THREE-DIMENSIONAL ALMOST KENMOTSU MANIFOLDS WITH $\eta$-PARALLEL RICCI TENSOR

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Abstract. In this paper, we prove that the Ricci tensor of a three-dimensional almost Kenmotsu manifold satisfying $\nabla_\xi h = 0$, $h \neq 0$, is $\eta$-parallel if and only if the manifold is locally isometric to either the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$ or a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure.

1. Introduction

Let us first recall some basics regarding the geometry of almost contact metric manifolds. Let $M^{2n+1}$ be a $(2n+1)$-dimensional smooth differentiable manifold on which there exists an almost contact structure $(\phi, \xi, \eta)$ defined by

$$\phi^2 = -\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where id denotes the identity map, $\phi$ a $(1,1)$-type tensor field, $\xi$ a global vector field tangent to $M^{2n+1}$ and $\eta$ a global one form. Then $M^{2n+1}$ is called an almost contact manifold and denoted by $(M^{2n+1}, \phi, \xi, \eta)$. If in addition there exists a Riemannian metric $g$ on $(M^{2n+1}, \phi, \xi, \eta)$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

we say that $g$ is compatible to the almost contact structure and $M^{2n+1}$ is called an almost contact metric manifold, denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$. Generally, $\xi$ is said to be the Reeb or characteristic vector field and $\eta$ is a contact 1-form. The fundamental 2-form $\Phi$ of an almost contact metric manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y)$$

for any vector fields $X, Y$.

An almost contact metric manifold is called a contact metric manifold if $d\eta = \Phi$ (see [1]) or an almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ (see [12]) or an almost cosymplectic (coKähler) manifold if both $\eta$ and $\Phi$ are $\eta$-parallel.
closed (see [1, 3]). On the Riemannian product of an almost contact manifold $M^{2n+1}$ and $\mathbb{R}$, there exists an almost complex structure defined by

$$J\left(X, f \frac{d}{dt}\right) = \left(\phi X - f \xi, \eta(X) \frac{d}{dt}\right),$$

where $X$ denotes a vector field tangent to $M^{2n+1}$, $t$ is the coordinate of $\mathbb{R}$ and $f$ is a $C^\infty$-function on $M^{2n+1} \times \mathbb{R}$. We denote by $[\phi, \phi]$ the Nijenhuis tensor of $\phi$, if $[\phi, \phi] = -2d\eta \otimes \xi$ (or equivalently, the almost complex structure $J$ is integrable), then the almost contact metric structure is said to be normal.

A normal contact metric (resp. almost Kenmotsu or almost coKähler) manifold is said to be a Sasakian (resp. Kenmotsu or coKähler) manifold. An almost Kenmotsu manifold but not Kenmotsu is called a strictly almost Kenmotsu manifold. Here we refer the reader to Blair [1] for more details on the geometry of almost contact manifolds.

It is known that a three-dimensional contact metric manifold is locally symmetric if and only if it is of constant sectional curvature 0 or 1 (see Blair and Sharma [2]). This implies that the local symmetry is a rather strong condition in contact metric manifolds. Thus, many new geometric conditions weaker than the local symmetry were considered. For example, Cho and Lee [6] proved that a three-dimensional contact metric manifold satisfying $\nabla_\xi h = 2ah\phi$, $a \in \mathbb{R}$, has an $\eta$-parallel Ricci tensor if and only if the manifold is locally isometric to a Sasakian $\phi$-symmetric space or a unimodular Lie group with a left invariant non-Sasakian contact metric structure. Recently, Perrone in [16] proved that a three-dimensional almost coKähler manifold is locally symmetric if and only if it is locally isometric to the flat Euclidean space $\mathbb{R}^3$ or the Riemannian product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$. Generalizing Perrone’s results, the present author in [18] proved that a three-dimensional almost coKähler manifold satisfying $\nabla_\xi h = 2f\phi h$, $f \in \mathbb{R}$ and having an $\eta$-parallel Ricci tensor is locally isometric to either the product space $\mathbb{R} \times N^2(c)$ or a simply connected unimodular Lie group equipped with a left invariant almost coKähler structure.

In 1972, Kenmotsu [13] proved that a locally symmetric Kenmotsu manifold is of constant sectional curvature $-1$. Generalizing this result for dimension three, recently, the present author [17] and Cho [4] independently obtained that a three-dimensional almost Kenmotsu manifold is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product space $\mathbb{H}^2(-4) \times \mathbb{R}$. This implies that the local symmetry is also a rather strong condition for almost Kenmotsu manifolds. Therefore, in this paper we aim to generalize Wang and Cho’s results to a special class of three-dimensional almost Kenmotsu manifolds under certain condition weaker than the local symmetry. Our main result is given as the following.

**Theorem 1.1.** Let $M^3$ be a three-dimensional almost Kenmotsu manifold satisfying $\nabla_\xi h = 0$ and $h \neq 0$. Then the Ricci tensor of $M^3$ is $\eta$-parallel if and
only if the manifold is locally isometric to a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure.

The present paper is arranged as follows. In Section 2, after presenting some necessary preliminaries used in proofs of our main results, we construct some examples of three-dimensional almost Kenmotsu manifolds satisfying $\nabla_{\xi}h = 0$. In Section 3, we present the detailed proofs of our main results with some corollaries. Moreover, we give some examples of three-dimensional almost Kenmotsu manifolds on which $\nabla_{\xi}h \neq 0$ but the Ricci tensor is $\eta$-parallel.

2. Three-dimensional almost Kenmotsu manifolds

In this paper, we denote by $(M^3, \phi, \xi, \eta, g)$ a three-dimensional almost Kenmotsu manifold and we set $l = R(\cdot, \xi)\xi$, $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$, where $\mathcal{L}$ denotes the Lie differentiation and $R$ is the Riemannian curvature tensor. From Dileo and Pastore [8, 9], we see that both $h$ and $h'$ are symmetric operators and we recall some properties of almost Kenmotsu manifolds as follows:

$$\nabla_{\xi} = h' + \text{id} - \eta \otimes \xi,$$

$$\phi l \phi - l = 2(h^2 - \phi^2),$$

$$\nabla_{\xi} h = -\phi - 2h - \phi h^2 - \phi l.$$

Throughout this paper, we denote by $\mathcal{D}$ the distribution $\mathcal{D} = \ker \eta$ which is of dimension $2n$. Then it is easy to see that each integral manifold of $\mathcal{D}$, with the restriction of $\phi$, admits an almost Kähler structure. If the associated almost Kähler structure is integrable, or equivalently (see [9]),

$$(\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX)$$

for any vector fields $X, Y$, then we say that $M^{2n+1}$ is CR-integrable. Following [12, Theorem 2.1], we obtain from (2.5) that an almost Kenmotsu manifold is Kenmotsu if and only if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

for any vector fields $X, Y$. Since a three-dimensional almost Kenmotsu manifold $M^3$ is CR-integrable, we obtain that $M^3$ is Kenmotsu if and only if $h$ vanishes.

Let $\mathcal{U}_1$ be the open subset of a 3-dimensional almost Kenmotsu manifold $M^3$ such that $h \neq 0$ and $\mathcal{U}_2$ the open subset of $M^3$ which is defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of $M^3$ and there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of $h$ for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we may set $he = \lambda e$ and hence $h\phi e = -\lambda \phi e$, where $\lambda$ is a positive function on $\mathcal{U}_1$. Notice that the eigenvalue function $\lambda$ is continuous on $M^3$ and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$. 


Lemma 2.1 ([5, Lemma 6]). On $U_1$ we have

\[ \nabla_{\xi} \xi = 0, \quad \nabla_{\xi} e = a \phi e, \quad \nabla_{\xi} \varphi e = -ae, \]

(2.7) \[ \nabla_{e} \xi = e - \lambda \phi e, \quad \nabla_{e} e = -\xi - b \phi e, \quad \nabla_{e} \varphi e = \lambda \xi + be, \]

where $a, b, c$ are smooth functions.

Applying Lemma 2.1 in the following Jacobi identity

\[ [[\xi, e], \phi e] + [[e, \phi e], \xi] + [[\phi e, \xi], e] = 0 \]

yields

(2.8) \[ \begin{cases} e(\lambda) - \xi(b) - c(a) + c(\lambda - a) - b = 0, \\ \phi e(\lambda) - \xi(c) + \phi e(a) + b(\lambda + a) - c = 0. \end{cases} \]

Moreover, applying Lemma 2.1 we have (see also [5]) the following.

Lemma 2.2. On $U_1$, the Ricci operator can be written as

\[ Q\xi = -2(\lambda^2 + 1)\xi - (\phi e(\lambda) + 2\lambda b)e - (e(\lambda) + 2\lambda c)\phi e, \]

\[ Qe = -(\phi e(\lambda) + 2\lambda b)\xi - (e(c) + \phi e(b) + b^2 + c^2 + 2\lambda a + 2)e + (\xi(\lambda) + 2\lambda)\phi e, \]

\[ Q\varphi e = -(e(\lambda) + 2\lambda c)\xi + (\xi(\lambda) + 2\lambda)e - (e(c) + \phi e(b) + b^2 + c^2 - 2\lambda a + 2)\phi e, \]

with respect to the local basis $\{\xi, e, \phi e\}$.

Following [9, Theorem 5.2] we now construct a left invariant almost Kenmotsu structure on a three-dimensional non-unimodular Lie group. A Lie group $G$ is said to be unimodular if its left invariant Haar measure is also right invariant. It is known that a Lie group $G$ is unimodular if and only if the endomorphism $\text{ad}_X : g \to g$ given by $\text{ad}_X(Y) = [X, Y]$ has trace equal to zero for any $X \in g$, where $g$ denotes the Lie algebra associated to $G$.

From Milnor [14], if $G$ is a three-dimensional non-unimodular Lie group, then there exists a left invariant local orthonormal frame fields $\{e_1, e_2, e_3\}$ satisfying

(2.9) \[ [e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = \gamma e_2 + \delta e_3 \]

and $\alpha + \delta = 2$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. We define a metric $g$ on $G$ by $g(e_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq 3$. We denote by $\xi = -e_1$ and by $\eta$ the dual 1-form of $\xi$. We define a $(1, 1)$-type tensor field $\phi$ by $\phi(\xi) = 0$, $\phi(e_2) = e_3$ and $\phi(e_3) = -e_2$. Then, one can check that $(G, \phi, \xi, \eta, g)$ admits a left invariant almost Kenmotsu structure. Using the Koszul formula and (2.9) we obtain

\[ \nabla_{\xi} \xi = 0, \quad \nabla_{e_2} \xi = (\beta + \gamma) e_2 + (2 - \alpha) e_3, \]

(2.10) \[ \nabla_{\xi} e_2 = \frac{1}{2}(\gamma - \beta) e_2, \quad \nabla_{e_2} e_2 = -\alpha \xi, \quad \nabla_{e_2} e_3 = -\frac{1}{2}(\beta + \gamma) \xi, \quad \nabla_{e_3} e_3 = (\alpha - 2) \xi. \]
By using (2.2), it follows from the first term of (2.10) that

\[(2.11) \quad he_2 = (\alpha - 1)e_3 - \frac{1}{2}(\beta + \gamma)e_2 \quad \text{and} \quad he_3 = \frac{1}{2}(\beta + \gamma)e_3 + (\alpha - 1)e_2.\]

Therefore, from (2.10) and (2.11) we have

\[(2.12) \quad \nabla_\xi h = (\beta - \gamma)h'.\]

**Example 2.1.** Let \(G\) be a three-dimensional non-unimodular Lie group with a left invariant local orthonormal frame fields \(\{e_1, e_2, e_3\}\) satisfying

\[
[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = \beta e_2 + (2 - \alpha)e_3
\]

for \(\alpha, \beta \in \mathbb{R}\). If either \(\alpha \neq 1\) or \(\beta \neq 0\), \(G\) admits a left invariant non-Kenmotsu almost Kenmotsu structure satisfying \(\nabla_\xi h = 0\).

### 3. \(\eta\)-parallel Ricci tensors

On a three-dimensional Riemannian manifold \((M, g)\), the curvature tensor \(R\) is given by using the Ricci operator \(Q\), the Ricci tensor \(S\) and the scalar curvature \(r\) as follows

\[
R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y
\]

\[-\frac{r}{2} \{g(Y, Z)X - g(X, Z)Y\}
\]

for any vector fields \(X, Y, Z\). It follows directly that

\[
(\nabla_V R)(X, Y)Z = g(Y, Z)(\nabla_V Q)X - g(X, Z)(\nabla_V Q)Y + g((\nabla_V Q)Y, Z)X
\]

\[-g((\nabla_V Q)X, Z)Y - \frac{1}{2} V(r)(g(Y, Z)X - g(X, Z)Y)
\]

for any vector field \(V\). From the above relation, it is easy to see that the local symmetry \((\nabla R = 0)\) and the Ricci parallelism \((\nabla Q = 0)\) are equivalent. Since the local classification problem of a three-dimensional almost Kenmotsu manifold \(M^3\) under the local symmetry condition was completed by the present author [17] and Cho [4], thus in this paper we study \(M^3\) under a weaker condition defined as follows.

**Definition 3.1.** An almost Kenmotsu manifold \((M^{2n+1}, \phi, \xi, \eta, g)\) is said to have an \(\eta\)-parallel Ricci tensor if there holds

\[(3.1) \quad g((\nabla_X Q)Y, Z) = 0\]

for any \(X, Y, Z\) orthogonal to the Reeb vector field \(\xi\).

If the Ricci tensor is parallel, then it must be \(\eta\)-parallel. However, the converse is not necessarily true.

**Theorem 3.1.** Let \(M^3\) be a three-dimensional strictly almost Kenmotsu manifold satisfying \(\nabla_\xi h = 0\). Then the Ricci tensor of \(M^3\) is \(\eta\)-parallel if and only if the manifold is locally isometric to either the Riemannian product \(\mathbb{H}^2(-4) \times \mathbb{R}\) or a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure.
Proof. Since a three-dimensional almost Kenmotsu manifold $M^3$ is assumed to be strictly, i.e., non-Kenmotsu, or equivalently, $h \neq 0$, then $U_1$ is a non-empty subset. By applying Lemma 2.1 and a direct calculation we obtain

\[(\nabla_\xi)e = \xi(\lambda)e + 2a\lambda\phi e \quad \text{and} \quad (\nabla_\xi h)\phi e = -\xi(\lambda)\phi e + 2a\lambda e.\]

By the assumption condition $\nabla_\xi h = 0$ and (3.2) we have

\[\xi(\lambda) = a = 0,\]

where we have used that $\lambda$ is positive on $U_1$. For simplicity, we denote $f$ by

\[f = e(c) + \phi e(b) + b^2 + c^2 + 2.\]

Then, using (3.3) and (3.4) we obtain from Lemmas 2.1 and 2.2 that

\[\langle \nabla_\xi Q \rangle = -\xi(\phi e(\lambda) + 2\lambda b)e - \xi(e(\lambda) + 2\lambda c)\phi e,\]

\[\langle \nabla_\xi Q \rangle e = -\xi(\phi e(\lambda) + 2\lambda b)\xi - \xi(f)e,\]

\[(\nabla_\xi Q)\phi e = -\xi(\phi e(\lambda) + 2\lambda c)\xi - \xi(f)\phi e,\]

\[\langle \nabla_\xi Q \rangle e = (f - 2 - e(\phi e(\lambda) + 2\lambda b) - b(e(\lambda) + 2\lambda c))\xi + (e(\lambda) + \lambda \phi e(\lambda) - 2\lambda c + 2\lambda^2 b - 2\lambda c)\phi e,\]

\[\langle \nabla_\xi Q \rangle e = (2\lambda^3 - f\lambda + b(\phi e(\lambda) + 2\lambda b) - e(\lambda) + 2\lambda c)\xi + (2\lambda(e(\lambda) + 2\lambda c) - e(f) - 4\lambda b)\phi e,\]

\[\langle \nabla_\xi Q \rangle e = (2\lambda^3 - f\lambda + c(e(\lambda) + 2\lambda c) - \phi e(\phi e(\lambda) + 2\lambda b))\xi - (\phi e(f) + 4\lambda c - 2\lambda(\phi e(\phi e(\lambda) + 2\lambda b)))e + (\phi e(\lambda) + \lambda e(\lambda) + 2\lambda^2 c - 2\lambda b)\phi e,\]

\[\langle \nabla_\xi Q \rangle e = (f - 2 - \phi e(\phi e(\lambda) + 2\lambda c) - c(\phi e(\phi e(\lambda) + 2\lambda b)))\xi + (\phi e(\lambda) + \lambda e(\lambda) + 2\lambda^2 c - 2\lambda b)e - (\phi e(f) + 2e(\lambda))\phi e,\]

\[\langle \nabla_\xi Q \rangle = 2(\phi e(\lambda) - 3\lambda e(\lambda) + 2\lambda b - 2\lambda^2 c)\xi + (f - 2 - e(\phi e(\lambda) + 2\lambda b) - b(e(\lambda) + 2\lambda c))e + (2\lambda^3 + b(\phi e(\phi e(\lambda) + 2\lambda b) - e(\lambda) + 2\lambda c) - \lambda f)\phi e,\]

\[\langle \nabla_\xi Q \rangle = 2(e(\lambda) - 3\lambda \phi e(\lambda) + 2\lambda c - 2\lambda^2 b)\xi + (2\lambda^3 + c(e(\lambda) + 2\lambda c) - \phi e(\phi e(\lambda) + 2\lambda b) - \lambda f)e + (f - 2 - \phi e(\phi e(\lambda) + 2\lambda b) - c(\phi e(\lambda) + 2\lambda b))\phi e.\]

If the Ricci tensor is $\eta$-parallel, then from (3.1) and (3.8) we have

\[\begin{cases}
   e(f) + 2\phi e(\lambda) = 0, \\
   e(\lambda) + \lambda \phi e(\lambda) + 2\lambda^2 b - 2\lambda c = 0.
\end{cases}\]
Moreover, it follows from (3.1) and (3.11) that

\begin{align}
\begin{cases}
\phi e(f) + 2e(\lambda) = 0, \\
\phi e(\lambda) + \lambda e(\lambda) + 2\lambda^2 c - 2\lambda b = 0.
\end{cases}
\end{align}

We remark that if (3.14) and (3.15) are true, then there hold

\[ g((\nabla \phi e) Q, X) = g((\nabla e) Q e, X) = 0 \]

for any vector field \( X \) orthogonal to \( \xi \). Notice that \( \lambda \) is assumed to be a positive eigenvalue function of \( h \). Thus, comparing the last term of (3.14) to that of (3.15) we see that either \( \lambda = 1 \) or \( e(\lambda) \) and \( \phi e(\lambda) \) are rewritten as follows.

\begin{align}
\begin{cases}
e(\lambda) = \frac{4\lambda^2 b - 2\lambda^3 c - 2\lambda c}{\lambda^2 - 1}, \\
\phi e(\lambda) = \frac{4\lambda^2 c - 2\lambda^3 b - 2\lambda b}{\lambda^2 - 1}.
\end{cases}
\end{align}

The following proof is divided into two cases.

**Case 1**: \( \lambda = 1 \). Using this in the last term of (3.14) or (3.15) gives \( b = c \). Then, it follows from Lemma 2.2 that

\begin{align}
\begin{cases}
Q \xi = -4\xi - 2be - 2b\phi e, \\
Q e = -2b\xi - fe + 2\phi e, \\
Q \phi e = -2b\xi + 2e - f\phi e.
\end{cases}
\end{align}

By using the above relations we directly obtain

\[ Qh\phi - h\phi Q + g(Q\xi, \cdot)\xi - \eta \otimes Q\xi = 0. \]

 Applying \( \lambda = 1, b = c \) and (3.3) in relation (2.8) implies \( \xi(b) = 0 \). Next, let us recall the well known formula \( \text{div} Q = \frac{1}{2} \text{grad} r \). By using (3.17) we obtain the scalar curvature \( r = -4 - 2f \). Therefore, using \( b = c, \lambda = 1 \), equations (3.4), (3.5), (3.8) and (3.11) in this formula we obtain \( \xi(r) = 2g((\nabla \xi) Q\xi + (\nabla e) Q e + (\nabla \phi e) Q \phi e, \xi) = 0 \) and hence we obtain \( \xi(f) = 0 \). Using this, \( \xi(b) = 0 \) and \( \lambda = 1 \) in (3.5)-(3.7) we have

\[ \nabla \xi Q = 0. \]

Combining (3.18) with (3.19) and applying (2.2) we see that the Ricci tensor is invariant along the Reeb flow, i.e., \( L_\xi Q = 0 \).

Cho in [5] proved that a three-dimensional non-Kenmotsu almost Kenmotsu manifold \( M^3 \) satisfies \( L_\xi Q = 0 \) if and only if the manifold is locally isometric to a non-unimodular Lie group with a left invariant almost Kenmotsu structure. Moreover, he obtained \( b = c = 0 \) on \( M^3 \) under the condition \( L_\xi Q = 0 \) (see [5, p. 272]). Thus, using \( \lambda = 1 \) and \( a = b = c = 0 \) in relations (3.5)-(3.13) we see that the Ricci tensor is parallel and hence the manifold is locally symmetric. Since the present author [17] and Cho [4] proved that a three-dimensional locally symmetric strictly almost Kenmotsu manifold is locally isometric to the product space \( \mathbb{H}^2(-4) \times \mathbb{R} \), then the proof for the first case follows.
Case ii: \( \lambda \neq 1 \). Now we suppose that \( \lambda \neq 1 \) holds on an open subset of \( \mathcal{U}_1 \) and hence (3.16) is true. From Lemma 2.2 we get

\[(3.20) \quad r = -2\lambda^2 - 2 - 2f.\]

Using (3.20) in the first terms of (3.14) and (3.15) we have

\[(3.21) \quad \begin{cases} e(r) = 4(\phi e(\lambda) - \lambda e(\lambda)), \\ \phi e(r) = 4(e(\lambda) - \lambda \phi e(\lambda)). \end{cases} \]

On the other hand, using again the formula \( \text{div} Q = \frac{1}{2} \text{grad} r \) and applying equations (3.5), (3.8) and (3.11) we obtain \( e(r) = -2\xi(\phi e(\lambda) + 2\lambda b) \). Putting the second term of relation (3.16) and the first term of (3.21) into this relation gives

\[(3.22) \quad \lambda \xi(c) - \xi(b) = 3\lambda^2 b + b - 3\lambda c - \lambda^3 c, \]

where we have used the first term of relation (3.3) and \( \lambda \neq 1 \).

Similarly, using the formula \( \text{div} Q = \frac{1}{2} \text{grad} r \) and applying equations (3.5), (3.8) and (3.11) we obtain \( \phi e(r) = -2\xi(e(\lambda) + 2\lambda c) \). Using (3.3) and putting the first term of (3.16) and the second term of (3.21) into this relation gives

\[(3.23) \quad \lambda \xi(b) - \xi(c) = 3\lambda^2 c + c - 3\lambda b - \lambda^3 b. \]

By a direct calculation, it follows from (3.22) and (3.23) that

\[(3.24) \quad \begin{cases} \xi(b) = 2\lambda c - (\lambda^2 + 1)b, \\ \xi(c) = 2\lambda b - (\lambda^2 + 1)c, \end{cases} \]

where we have used the second terms of (3.14) and (3.15) and that \( \lambda \) is a positive function \( \neq 1 \).

Moreover, by using (3.3) in (2.8) we have

\[\begin{cases} e(\lambda) = \xi(b) + b - \lambda c, \\ \phi e(\lambda) = \xi(c) + c - \lambda b. \end{cases} \]

Putting (3.24) into the above relation we get

\[(3.25) \quad \begin{cases} e(\lambda) = -\lambda^2 b + \lambda c, \\ \phi e(\lambda) = -\lambda^2 c + \lambda b. \end{cases} \]

For simplicity, we set

\[\theta = -\lambda(3\lambda^2 + 1), \quad \rho = \lambda^2(\lambda^2 + 3). \]

Thus, comparing (3.25) with (3.16) we get

\[\begin{pmatrix} \theta & \rho \\ \rho & \theta \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Since we have assumed that \( \lambda \neq 1 \) holds on an open subset of \( \mathcal{U}_1 \) and that \( \lambda \) is positive, then it is easy to see that \( \theta^2 - \rho^2 \neq 0 \) and hence there exists a unique solution for the above equation, i.e., \( b = c = 0 \). Using this in (3.25)
implies that $e(\lambda) = \phi e(\lambda) = 0$. Since $\lambda$ is continuous on $U_1$, then by (3.3) we conclude that $\lambda$ is a global positive constant on $M$. In this context, from (2.7) we have

\[ [\xi, e] = \lambda \phi e - e, \quad [e, \phi e] = 0, \quad [\phi e, \xi] = -\lambda e + \phi e. \]  

Following Milnor [14], here we state that $M$ is locally isometric to a three-dimensional non-unimodular Lie group. Actually, from (3.26) we observe that its unimodular kernel \( \{ X \in g : \text{trace}(\text{ad}_X) = 0 \} \) is commutative and of 2-dimension and $\text{trace}(\text{ad}_\xi) = -2$.

Conversely, since the product space $H^2(-4) \times \mathbb{R}$ is locally symmetric then it has a $\eta$-parallel Ricci tensor. Finally, we need only to show that the Ricci tensor of a non-Kenmotsu almost Kenmotsu structure defined on any three-dimensional non-unimodular Lie group satisfying $\nabla_X h = 0$ is $\eta$-parallel. In this context, from (2.12) we have $\beta = \gamma$. By a direct calculation, we obtain from (2.10) that

\[
R(e_2, \xi)e_2 = -(\alpha^2 + \beta^2)e_2 - 2\beta e_3, \\
R(e_3, \xi)e_2 = -2\beta e_2 - (\alpha - 2)^2 + \beta^2)e_3, \\
R(\xi, e_2)e_2 = -(\alpha^2 + \beta^2)e_2, \\
R(\xi, e_3)e_3 = -((\alpha - 2)^2 + \beta^2)e_3, \\
R(e_2, e_3)e_3 = (\alpha^2 - 2\alpha + \beta^2)e_2, \\
R(e_3, e_2)e_2 = (\alpha^2 - 2\alpha + \beta^2)e_3.
\]

It follows directly that

\[
Q\xi = -2(\alpha^2 - 2\alpha + 2 + \beta^2)e_3, \\
Qe_2 = -2\alpha e_2 - 2\beta e_3, \\
Qe_3 = -2\beta e_2 + 2(\alpha - 2)e_3.
\]

Therefore, using relation (2.10), by a direct calculation we obtain that $(\nabla e_i Q)e_j$ is colinear with the Reeb vector field $\xi$ for any $i, j \in \{2, 3\}$ and hence the Ricci tensor is $\eta$-parallel. This completes the proof. \( \square \)

**Remark 3.1.** An almost Kenmotsu structure defined on a three-dimensional non-unimodular Lie group whose Lie algebra admits a local orthonormal basis satisfying

\[ [e_1, e_2] = 2e_2, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0 \]

is in fact locally a Riemannian product space $H^2(-4) \times \mathbb{R}$ (see also [9, pp. 59–60]). Then Theorem 1.1 follows.

In what follows, we say that the Ricci tensor of an almost contact metric manifold is strong $\eta$-parallel if it satisfies

\[ g((\nabla_X Q)Y, Z) = 0 \]
for any vector field $X$ and any vector fields $Y, Z$ orthogonal to $\xi$. In view of the almost Kenmotsu structures considered in Theorem 3.1 being non-Kenmotsu, then the theorem can be rewritten as the following.

**Corollary 3.1.** Let $M^3$ be a three-dimensional almost Kenmotsu manifold satisfying $\nabla_\xi h = 0$. Then the Ricci tensor of $M^3$ is strong $\eta$-parallel if and only if the manifold is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$, the product space $\mathbb{H}^2(-4) \times \mathbb{R}$ or a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure.

**Proof.** Notice that a three-dimensional almost Kenmotsu manifold is Kenmotsu if and only if $h$ vanishes. It is proved by De and Pathak [7, Proposition 5.1] that a three-dimensional Kenmotsu manifold having a strong $\eta$-parallel Ricci tensor is of constant scalar curvature. Inoguchi [11] proved that a three-dimensional Kenmotsu manifold with constant scalar curvature is of constant sectional curvature $-1$. Thus, the proof for Kenmotsu case follows.

Finally, we consider the non-Kenmotsu cases. Obviously, a strong $\eta$-parallel Ricci tensor must be $\eta$-parallel. Thus, the proof follows immediately from Theorem 3.1. This completes the proof. □

Since on a locally symmetric almost Kenmotsu manifold there holds $\nabla_\xi h = 0$ (see [8, Proposition 6]), therefore, the following corollary follows from Theorem 3.1 and [13, Corollary 6].

**Corollary 3.2 ([4, 17]).** A three-dimensional almost Kenmotsu manifold is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product space $\mathbb{H}^2(-4) \times \mathbb{R}$.

**Proof.** In [17, p. 83], the present author proved that on any three-dimensional locally symmetric non-Kenmotsu almost Kenmotsu manifold there holds $h^2 = -\phi^2$. Therefore, in this context we observe that Case ii in proof of Theorem 3.1 can not occur. For the Kenmotsu case, the proof follows from [13, Corollary 6], i.e., a locally symmetric Kenmotsu manifold is of constant sectional curvature $-1$. This completes the proof. □

An almost Kenmotsu manifold is called a $(k, \mu, \nu)$-almost Kenmotsu manifold if the Reeb vector field $\xi$ satisfies the $(k, \mu, \nu)$-nullity condition, that is,

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)hX' - \eta(X)hY')$$

for any vector fields $X, Y, Z$ and some smooth functions $k, \mu, \nu$. Putting $Y = \xi$ into (3.27) gives

$$l = -k\phi^2 + \mu h + \nu h'.$$

Using the above relation in equation (2.3) we obtain $h^2 = (k + 1)\phi^2$.

**Remark 3.2.** By relation (2.4) we obtain that on a three-dimensional $(k, \mu, \nu)$-almost Kenmotsu manifold there holds $\nabla_\xi h = \mu h' - (\nu + 2)h$. Thus, one can find
many examples of three-dimensional non-Kenmotsu \((k, \mu, \nu)\)-almost Kenmotsu manifolds with either \(\mu \neq 0\) or \(\nu \neq -2\) on which \(\nabla \xi h \neq 0\). For these examples, we refer the reader to [15, Examples 6.3 and 6.4].

Finally, we point out that on [15, Example 6.3] the Ricci tensor of the manifold is \(\eta\)-parallel in spite of \(\nabla \xi h \neq 0\).

**Example 3.1.** Let \((x, y, z)\) be the canonical coordinates on \(\mathbb{R}^3\). We now consider a three-dimensional manifold \(M^3 \subset \mathbb{R}^3\) defined by

\[
M^3 := \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}.
\]

In what follows, on \(M^3\) we set

\[
\xi := \frac{\partial}{\partial z}, \quad \eta := dz, \quad \phi \left( \frac{\partial}{\partial x} \right) = z \frac{\partial}{\partial y}, \quad \phi \left( \frac{\partial}{\partial y} \right) = -\frac{1}{z} \frac{\partial}{\partial x},
\]

\[
g = ze^{2z} dx^2 + \frac{e^{2z}}{z} dy^2 + dz^2.
\]

Saltarelli [15] proved that \((M^3, \phi, \xi, \eta, g)\) is an almost Kenmotsu manifold for which \(\xi\) belongs to the \((-\frac{1}{4z^2}, -2 + \frac{1}{z}, 0)\)-nullity distribution. Then it follows from (3.27) that

\[
Q\xi = -\left(2 + \frac{1}{2z^2}\right)\xi.
\]

Moreover, using the Koszul formula we have

\[
\nabla_\xi \xi = 0, \quad \nabla_\xi \frac{\partial}{\partial x} = \left(1 + \frac{1}{2z}\right) \frac{\partial}{\partial x},
\]

\[
\nabla_\xi \frac{\partial}{\partial y} = \left(1 - \frac{1}{2z}\right) \frac{\partial}{\partial y}, \quad \nabla_\phi \phi \xi = \left(1 + \frac{1}{2z}\right) \frac{\partial}{\partial x},
\]

\[
\nabla_\phi \xi = -\frac{1 + z^2}{z} e^{2z} \xi, \quad \nabla_\phi \frac{\partial}{\partial y} = \nabla_\phi \frac{\partial}{\partial x} = 0,
\]

\[
\nabla_\phi \phi \xi = \left(1 - \frac{1}{2z}\right) \frac{\partial}{\partial y}, \quad \nabla_\phi \frac{\partial}{\partial y} = \frac{1 - 2z}{2z^2} e^{2z} \xi,
\]

where \(\nabla\) denotes the Levi-Civita connection of the Riemannian metric \(g\). Using (2.2) we also have from the above relation that

\[
h \frac{\partial}{\partial x} = \frac{1}{2} \frac{\partial}{\partial y}, \quad h \frac{\partial}{\partial y} = \frac{1}{2z^2} \frac{\partial}{\partial x}.
By a straightforward calculation we have

\[ R\left(\frac{\partial}{\partial x}, \xi \right) \xi = \left(1 - \frac{3}{4} z^2 + \frac{1}{z} - 1 \right) \frac{\partial}{\partial x}, \]

\[ R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \frac{\partial}{\partial y} = \left(1 - \frac{3}{4} z^2 + \frac{1}{z} - 1 \right) \frac{\partial}{\partial y}, \]

\[ R\left(\frac{\partial}{\partial y}, \xi \right) \xi = \left(\frac{1 + z^2}{2 z} \right) \frac{\partial}{\partial y}, \]

and hence we get

\[ Q \frac{\partial}{\partial x} = \left(1 - \frac{1}{2} z - 2 \right) \frac{\partial}{\partial x}, \]

and

\[ Q \frac{\partial}{\partial y} = \left(1 + z^2 - \frac{7}{4} + \frac{3}{2} - 2 \right) \frac{\partial}{\partial y}, \]

where \( Q \) is the Ricci operator. Finally, by a direct calculation we obtain that \( (\nabla \frac{\partial}{\partial x} Q) \frac{\partial}{\partial x}, (\nabla \frac{\partial}{\partial x} Q) \frac{\partial}{\partial y}, (\nabla \frac{\partial}{\partial y} Q) \frac{\partial}{\partial x} \) and \( (\nabla \frac{\partial}{\partial y} Q) \frac{\partial}{\partial y} \) are all colinear with the Reeb vector field \( \xi \). This means that the Ricci tensor is \( \eta \)-parallel, but not parallel.

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