.1. *Let $H = \{h_n\}$ be a higher derivation of any rank on a unital Banach algebra A. Then $H(A) \subseteq rad(A)$ if and only if H is spectrally bounded on A.*

Proof. One way implication is obvious. Conversely, we suppose that H is a spectrally bounded higher derivation of any rank on A. By induction, we prove that $h_n(A) \subseteq rad(A)$ for all $n \in \mathbb{N}$. Since h_1 is a spectrally bounded derivation on A, we see that $h_1(A) \subseteq rad(A)$ from [2, Theorem 2.5]. Now we assume that $h_n(A) \subseteq rad(A)$ for $1 \leq n \leq k-1$. Let $\pi : A \to A/rad(A)$ be the canonical epimorphism which is spectrally isometry. Then for each $x \in A$, each $y \in rad(A)$

and each $k \in \mathbb{N}$, we get

$$\begin{aligned} r(xh_k(y)) &= r\Big(h_k(xy) - h_k(x)y - \sum_{i=1}^{k-1} h_i(x)h_{k-i}(y)\Big) \\ &= r\Big(\pi\Big(h_k(xy) - h_k(x)y - \sum_{i=1}^{k-1} h_i(x)h_{k-i}(y)\Big)\Big) \\ &= r(\pi(h_k(xy))) \\ &= r(h_k(xy)) \leq M_k r(xy) = 0 \end{aligned}$$

for some $M_k > 0$. Therefore, we obtain that $h_k(rad(A)) \subseteq rad(A)$. Let us define a mapping $\tilde{h}_k$ on a unital semisimple Banach algebra $A/rad(A)$ by $\tilde{h}_k(x + rad(A)) = h_k(x) + rad(A)$ for all $x \in A$. Then $\tilde{h}_k$ becomes a derivation on $A/rad(A)$ since $h_n(A) \subseteq rad(A)$ for $1 \leq n \leq k-1$. By the hypothesis, since h_k is spectrally bounded on A, we see that $\tilde{h}_k$ also is spectrally bounded on $A/rad(A)$. That is, $\tilde{h}_k$ is a spectrally bounded derivation on $A/rad(A)$. Therefore it follows from [2, Theorem 2.5] that $\tilde{h}_k = 0$ on $A/rad(A)$. This yields that $h_k(A) \subseteq rad(A)$. The proof of theorem is complete. □

Theorem 2.2. *Let $H = \{h_n\}$ be a higher derivation of any rank on a unital Banach algebra A with unit e. Then $H(A) \subseteq rad(A)$ if and only if for each $n \in \mathbb{N}$, $\sup\{r(x^{-1}h_n(x)) : x \in A$ invertible$\} < \infty$.*

Proof. Let $s_n = \sup\{r(x^{-1}h_n(x)) : x \in A \text{ invertible}\}$ for each $n \in \mathbb{N}$ and let $\pi : A \to A/rad(A)$ be the canonical epimorphism. If H is a higher derivation of any rank on A such that $H(A) \subseteq rad(A)$, then $s_n = 0$ for each $n \in \mathbb{N}$. Suppose conversely that $s_n < \infty$ for each $n \in \mathbb{N}$. As in the proof of Theorem 2.1, it needs to prove that $h_n(A) \subseteq rad(A)$ for all $n \in \mathbb{N}$. We use the induction. Since h_1 is a derivation on A and $s_1 < \infty$, we see, by [2, Theorem 2.6], that $h_1(A) \subseteq rad(A)$. Assume that $h_n(A) \subseteq rad(A)$ for $1 \leq n \leq k-1$. It needs to show that $h_k(rad(A)) \subseteq rad(A)$. Given $x \in rad(A)$, we have $(e+x)^{-1} = e - x(e+x)^{-1} \in e + rad(A)$ and so, considering $h_n(e) = 0$ for all $n \in \mathbb{N}$,

$$\begin{aligned} r((e+x)^{-1}h_k(e+x)) &= r((e - x(e+x)^{-1})h_k(x)) \\ &= r(h_k(x) - x(e+x)^{-1}h_k(x)) \\ &= r(\pi(h_k(x) - x(e+x)^{-1}h_k(x))) \\ &= r(\pi(h_k(x))) \\ &= r(h_k(x)). \end{aligned}$$

From the assumption, it follows that $r(h_k(x)) \leq s_k < \infty$ for all $x \in rad(A)$. This yields $r(h_k(x)) = 0$ for all $x \in rad(A)$. Consequently, for each $x \in A$ and each

$y \in rad(A)$, we get

$$\begin{aligned} r(xh_k(y)) &= r\bigg(h_k(xy) - h_k(x)y - \sum_{i=1}^{k-1} h_i(x)h_{k-i}(y)\bigg) \\ &= r\bigg(\pi\bigg(h_k(xy) - h_k(x)y - \sum_{i=1}^{k-1} h_i(x)h_{k-i}(y)\bigg)\bigg) \\ &= r(\pi(h_k(xy))) \\ &= r(h_k(xy)) = 0. \end{aligned}$$

Therefore, we see that $h_k(rad(A)) \subseteq rad(A)$, as claimed. Define a mapping $\tilde{h}_k$ on a unital semisimple Banach algebra $A/rad(A)$ by $\tilde{h}_k(x + rad(A)) = h_k(x) + rad(A)$ for all $x \in A$. Then $\tilde{h}_k$ becomes a derivation on $A/rad(A)$ since $h_n(A) \subseteq rad(A)$ for $0 \leq n \leq k-1$. Since

$$\sup\{r((x + rad(A))^{-1}\tilde{h}_k(x + rad(A))) : x \in A \text{ invertible}\} = s_k < \infty,$$

it follows from [2, Theorem 2.6] that $\tilde{h}_k = 0$ on $A/rad(A)$. Hence we have $h_k(A) \subseteq rad(A)$. This complete the proof. □

An additive mapping $\varphi : A \to A$ is called a *left multiplier* if $\varphi(xy) = \varphi(x)y$ holds for all $x, y \in A$. In [1], M. Brešar defined the following concept. Let $\delta : A \to A$ be a derivation. An additive mapping $f : A \to A$ is called a *generalized derivation* associated with δ if $f(xy) = x\delta(y) + f(x)y$ holds for all $x, y \in A$. This notion is a generalization of both derivations and multipliers. Note that $f = L_a + \delta$ on a unital Banach algebra with unit e, where L_a is a left multiplication by $a = f(e)$ and that f need not map into the Jacobson radical [2]. M. Brešar and M. Mathieu [2] proved that the following conditions are equivalent on a unital Banach algebra.

(i) f is spectrally bounded.

(ii) Both L_a and δ are spectrally bounded.

Here we consider the extended concepts of generalized derivations and left multipliers, respectively.

Let A be a Banach algebra, $m \in \mathbb{N}$ and $H = \{h_1, \cdots, h_m\}$ (resp. $H = \{h_1, \cdots, h_n, \cdots\}$) a higher derivation of rank m (resp. infinite rank) on A. A sequence $F = \{f_1, \cdots, f_m\}$ (resp. $F = \{f_1, \cdots, f_n, \cdots\}$) of linear mappings on A is said to be a *generalized higher derivation* of rank m (resp. infinite rank) on A associated with H if the functional equation

$$f_n(xy) = \sum_{i=0}^{n} f_i(x)h_{n-i}(y)$$

holds for all $x, y \in A$, where f_0 and h_0 are identity mappings on A. Of course, a generalized higher derivation of rank 1 is a generalized derivation on A.

Also, let $G = \{g_1, \cdots, g_m\}$ (resp. $G = \{g_1, \cdots, g_n, \cdots\}$) be a sequence of mappings on A. A sequence $\Phi = \{\varphi_1, \cdots, \varphi_m\}$ (resp. $\Phi = \{\varphi_1, \cdots, \varphi_n, \cdots\}$) of linear mappings on A is called a *higher left multiplier* of rank m (resp. infinite rank) on A associated with G if the functional equation

$$\varphi_n(xy) = \sum_{i=0}^{n} \varphi_i(x) g_{n-i}(y)$$

is valid for all $x, y \in A$, where $\varphi_0 = 0$ on A and g_0 is an identity mapping on A. A higher left multiplier of rank 1 is just a left multiplier on A.

Lemma 2.3. *Let $H = \{h_n\}$ be a higher derivation of any rank on a unital Banach algebra A with unit e. A sequence $F = \{f_n\}$ is a generalized higher derivation of the same rank with H on A associated with H if and only if there exists a higher left multiplier $\Phi = \{\varphi_n\}$ of the same rank with H on A associated with H such that $F = \Phi + H$ on A.*

Proof. Suppose that $F = \{f_n\}$ be a generalized higher derivation of the same rank with H on A associated with H. Then

$$f_n(xy) = \sum_{i=0}^{n} f_i(x) h_{n-i}(y)$$

holds for all $x, y \in A$, where f_0 and h_0 are identity mappings on A. First, we see that $\varphi_1 = f_1 - h_1$, where $\varphi_1 = L_a$ is a left multiplication by $a = f_1(e)$. Put $\varphi_n = f_n - h_n$ for all $n \geq 2$. Then for each $n \geq 2$, φ_n is a linear mapping on A and we have

$$\begin{aligned} \varphi_n(xy) &= f_n(xy) - h_n(xy) \\ &= \sum_{i=0}^{n} f_i(x) h_{n-i}(y) - \sum_{i=0}^{n} h_i(x) h_{n-i}(y) \\ &= \sum_{i=0}^{n} [f_i(x) - h_i(x)] h_{n-i}(y) \\ &= \sum_{i=0}^{n} \varphi_i(x) h_{n-i}(y). \end{aligned}$$

for all $x, y \in A$. Therefore $\Phi = \{\varphi_n\}$ is a higher left multiplier associated with H and $f_n = \varphi_n + h_n$ for all $n \in \mathbb{N}$.

Conversely, let $\Phi = \{\varphi_n\}$ be a higher left multiplier of the same rank with H on A associated with H. Since $f_1 = \varphi_1 + h_1$, where $\varphi_1 = L_a$ is a left multiplication by $a = f_1(e)$, we assume that $f_n = \varphi_n + h_n$ for all $n \geq 2$. Then for each $n \in \mathbb{N}$, f_n

is a linear mapping on A and we get

$$\begin{aligned} f_n(xy) &= \varphi_n(xy) + h_n(xy) \\ &= \sum_{i=0}^{n} \varphi_i(x)h_{n-i}(y) + \sum_{i=0}^{n} h_i(x)h_{n-i}(y) \\ &= \sum_{i=0}^{n} [\varphi_i(x) + h_i(x)]h_{n-i}(y) \\ &= \sum_{i=0}^{n} f_i(x)h_{n-i}(y) \end{aligned}$$

for all $x, y \in A$. Thus $F = \{f_n\}$ is a generalized higher derivation on A associated with H. □

Lemma 2.4. *Let $H = \{h_n\}$ be a higher derivation of any rank on a unital Banach algebra A with unit e. Suppose that $F = \{f_n\}$ is a spectrally bounded generalized higher derivation of the same rank with H on A associated with H. Then F leaves the Jacobson radical invariant, that is, $f_n(rad(A)) \subseteq rad(A)$ for each $n \in \mathbb{N}$.*

Proof. Suppose that F is spectrally bounded on A. Let $\Phi = \{\varphi_n\}$ be a higher left multiplier of the same rank with H on A associated with H, where $\varphi_1 = L_a$ is a left multiplication with $a = f_1(e)$. Then it follows from Lemma 2.3 that $f_n = \varphi_n + h_n$ for all $n \in \mathbb{N}$. By [2, Lemma 2.7], we have $f_1(rad(A)) \subseteq rad(A)$ and so $h_1(rad(A)) \subseteq rad(A)$.

Assume that $h_n(rad(A)) \subseteq rad(A)$ for $1 \leq n \leq k-1$. Then for each $x \in rad(A)$, we obtain that

$$\varphi_n(x) = \sum_{i=0}^{n} \varphi_i(e)h_{n-i}(x) \in rad(A),$$

whence $\varphi_n(rad(A)) \subseteq rad(A)$. Therefore for each $x \in rad(A)$, we get

$$\varphi_k(x) = \sum_{i=0}^{k} \varphi_i(e)h_{k-i}(x) = \sum_{i=0}^{k-1} \varphi_i(e)h_{k-i}(x) + \varphi_k(e)x \in rad(A),$$

which yields that $\varphi_k(rad(A)) \subseteq rad(A)$.

Since f_k is spectrally bounded, we have for each $x \in rad(A)$,

$$\begin{aligned} r(h_k(x)y) &= r\left(h_n(xy) - \sum_{i=0}^{k-1} h_i(x)h_{k-i}(y)\right) \\ &= r\left(\pi\left(h_k(xy) - \sum_{i=0}^{k-1} h_i(x)h_{k-i}(y)\right)\right) \\ &= r(\pi(h_k(xy))) = r(h_k(xy)) \\ &= r(f_k(xy) - \varphi_k(xy)) \\ &= r(\pi(f_k(xy) - \varphi_k(xy))) \\ &= r(\pi(f_k(xy))) = r(f_k(xy)) \leq M_k r(xy) = 0 \end{aligned}$$

for some $M_k > 0$ and all $y \in A$. Thus we see that

$$h_k(rad(A)) \subseteq rad(A).$$

The induction means that

$$\varphi_n(rad(A)) \subseteq rad(A) \quad \text{and} \quad h_n(rad(A)) \subseteq rad(A)$$

for all $n \in \mathbb{N}$.

Since $f_n = \varphi_n + h_n$, it follows that

$$f_n(rad(A)) \subseteq rad(A)$$

for all $n \in \mathbb{N}$ which is a conclusion. □

As our final result, we extend [2, Theorem 2.8] to higher generalized derivations.

Theorem 2.5. *Let $H = \{h_n\}$ a higher derivation of any rank on a unital Banach algebra A with unit e. Suppose that $F = \{f_n\}$ is a generalized higher derivation of the same rank with H on A associated with H and that $\Phi = \{\varphi_n\}$ is a higher left multiplier of the same rank with H on A associated with H, where $\varphi_1 = L_a$ is a left multiplication by $a = f_1(e)$. The following conditions are equivalent.*

(a) *F is spectrally bounded on A.*

(b) *Both Φ and H are spectrally bounded on A.*

Proof. (b) ⇒ (a) By Theorem 2.1, we have $H(A) \subseteq rad(A)$ and hence for each $n \in \mathbb{N}$,

$$\begin{aligned} r(f_n(x)) &= r(\varphi_n(x) + h_n(x)) \\ &= r(\pi(\varphi_n(x) + h_n(x))) \\ &= r(\pi(\varphi_n(x))) \\ &= r(\varphi_n(x)) \leq M_n r(x) \end{aligned}$$

for some $M_n > 0$ and all $x \in A$. Therefore, F is spectrally bounded on A.

(a) ⇒ (b) From Theorem 2.1, it suffices to show that H is spectrally bounded, i.e., $H(A) \subseteq rad(A)$. For then, $H(A) \subseteq rad(A)$ and for each $n \in \mathbb{N}$, $r(\varphi_n(x)) = r(f_n(x))$ for all $x \in A$ yield that φ_n is spectrally bounded with the same constant with f_n, that is, Φ is spectrally bounded.

Since f_1 is spectrally bounded, it follows from [2, Theorem 2.8] that h_1 is spectrally bounded which implies that $h_1(A) \subseteq rad(A)$. Assume that $h_n(A) \subseteq rad(A)$ for $1 \leq n \leq k-1$ and so $h_n(rad(A)) \subseteq rad(A)$. As in the proof of Lemma 2.4, we see that $h_k(rad(A)) \subseteq rad(A)$. Then a mapping $\tilde{h}_k : A/rad(A) \to A/rad(A)$ defined by $\tilde{h}_k(x + rad(A)) = h_k(x) + rad(A)$ for all $x \in A$ becomes a derivation on a unital semisimple Banach algebra $A/rad(A)$ because $h_n(A) \subseteq rad(A)$ for $1 \leq n \leq k-1$.

Since $f_k(rad(A)) \subseteq rad(A)$ by Lemma 2.4, we define a mapping $\tilde{f}_k : A/rad(A) \to A/rad(A)$ by $\tilde{f}_k(x + rad(A)) = f_k(x) + rad(A)$ for all $x \in A$. Since $h_n(A) \subseteq rad(A)$ for $1 \leq n \leq k-1$, it follows that $\tilde{f}_k$ is a generalized derivation on $A/rad(A)$ associated to $\tilde{h}_k$ and $\tilde{f}_k = L_{\tilde{a}_k} + \tilde{h}_k$, where $\tilde{a}_k = \tilde{f}_k(e + rad(A))$. The fact that $r(\pi(z)) = r(z)$ for all $z \in A$ and f_k is spectrally bounded on A, tells us that $\tilde{f}_k$ is spectrally bounded on $A/rad(A)$. From [2, Theorem 2.8], we now see that $\tilde{h}_k$ is a spectrally bounded derivation on $A/rad(A)$. In view of Theorem 2.1, we obtain that $\tilde{h}_k = 0$ on $A/rad(A)$. This means that $h_k(A) \subseteq rad(A)$. Consequently, $h_n(A) \subseteq rad(A)$ for all $n \in \mathbb{N}$ by induction which completes the proof. □

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