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Polynomials and Homotopy of Virtual Knot Diagrams

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Abstract. If a virtual knot diagram can be transformed to another virtual one by a finite sequence of crossing changes, Reidemeister moves and virtual moves then the two virtual knot diagrams are said to be *homotopic*. There are infinitely many homotopy classes of virtual knot diagrams.

We give necessary conditions by using polynomial invariants of virtual knots for two virtual knots to be homotopic. For a sequence S of crossing changes, Reidemeister moves and virtual moves between two homotopic virtual knot diagrams, we give a lower bound for the number of crossing changes in S by using the affine index polynomial introduced in [13]. In [10], the first author gave the q-polynomial of a virtual knot diagram to find Reidemeister moves of virtually isotopic virtual knot diagrams. We find how to apply Reidemeister moves by using the q-polynomial to show homotopy of two virtual knot diagrams.

1. Introduction

Any classical knot diagram can be unknotted by a sequence of Reidemeister moves and crossing changes. So crossing change is called an unknotting operation

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for classical knot diagrams. There was an open question whether one can get the minimal number of crossing changes from a knot diagram with a minimal crossing number to get the trivial knot diagram. For this question S. Bleiler [1] and Y. Nakanishi [19] gave a counterexample $C(5,1,4)$ in Conway notation in Figure 1. Crossing change is no more an unknotting operation for virtual knot diagrams. Two virtual knot diagrams are *homotopic* if there is a sequence of Reidemeister moves, virtual moves and crossing changes. We will give necessary conditions for two given virtual knot diagrams to be homotopic. To show homotopy of virtual knot diagrams, we investigate how to apply crossing changes by using an affine index polynomial $P_K(t)$ of a virtual knot diagram K and how to apply Reidemeister moves by using a polynomial $q_K(t)$ of a virtual knot diagram K.

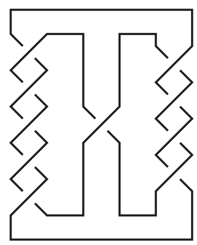

Figure 1: A knot $C(5,1,4)$.

In 1996 L. H. Kauffman introduced virtual knots which generalize classical knots [12]. A *virtual knot diagram* is a knot diagram allowed to have virtual crossings. We denote a virtual crossing by an encircled crossing as shown in Figure 2.

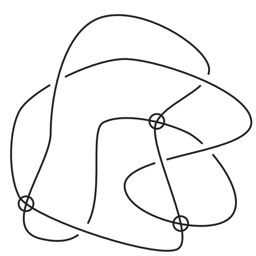

Figure 2: A virtual knot diagram.

If a knot K is isotopic to another knot K' then there is a sequence of moves from a diagram of K to a diagram of K' as shown in Figure 3. These moves are called *Reidemeister moves*. The moves of diagrams shown in Figure 4 are called *virtual moves*. Two virtual knot diagrams K_1 and K_2 are said to be *virtually isotopic* if there is a sequence of Reidemeister moves and virtual moves from K_1 to K_2. A *virtual knot* is defined to be the virtual isotopy class of a virtual knot diagram.

From now on all virtual knot diagrams are assumed to be oriented. We define

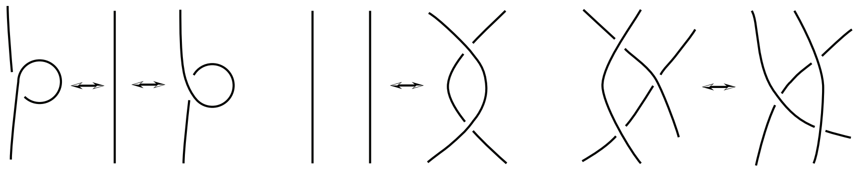

Figure 3: Reidemeister moves.

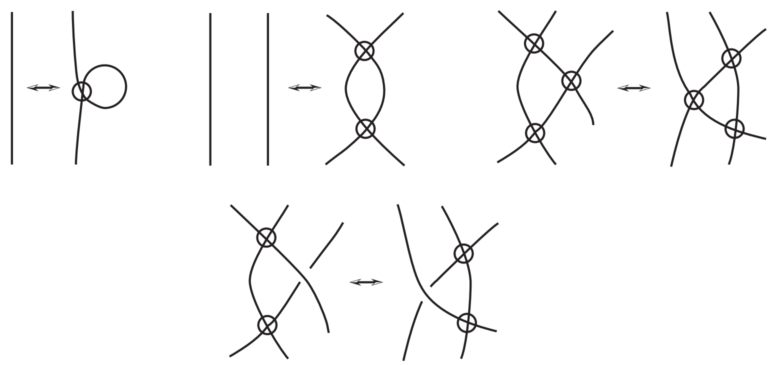

Figure 4: Virtual moves.

the *sign* $\mathrm{sgn}(c)$ of a crossing c of a virtual knot diagram as shown in Figure 5. The *writhe* $w(K)$ of a virtual knot diagram K is defined to be the sum of signs of all crossings of K.

Figure 5: The sign of a crossing.

M. Goussarov, M. Polyak and O. Viro defined finite type invariants of virtual knots and gave some combinatorial representations of finite type invariants of low degree by using Gauss diagrams [6]. The *Gauss diagram* of a virtual knot diagram K is an oriented circle with chords corresponding to crossings. The two endpoints of a chord correspond to the preimages of the crossing of K. A chord corresponding to a crossing c is oriented from the preimage of the over crossing point of c to the preimage of the under crossing point of c. A chord is assumed to have the sign of the crossing corresponding to the chord. See Figure 6. We denote the Gauss diagram of K by $G(K)$.

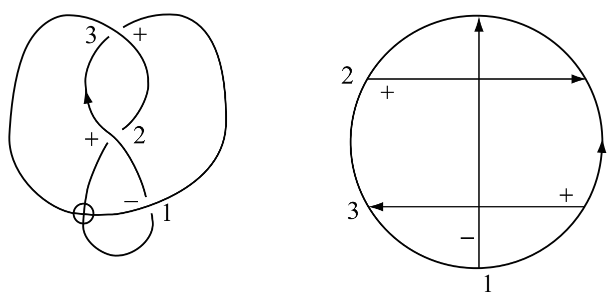

Figure 6: A Gauss diagram of a virtual knot diagram.

A virtual knot can be represented as a Gauss diagram and vice versa [6, 12]. For a Gauss diagram G, there are many virtual knot diagrams whose Gauss diagrams are the same G. All these virtual knot diagrams are virtually isotopic [6]. In Gauss diagrams, the Reidemeister moves can be described as a sequence of moves shown in Figure 7 [6, 21]. In the figure ϵ denotes the sign of a chord which can be either $+$ or $-$.

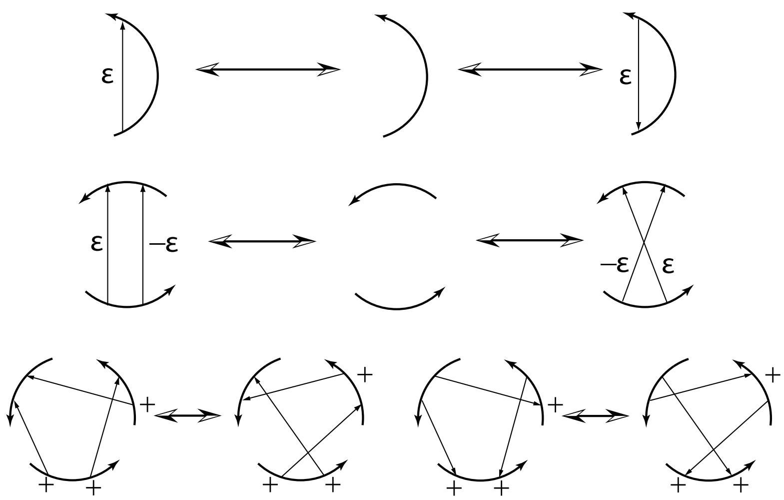

Figure 7: Moves of Gauss diagrams.

A local move of virtual knot diagrams shown in Figure 8 is called a *crossing change* or a *CC-move*. Two virtual knot diagrams are said to be *homotopic* if they are related by a sequence of CC-moves, Reidemeister moves, and virtual moves. If two virtual knot diagrams K_1 and K_2 are homotopic then the *Gordian distance*

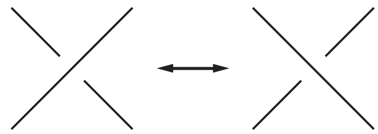

Figure 8: A crossing change.

$d_G(K_1, K_2)$ between K_1 and K_2 is defined to be the minimal number of CC-moves needed to transform K_1 to K_2, up to Reidemeister moves and virtual moves. Although any two classical knot diagrams are homotopic, we will see that there are infinitely many non-homotopic pairs of virtual knot diagrams.

A. Kawauchi gave a condition on a pair of the Alexander polynomials of knots which are realizable by a pair of knots with Gordian distance one [15]. He showed that there are infinitely many mutually disjoint infinite subsets of the set of the Alexander polynomials of knots such that every pair of distinct polynomials in each subset is not realizable by any pair of knots with Gordian distance one [15]. Y. Miyazawa gave several evaluations of the Gordian distance of two classical knots by using the HOMFLY polynomial, the Jones polynomial and the Q-polynomial [18]. Similarly to the classical case, we can extend virtual knot diagrams to virtual link diagrams with multiple components naturally [12]. Two virtual link diagrams are said to be *welded equivalent* if one can be transformed to the other by a sequence of Reidemeister moves, virtual moves and the upper forbidden move shown in Figure 9. Two welded link diagrams are said to be *homotopic* if one can be transformed to the other by a sequence of Reidemeister moves, virtual moves and self crossing changes. H. A. Dye and L. H. Kauffman extended Milnor's μ and $\bar{\mu}$ invariants to welded and virtual links [5]. Both the upper forbidden move and the lower forbidden move in Figure 9 are called *forbidden moves* which unknot all virtual knots [8, 14, 20]. A Δ-*move* is a local move of virtual knot diagrams as shown in Figure 10. T. Kanenobu showed that a Δ-move can be realized by a finite sequence of the Reidemeister moves, the virtual moves and the forbidden moves [14]. The first author showed that the values of a Vassiliev invariant of degree 2 for two virtual knots related by a Δ-move differ by 48 [9].

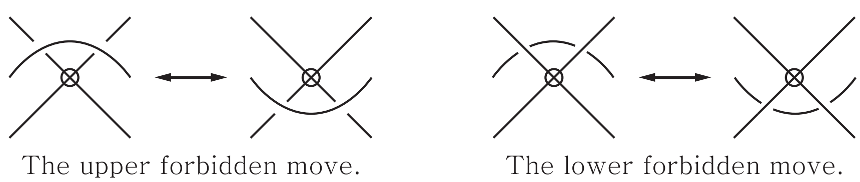

Figure 9: Forbidden moves.

A. Henrich defined a polynomial invariant of virtual knots by using weight of a crossing. Z. Cheng [2] introduced an odd writhe polynomial $f_K(t)$ of virtual knots

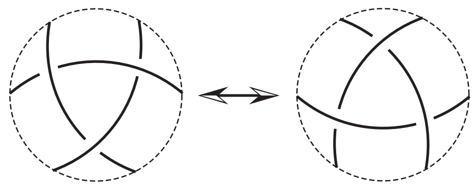

Figure 10: A Δ-move.

K considering the Gauss diagram of a virtual knot and coloring of arcs. Z. Cheng and H. Gao [3] generalized the odd writhe polynomial to a sequence of polynomials from which Henrich's polynomial can be recovered. L. H. Kauffman introduced the affine index polynomial of a virtual knot based on a Cheng coloring of a virtual knot diagram [13]. An affine index polynomial $P_K(t)$ is given in the form

$$P_K(t) = \sum_c \text{sgn}(c)(t^{W_K(c)} - 1),$$

where sgn(c) and $W_K(c)$ are the sign and the weight of a crossing c respectively. We will show that $P_{K_2}(t) - P_{K_1}(t) = \pm(t^n + t^{-n} - 2)$ for some integer n if K_2 is obtained from K_1 by applying a CC-move.

The first author defined a polynomial $q_K(t)$ of a virtual knot diagram K in [10]. If K_1 and K_2 are related by a second Reidemeister move then $q_{K_2}(t) - q_{K_1}(t) = \pm 1$ or 0. If K_1 and K_2 are related by a third Reidemeister move then $q_{K_2}(t) - q_{K_1}(t) = \pm(t^{\alpha+\beta} - t^{\alpha} - t^{\beta})$ for some $\alpha, \beta \in \mathbb{Z}$. To show that two virtual knot diagrams K_1 and K_2 are homotopic, basically we find a sequence of CC-moves, Reidemeister moves and virtual moves between K_1 and K_2. The affine index polynomial and q-polynomial can shed light on finding such a sequence.

In Section 2, we give a sequence of virtual knots every pair of which is non-homotopic and give necessary condition for two virtual knot diagrams to be homotopic by using the affine index polynomial. Moreover we give a lower bound for the number of CC-moves needed to transform a virtual knot diagram to a homotopic one by comparing their affine index polynomials. In Section 3, we give a relationship between $q_{K_1}(t)$ and $q_{K_2}(t)$ if K_1 and K_2 differ by a CC-move. When two virtual knot diagrams K_1 and K_2 are homotopic, their Gauss diagrams are related by the moves corresponding to CC-moves and Reidemeister moves. $q_K(t)$ is defined from the Gauss diagram of K and it is sensitive to Reidemeister moves. So we can predict how virtual knot diagrams K_1 and K_2 are related by Reidemeister moves by comparing $q_{K_1}(t)$ and $q_{K_2}(t)$ if K_1 and K_2 are virtually isotopic. We can map out a sequence of Reidemeister moves and CC-moves from K_1 to K_2 by using the affine index polynomial and the q-polynomial for homotopic virtual knot diagrams K_1 and K_2. We give an example illustrating that $P_K(t)$ and $q_K(t)$ are useful to show that two virtual knot diagrams are homotopic.

2. Polynomial Invariants and CC-moves of Virtual Knot Diagrams

Y. Miyazawa found a 2-variable polynomial invariant of virtual links [16] and generalized it by constructing a multi-variable polynomial invariant of virtual links [17]. He gave a lower bound on the virtual crossing number by using the multi-variable polynomial.

H. A. Dye and L. H. Kauffman defined the arrow polynomial of a virtual link which is a generalization of the bracket polynomial [4]. By normalizing the arrow polynomial we get an invariant of virtual knots and links. By changing variables suitably, we may get the normalized arrow polynomial from the Miyazawa's multi-variable polynomial and vice versa. By using the normalized arrow polynomial we will get a necessary condition for two given virtual knot diagrams to be homotopic.

Definition 2.1. The *arrow polynomial* of a virtual link diagram is defined by using the following relations.

(1) $\langle L_+\rangle = A\langle L_0\rangle + A^{-1}\langle L_\infty\rangle$ and
$\langle L_-\rangle = A^{-1}\langle L_0\rangle + A\langle L_\infty\rangle$, where L_+, L_-, L_0, and L_∞ are virtual link diagrams which differ as shown in Figure 11 and A is an indeterminate.

(2) $\langle C_1\rangle = \langle C_2\rangle$ and $\langle C_3\rangle = \langle C_4\rangle$, where C_1, C_2, C_3 and C_4 are virtual link diagrams which differ as shown in Figure 12.

(3) $\langle O\rangle = 1$ and $\langle L \cup O_m\rangle = (-A^2 - A^{-2})X_m\langle L\rangle$, where X_m's are indeterminates, O is the trivial knot diagram with no crossings and no poles and O_m is the diagram with $2m$ poles as shown in Figure 12.

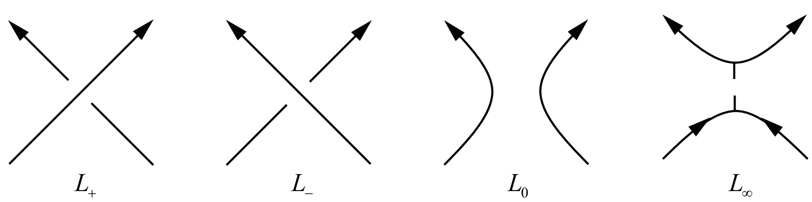

Figure 11:

The arrow polynomial is invariant under Reidemeister moves and virtual moves except for the first Reidemeister moves [4]. Similarly to the bracket polynomial we can normalize the arrow polynomial to get an invariant of virtual links. The *normalized arrow polynomial* $\langle L\rangle_{NA}$ is defined by the formula

$$\langle L\rangle_{NA} = (-A^3)^{-w(L)}\langle L\rangle.$$

The arrow polynomial and normalized arrow polynomial take values in the polynomial ring $\mathbb{Z}[A, A^{-1}, X_1, X_2, \cdots]$.

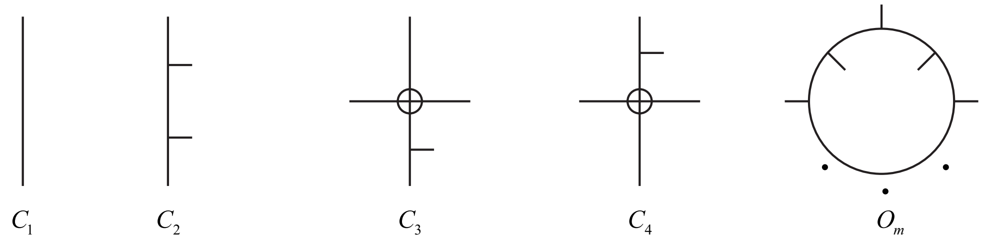

Figure 12:

The first and second authors defined polar link diagrams, a generalization of virtual link diagrams, and extended the normalized arrow polynomial to polar link diagrams. We gave the following lemma, which is useful to see whether two virtual knot diagrams are homotopic or not [11].

Lemma 2.2.([11]) *If L and L' are homotopic polar link diagrams then*

$$\langle L \rangle_{NA} \equiv \langle L' \rangle_{NA} \mod A^4 - 1.$$

In particular, if two virtual knot diagrams K and K' are homotopic then their normalized arrow polynomials $\langle K \rangle_{NA}$ and $\langle K' \rangle_{NA}$ are congruent modulo $A^4 - 1$. In the following example, we show that there are infinitely many homotopy classes of virtual knot diagrams.

Example 2.3. Consider a sequence of virtual knot diagrams $\{K_n\}_{n=0}^{\infty}$ as shown in Figure 13. Inductively we can show that the degree of X_1 of $\langle K_n \rangle_{NA}$ modulo $A^4 - 1$ is n. Therefore any two virtual knot diagrams in the sequence are not homotopic by Lemma 2.2.

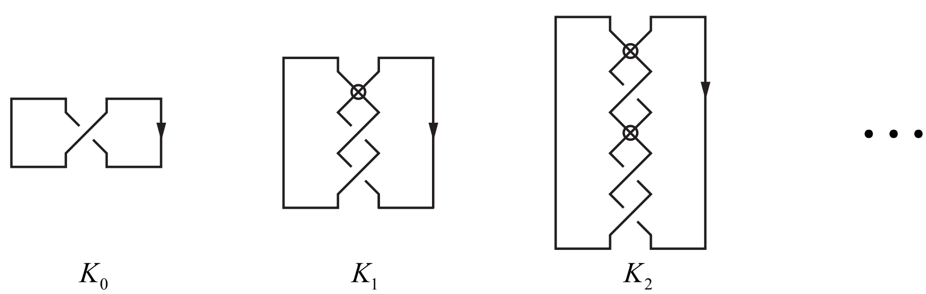

Figure 13: A sequence of virtual knot diagrams with distinct homotopy classes.

A *flat crossing* is an intersecting line segments with no information about over and under strand. A *flat virtual knot diagram* is a virtual knot diagram whose crossings are replaced with flat crossings. From a virtual knot diagram K we get a flat virtual knot diagram $F(K)$ by taking its shadow of K. Figure 14 shows a flat virtual knot diagram with two flat crossings.

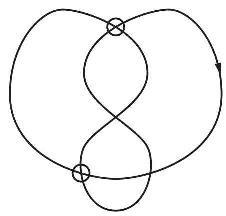

Figure 14: A flat virtual knot.

By replacing crossings in Reidemeister moves and virtual moves with flat crossings, we get *flat Reidemeister moves* and *flat virtual moves* respectively. A sequence of flat Reidemeister moves and flat virtual moves is called a *flat virtual isotopy*. A *flat virtual knot* is defined as the flat virtual isotopy class of a flat virtual knot diagram.

Let K be a virtual knot diagram. Then $F(K)$ can be regarded as a 4-regular graph G by considering a flat crossing as a vertex. An *arc* of $F(K)$ is an edge of G that it represents. Label each arc of $F(K)$ with an integer so that it satisfies the rule in Figure 15. L. H. Kauffman proved that such labeling exists always [13]. Such labeling of arcs for a flat virtual knot diagram is called a *Cheng coloring*. See Figure 16.

Let c be a flat crossing of a labeled flat virtual knot diagram. If arcs near c is labeled as shown in the left part of Figure 15, then we define $W_+(c) = m - n - 1$ and $W_-(c) = n - m + 1$. $W_+(c)$ and $W_-(c)$ are independent of Cheng colorings of $F(K)$ [3, 13].

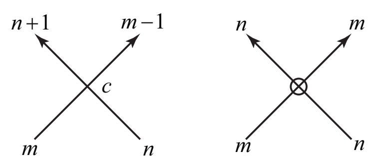

Figure 15: Labeling of arcs.

For each crossing c of K, there is a corresponding flat crossing c' in $F(K)$. If $F(K)$ is labeled as previously described, then define $W_K(c) = W_{\mathrm{sgn}(c)}(c')$. For simplicity sake we often denote $W_K(c)$ by $W(c)$. We denote the set of all crossings of K by $C(K)$. The *affine index polynomial* $P_K(t)$ of K is defined by the equation

$$P_K(t) = \sum_{c \in C(K)} \mathrm{sgn}(c)(t^{W_K(c)} - 1).$$

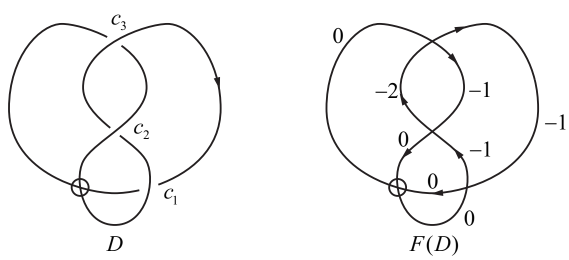

Figure 16: A Cheng coloring.

L. H. Kauffman showed that $P_K(t)$ is an invariant of virtual knots [13]. For example the affine index polynomial of a virtual knot diagram D in Figure 16 can be calculated as following

$$\begin{aligned} P_D(t) &= \operatorname{sgn}(c_1)(t^{W_K(c_1)}-1)+\operatorname{sgn}(c_2)(t^{W_K(c_2)}-1)+\operatorname{sgn}(c_3)(t^{W_K(c_3)}-1) \\ &= -t-t^{-1}+2. \end{aligned}$$

We give a necessary condition for two given virtual knot diagrams K_1 and K_2 to be homotopic by using the affine index polynomials of K_1 and K_2.

Lemma 2.4. *Let K_1 be a virtual knot diagram and $c_1 \in C(K_1)$. If K_2 is the virtual knot diagram obtained from K_1 by changing the crossing c_1 then*

$$P_{K_2}(t)-P_{K_1}(t) = -\operatorname{sgn}(c_1)(t^{W_{\operatorname{sgn}(c_1)}(c_1)}+t^{-W_{\operatorname{sgn}(c_1)}(c_1)}-2).$$

Proof. Assume that a crossing c_1 of K_1 is changed to a crossing c_2 of K_2 by the CC-move. Since $F(K_1)=F(K_2)$, $\operatorname{sgn}(c_1)=-\operatorname{sgn}(c_2)$ and $W_+(c)=-W_-(c)$, we see that

$$\begin{aligned} P_{K_2}(t)-P_{K_1}(t) &= \operatorname{sgn}(c_2)(t^{W_{\operatorname{sgn}(c_2)}(c_2)}-1)-\operatorname{sgn}(c_1)(t^{W_{\operatorname{sgn}(c_1)}(c_1)}-1) \\ &= -\operatorname{sgn}(c_1)(t^{W_{\operatorname{sgn}(c_1)}(c_1)}+t^{-W_{\operatorname{sgn}(c_1)}(c_1)}-2). \qquad \square \end{aligned}$$

Now we see that $P_{K_2}(t)-P_{K_1}(t)=\pm(t^{-n}+t^n-2)$ for some integer n if two virtual knot diagrams K_1 and K_2 are related by a single CC-move. Moreover Either n or $-n$ is the weight of the crossing which was changed by the CC-move. Therefore we get the following

Theorem 2.5. *Let K_1 and K_2 be homotopic virtual knot diagrams. Then*

$$P_{K_1}(t)-P_{K_1}(t^{-1}) = P_{K_2}(t)-P_{K_2}(t^{-1}).$$

Moreover

$$d_G(K_1,K_2) \geq \sum_{i=1}^{m} |a_i|,$$

where $P_{K_2}(t) - P_{K_1}(t) = \sum_{i=1}^{m} a_i(t^{-i} + t^i - 2)$.

Example 2.6. Let $\{K_n\}_{n=0}^{\infty}$ be a sequence of virtual knot diagrams as shown in Figure 17. We can easily see that $P_{K_n}(t) = -n(t^{-1} + t^1 - 2)$ for each $n = 0, 1, \cdots$. Therefore $d_G(K_n, O) \geq n$ by Theorem 2.5, where O is the trivial virtual knot diagram. Since K_n can be transformed to O by changing n crossings, we see that $d_G(K_n, O) = n$.

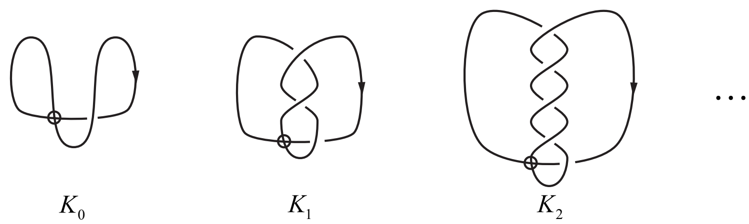

Figure 17:

3. CC-moves and Reidemeister Moves

If two virtual knot diagrams K_1 and K_2 are supposed to be homotopic, then we usually want to find a sequence of Reidemeister moves, virtual moves and CC-moves. This can be done through their Gauss diagrams. In Gauss diagrams we need not to consider virtual moves by its definition. So we are interested in finding a sequence of Reidemeister moves and CC-moves in a Gauss diagrammatic approach. The affine index polynomial gives us information on CC-moves but it does not give us any information about Reidemeister moves as it is an invariant. In this section we find how to apply Reidemeister moves to show homotopy of K_1 and K_2 by using q-polynomials.

A q-polynomial $q_K(t)$ of a virtual knot diagram K is defined to study Reidemeister moves [10]. It is invariant under the first Reidemeister move but not invariant under the second Reidemeister move and the third Reidemeister move. It is defined by using mixed pairs of crossings of a virtual knot diagram and degrees of its crossings. In [10], the first author defined the degree $\deg(c)$ of a crossing c by using Gauss diagram. A Cheng coloring of a virtual knot diagram K can be represented in a Gauss diagram [3] and we can see that $\deg(c) = -W(c)$ for any crossing c. Refer [3, 10] for more details.

Let K be a virtual knot diagram with crossings c_1 and c_2. If the two chords corresponding to c_1 and c_2 appear as shown in Figure 18, then the ordered pair (c_1, c_2) is called a *mixed pair of crossings*. For example the knot in Figure 6 has two mixed pairs $(c_1, c_3), (c_2, c_1)$ of crossings.

Denote the set of all mixed pairs of crossings of K by $M(K)$. The two chords of a Gauss diagram $G(K)$ corresponding to crossings c_1 and c_2 of K intersect if $(c_1, c_2) \in M(K)$.

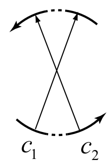

Figure 18:

Let c be a crossing of a virtual knot diagram K. For simplicity sake we will denote the chord of the Gauss diagram $G(K)$ corresponding to the crossing c by c. For a chord c of a Gauss diagram, assume that we walk on the chord c from the starting point of c to the ending point of c. Let r_+ and r_- be the numbers of positive chords and negative chords from left to right respectively and let l_+ and l_- be the numbers of positive chords and negative chords from right to left respectively. We define the *degree* $\deg(c)$ of a crossing c by the equation

$$\deg(c) = r_+ - r_- - l_+ + l_-.$$

For $c_1, c_2 \in C(K)$ we also define $\deg(c_1, c_2)$ by the formula

$$\deg(c_1, c_2) = \deg(c_1) - \deg(c_2).$$

For example, if c_1, c_2, c_3 and c_4 are crossings of a virtual knot diagram whose Gauss diagrams are as shown in Figure 19, then $\deg(c_1) = 1$, $\deg(c_2) = 0$, $\deg(c_3) = -2$, $\deg(c_4) = 1$, $\deg(c_1, c_2) = 1$, $\deg(c_2, c_3) = 2$ and $\deg(c_3, c_4) = -3$.

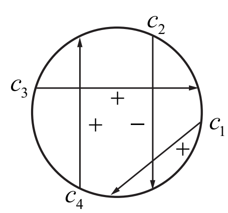

Figure 19:

Now we define a polynomial $q_K(t)$ associated to K by the equation

$$q_K(t) = \sum_{(c_1,c_2)\in M(K)} \operatorname{sgn}(c_1)\operatorname{sgn}(c_2)t^{\deg(c_1,c_2)}.$$

Lemma 3.1.([10]) *Let K_1 and K_2 be related by a second Reidemeister move and their Gauss diagrams be as shown in Figure 20. Then*

$$q_{K_1}(t) - q_{K_2}(t) = -1.$$

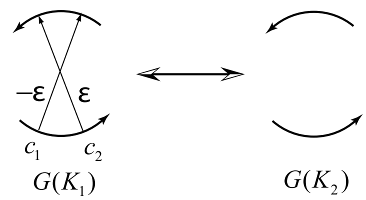

Figure 20:

Lemma 3.2.([10]) *Let K_1 and K_2 be related by a second Reidemeister move and their Gauss diagrams be as shown in Figure 21. Then*

$$q_{K_1}(t) = q_{K_2}(t).$$

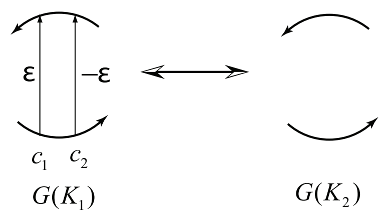

Figure 21:

Lemma 3.3.([10]) *Let two virtual knot diagrams K_1 and K_2 be related by a third Reidemeister move. Then there exist $\alpha, \beta \in \mathbb{Z}$ such that*

$$q_{K_1}(t) - q_{K_2}(t) = \pm(t^{\alpha+\beta} - t^{\alpha} - t^{\beta}).$$

Let K_1 and K_2 be virtual knot diagrams related by a third Reidemeister move and c_1, c_2, c_3 be the crossings of K_1 involved with the third Reidemeister move. Then α, β and $\alpha + \beta$ are given in the form $\deg(c_i) - \deg(c_j)$ for some distinct elements i, j of the set $\{1, 2, 3\}$ [10].

Let K_1 and K_2 be virtually isotopic virtual knot diagrams. Then there exists a sequence S of Reidemeister moves transforming K_1 to K_2. We denote the numbers of the first Reidemeister moves, the second Reidemeister moves and the third Reidemeister moves in the sequence S by $n_1(S)$, $n_2(S)$ and $n_3(S)$ respectively. A lower bound for $n_1(S)$ is the difference $|w(K_1) - w(K_2)|$ of writhes of K_1 and K_2. By using Lemma 3.1, Lemma 3.2 and Lemma 3.3, we get the following theorems for the numbers of the second Reidemeister moves and the third Reidemeister moves in S.

Theorem 3.4.([10]) *Let K_1 and K_2 be two virtually isotopic virtual knot diagrams and S be a sequence of Reidemeister moves deforming K_1 to K_2. Let $g_K(t) = q_K(t) - q_K(1)$ and $g_{K_1}(t) - g_{K_2}(t) = \sum_{i=m}^{l} a_i(t^i - 1)$. Then we get inequalities*

$$n_3(S) \geq \frac{\sum_{i=m}^{l} |a_i|}{3} \text{ and } n_3(S) \geq \left| \sum_{i=m}^{l} a_i \right|.$$

Theorem 3.5.([10]) *Let K_1 and K_2 be two virtually isotopic knot diagrams and S be a sequence of Reidemeister moves deforming K_1 to K_2. Let $q_{K_1}(t) - q_{K_2}(t) = \sum_{i=m}^{l} a_i t^i$. Then we get an inequality*

$$n_2(S) + 3n_3(S) \geq \sum_{i=m}^{l} |a_i|.$$

There are four versions of second Reidemeister moves for oriented virtual knot diagrams as shown in Figure 22. For the numbers of second Reidemeister moves and third Reidemeister moves, C. Hayashi introduced cowrithe of a knot which can be given as $q_K(1)$ and gave the following theorem.

Theorem 3.6.([7]) *Let K be a virtual knot diagram. The first Reidemeister move does not change $q_K(1)$. The second Reidemeister moves $\Omega 2a$ and $\Omega 2b$ change $q_K(1)$ by ± 1. The second Reidemeister moves $\Omega 2c$ and $\Omega 2d$ do not change $q_K(1)$. The third Reidemeister move changes $q_K(1)$ by ± 1.*

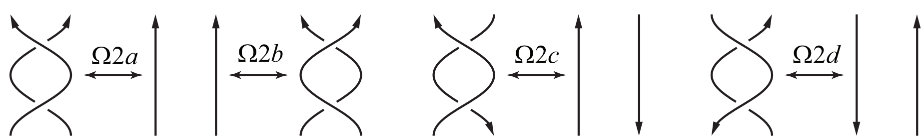

Figure 22: The second Reidemeister moves.

Let K_1 and K_2 be related by a CC-move. Assume that a crossing $c_1 \in C(K_1)$ is changed to $c_2 \in C(K_2)$ by the CC-move. Let

$$q_{K_1}(t) = \sum_{(c,c') \in M(K_1)} \text{sgn}(c)\text{sgn}(c') t^{\deg(c,c')} = g(t) + t^{\deg(c_1)} h(t) + t^{-\deg(c_1)} i(t),$$

where $g(t)$, $h(t)$ and $i(t)$ come from the sums for $c, c' \neq c_1$, for $c = c_1$ and for $c' = c_1$ respectively. Note that if $(c_1, c') \in M(K_1)$ then $(c', c_2) \in M(K_2)$. Similarly, if $(c, c_1) \in M(K_1)$ then $(c_2, c) \in M(K_2)$. Since $\text{sgn}(c_2) = -\text{sgn}(c_1)$ we have

$$q_{K_2}(t) = g(t) - t^{\deg(c_1)} h(t^{-1}) - t^{-\deg(c_1)} i(t^{-1}).$$

Now we get the following

Lemma 3.7. *Let K_2 be a virtual knot diagram obtained from K_1 by changing a crossing c_1 of K_1. Let $h(t) = t^{-\deg(c_1)} \sum_{(c_1,c')\in M(K_1)} t^{\deg(c_1,c')}$ and $i(t) = t^{\deg(c_1)} \sum_{(c,c_1)\in M(K_1)} t^{\deg(c,c_1)}$. Then*

$$q_{K_2}(t) - q_{K_1}(t) = -t^{\deg(c_1)}(h(t) + h(t^{-1})) - t^{-\deg(c_1)}(i(t) + i(t^{-1})).$$

Note that the coefficients of t^k and t^{-k} are the same for $h(t) + h(t^{-1})$ and $i(t) + i(t^{-1})$ for all $k = 1, 2, \cdots$.

Example 3.8. Let K_1 and K_2 be the virtual knot diagrams as shown in Figure 23. Since $P_{K_1}(t) = t^{-2} + t^{-1} - 4 + t + t^2$ and $P_{K_2}(t) = t^{-1} - 2 + t$, we have

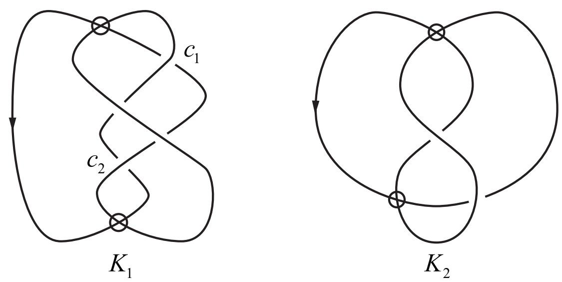

Figure 23:

$P_{K_2}(t) - P_{K_1}(t) = -t^{-2} + 2 - t^2$. If K_1 and K_2 are homotopic then we need at least 1 CC-move which changes a crossing of K_1 with weight ± 2 and with sign $+$ by Theorem 2.5. There are two crossings c_1 and c_2 in K_1 with weight ± 2 and with sign $+$. If we change the crossing c_1 then we get a diagram K_3 in Figure 24. Since $q_{K_3}(t) = t^{-3} - t^{-1} - 1$ and $q_{K_2}(t) = t^{-2}$, we have $q_{K_2}(t) - q_{K_3}(t) = -t^{-3} + t^{-2} + t^{-1} + 1$. By Theorem 3.4 we see that we need a third Reidemeister move at least

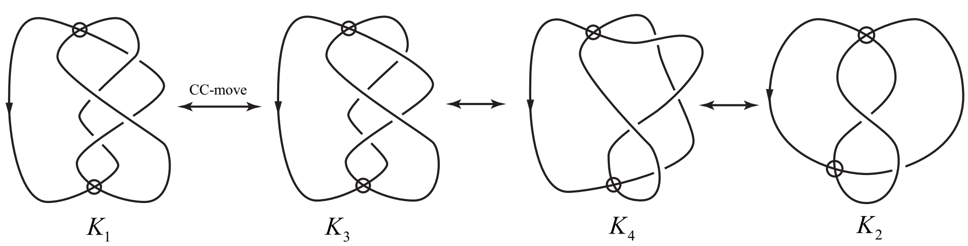

Figure 24:

once. By Theorem 3.5, we also see that we need another third Reidemeister move or a second Reidemeister move. By comparing $w(K_3)$ and $w(K_2)$, we see that we may not need a first Reidemeister move. By applying a third Reidemeister move to K_3 we get K_4 as shown in Figure 24. Now $q_{K_4}(t) = t^{-2} - 1$ and $q_{K_2}(t) - q_{K_4}(t) = 1$. From Lemma 3.1, we see that K_4 may be transformed to K_2 by a suitable second Reidemeister move. Actually we get K_1 by applying a second Reidemeister move to K_4 as shown in Figure 24.

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