STRONG CONVERGENCE OF GENERAL ITERATIVE ALGORITHMS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce two general iterative algorithms (one implicit algorithm and other explicit algorithm) for nonexpansive mappings in a reflexive Banach space with a uniformly G\'ateaux differentiable norm. Strong convergence theorems for the sequences generated by the proposed algorithms are established.

1. Introduction

Let $E$ be a real Banach space with the norm $\| \cdot \|$, and let $E^*$ be the dual space of $E$. Let $J$ denote the normalized duality mapping from $E$ into $2^{E^*}$ defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \| x \| \| f \|, \| f \| = \| x \| \}, \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pair between $E$ and $E^*$. Let $C$ be a nonempty closed convex subset of $E$. For the mapping $T : C \to C$, we denote the fixed point set of $T$ by $Fix(T)$, that is, $Fix(T) = \{ x \in C : Tx = x \}$. Recall that the mapping $T : C \to C$ is said to be nonexpansive if

$$\| Tx - Ty \| \leq \| x - y \|, \forall x, y \in C.$$

In a Banach space $E$ having a single-valued normalized duality mapping $J$, we say that an operator $A$ is strongly positive on $E$ if there exists a $\varpi > 0$ with the property

$$(1.1) \quad \langle Ax, J(x) \rangle \geq \varpi \| x \|^2$$

and

$$\| aI - bA \| = \sup_{\| x \| \leq 1} | \langle (aI - bA)x, J(x) \rangle |, \quad a \in [0, 1], \ b \in [-1, 1],$$

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for all $x \in E$, where $I$ is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (1.1) reduce to

$$(Ax, x) \geq \gamma \|x\|^2, \quad \forall x \in H.$$  

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : E \to E$ by

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in E,$$

where $u \in E$ is an arbitrarily chosen point. Banach’s contraction mapping principle guarantees that $T_t$ has unique a fixed point $x_t$ in $E$, which uniquely solves the following fixed point equation:

$$x_t = tu + (1 - t)Tx_t.$$

(Such a path $\{x_t\}$ is said to be an approximating fixed point of $T$ since it possesses the property that if $\{x_t\}$ is bounded, then $\lim_{t \to 0} \|Tx_t - x_t\| = 0$.) It is unclear, in general, what is the behavior of $x_t$ as $t \to 0$, even if $T$ has a fixed point. However, in the case of $T$ having a fixed point, Browder [3] proved that if $E$ is a Hilbert space, then $x_t$ converges strongly to a fixed point of $T$. Reich [10] extended Browder’s result to the setting of Banach spaces and proved that if $E$ is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of $T$ and the limit defines the (unique) sunny nonexpansive retraction from $E$ onto $\text{Fix}(T)$. Xu [16] proved Reich’s results hold in reflexive Banach space having a weakly continuous duality mapping.

In a real Hilbert space $H$, in 2000, Moudafi [9] introduced the following viscosity approximation methods for nonexpansive mapping $T$ on $C$ in an implicit way and an explicit way, respectively:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0,$$

and

$$(1.2) \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$; and $f : C \to C$ is a contractive mapping (i.e., there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $\forall x, y \in H$).

In 2006, Marino and Xu [8] considered the following general iterative algorithm for nonexpansive mapping $T$ on $H$ in an implicit way:

$$(1.3) \quad x_t = \alpha_t f(x_t) + (I - tA)Tx_t, \quad \forall t \in (0, \min\{1, \|A\|^{-1}\}),$$

where $A : H \to H$ is a strongly positive linear bounded operator with a coefficient $\gamma > 0$; $f : H \to H$ is a contractive mapping; and $\gamma > 0$. In 2011, Wangkeeree et al. [13] extended the result of Marino and Xu [8] to a reflexive Banach space having a weakly continuous duality mapping. The results of Marino and Xu [8] and Wangkeeree et al. [13] improved upon the corresponding results of Browder [3], Moudafi [9], Reich [10] and Xu [16] to a general approximating fixed point $\{x_t\}$ defined by (1.3). Combining the Moudafi’s method...
(1.2) with Xu’s method [15], Marino and Xu [8] also considered the following general iterative algorithm for a nonexpansive mapping $T$ in an explicit way:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0,$$

where $f$ is a contractive mapping on $H$; and $\gamma > 0$. They proved that if the sequence $\{\alpha_n\}$ in $(0, 1)$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of a certain variational inequality related to $A$.

In this paper, as a continuation of study in this direction, we present new general iterative algorithms for the nonexpansive mapping in a reflexive Banach space with a uniformly Gâteaux differentiable norm. First, we introduce a general implicit iterative algorithm. Consequently, by discretizing the continuous implicit method, we provide a general explicit iterative algorithm for finding a fixed point of the nonexpansive mapping. Under some control conditions, we establish the strong convergence of the proposed explicit algorithm to a fixed point of the mapping, which solves a certain variational inequality.

2. Preliminaries and lemmas

Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $E^*$ be its dual.

A Banach space $E$ is called strictly convex if its unit sphere $U = \{x \in E : \|x\| = 1\}$ does not contain any linear segment. For every $\varepsilon$ with $0 \leq \varepsilon \leq 2$, the modulus $\delta(\varepsilon)$ of convexity of $E$ is defined by

$$\delta(\varepsilon) = \inf \{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \}.$$

$E$ is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If $E$ is uniformly convex, then $E$ is reflexive and strictly convex.

The norm of $E$ is said to be Gâteaux differentiable (and $E$ is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Gâteaux differentiable if for each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$. Since the dual $E^*$ of $E$ is uniformly convex if and only if the norm of $E$ is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The converse implication is false. A discussion of these and related concepts may be found in [5].

Let $J$ be the normalized duality mapping from $E$ into $2^{E^*}$. It is well-known that $J$ is single valued if and only if $E$ is smooth, and that if $E$ has a uniformly Gâteaux differentiable norm, $J$ is uniformly continuous on bounded subsets of
Let $LIM$ be a linear continuous functional on $\ell^\infty$. According to time and circumstances, we use $LIM_n(a_n)$ instead of $LIM(a)$ for every $a = \{a_n\} \in \ell^\infty$. $LIM$ is called a Banach limit if $\|LIM\| = LIM(1) = 1$ and $LIM_n(a_{n+1}) = LIM_n(a_n)$ for every $a = \{a_n\} \in \ell^\infty$.

Recall that a closed convex subset $C$ of $E$ is said to have the fixed point property for nonexpansive self-mappings (FPP for short) if every nonexpansive mapping $T : C \rightarrow C$ has a fixed point, that is, there is a point $p \in C$ such that $Tp = p$. It is well-known that every bounded closed convex subset of a uniformly smooth Banach space has the FPP ([7, p. 45]).

The mapping $T : C \rightarrow C$ is said to be pseudocontractive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2, \quad \forall x, y \in C,$$

and $T$ is said to be strongly pseudocontractive if there exists a constant $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2, \quad \forall x, y \in C.$$

We need the following lemmas for the proof of our main results.

**Lemma 2.1** ([5]). Let $E$ be a Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $T : C \rightarrow C$ be a continuous strongly pseudocontractive mapping. Then $T$ has a fixed point in $C$.

**Lemma 2.2** ([4]). Assume that $A$ is a strongly positive linear bounded operator on a smooth Banach space $E$ with coefficient $\gamma > 0$ and $0 < \rho < \|A\|^{-1}$. Then

$$\|I - \rho A\| \leq 1 - \rho \gamma.$$

**Lemma 2.3** ([14]). Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n \delta_n + \omega_n, \quad \forall n \geq 1,$$

where $\{\lambda_n\}, \{\delta_n\}$ and $\omega_n$ satisfy the following conditions:

1. $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=1}^\infty \lambda_n = \infty$ or, equivalently, $\prod_{n=1}^\infty (1 - \lambda_n) = 0$;
2. $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=1}^\infty \lambda_n |\delta_n| < \infty$;
3. $\omega_n \geq 0$ and $\sum_{n=1}^\infty \omega_n < \infty$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

**Lemma 2.4** ([11]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $E$ such that

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)y_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ is a sequence in $[0, 1]$ such that

$$0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 1.$$
Assume that
\[
\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]
Then \(\lim_{n \to \infty} \|y_n - x_n\| = 0\).

**Lemma 2.5** ([1, 2]). Let \(C\) be a closed convex of a reflexive and strictly convex Banach space \(E\). Then \(C^0 = \{x \in C : \|x\| = \inf \{\|y\| : y \in C\}\}\) is a singleton.

**Lemma 2.6.** Let \(E\) be a smooth Banach space. Then there holds
\[
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, J(x + y) \rangle, \quad \forall x, y \in E.
\]

3. Main results

Throughout the rest of this paper, we always assume the following:

- \(E\) is a real smooth Banach space;
- \(C\) is a nonempty closed subspace of \(E\);
- \(A : C \to C\) is a strongly positive linear bounded operator with a constant \(\gamma > 0\);
- \(h : C \to C\) is a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient \(k \in (0, 1)\);
- The constant \(\gamma > 0\) satisfies \(0 < \gamma < \frac{1}{k}\);
- \(T : C \to C\) is a nonexpansive mapping with \(\text{Fix}(T) \neq \emptyset\).

In this section, first, we introduce the following general iterative algorithm that generates a net \(\{x_t\}, t \in (0, \min\{1, \|A\|^{-1}\})\) in an implicit way:

\[
x_t = t\gamma h(x_t) + (I - tA)Tx,
\]

Now, for \(t \in (0, \min\{1, \|A\|^{-1}\})\), consider the mapping \(G_t : C \to C\) defined by
\[
G_t(x) := t\gamma h(x) + (I - tA)Tx, \quad x \in C.
\]

Then \(G_t\) is a continuous strongly pseudocontractive mapping with a pseudocontractive coefficient \(1 - t(\overline{\gamma} - \gamma k) \in (0, 1)\). Indeed, from Lemma 2.2 we have for each \(x, y \in C\),
\[
\langle G_t x - G_t y, J(x - y) \rangle = t\gamma \langle h(x) - h(y), J(x - y) \rangle + \langle (I - tA)(Tx - Ty), J(x - y) \rangle \\
\leq t\gamma k \|x - y\|^2 + \|I - tA\| \|Tx - Ty\| \|x - y\| \\
\leq t\gamma k \|x - y\|^2 + (1 - t\overline{\gamma}) \|x - y\|^2 \\
= (1 - t(\overline{\gamma} - \gamma k)) \|x - y\|^2.
\]

Thus, by Lemma 2.1, \(G_t\) has a unique fixed point, denoted by \(x_t\), which uniquely solves the fixed point equation \(3.1\).

We summarize the basic properties of \(\{x_t\}\).

**Proposition 3.1.** Let \(\{x_t\}\) be defined via \(3.1\). Then the following hold:

(a) \(x_t\) is a unique path \(t \mapsto x_t \in C, t \in (0, \min\{1, \|A\|^{-1}\})\).
(b) If \( v \) is a fixed point of \( T \), then for each \( t \in (0, \min\{1, \|A\|^{-1}\}) \)
\[
\langle (A - \gamma h)x_t, J(x_t - v) \rangle \leq \langle A(I - T)x_t, J(x_t - v) \rangle.
\]

(c) If \( T \) has a fixed point in \( C \), then the path \( \{x_t\} \) is bounded and \( \|x_t - Tx_t\| \to 0 \) as \( t \to 0 \).

Proof. (a) To see the continuity of \( x_t \), let \( t, t_0 \in (0, \min\{1, \|A\|^{-1}\}) \). Then we get
\[
\|x_t - x_{t_0}\| \leq \frac{\gamma \|h(x_t)\| + \|ATx_t\|}{t_0(\gamma - \gamma k)}|t - t_0|.
\]
This shows that \( x_t \) is locally Lipschitzian and hence continuous.

(b) Suppose that \( v \) is a fixed point of \( T \). Since \( T \) is nonexpansive, we have for all \( x, y \in C \)
\[
\langle (I - T)x - (I - T)y, J(x - y) \rangle = \|x - y\|^2 - \langle Tx - Ty, J(x - y) \rangle
\geq \|x - y\|^2 - \|x - y\|^2 = 0.
\]

Thus, from (3.1) we obtain
\[
\langle (A - \gamma h)x_t, J(x_t - v) \rangle = -\frac{1}{t}\langle (I - tA)(I - T)x_t, J(x_t - v) \rangle
+ (A(I - T)x_t, J(x_t - v))
\leq \langle A(I - T)x_t, J(x_t - v) \rangle.
\]

(c) Let \( v \in \text{Fix}(T) \). From strong pseudocontractivity of \( h \), it follows that
\[
\langle h(x_t) - h(v), J(x_t - v) \rangle \leq k\|x_t - v\|^2.
\]
Thus we have
\[
\|x_t - v\|^2 = \langle (I - tA)(Tx_t - v) + t(\gamma h(x_t) - Av), J(x_t - v) \rangle
\leq (1 - \gamma t\|x_t - v\|^2 + t\gamma (\|h(x_t) - h(v)\|, J(x_t - v) - Av, J(x_t - v))
\leq (1 - \gamma t\|x_t - v\|^2 + t\gamma k\|x_t - v\|^2 + t\|h(x_t) - Av, J(x_t - v) \|)
\leq (1 - \gamma t\|x_t - v\|^2 + t\gamma k\|x_t - v\|^2 + t\|h(x_t) - Av, J(x_t - v) \|) \to 0 \text{ as } t \to 0.
It follows that
\[ \|x_t - v\| \leq \frac{\|\gamma h(v) - Ax_t\|}{\gamma - \gamma k}. \]
Hence \( \{x_t\} \) is bounded for \( t \in (0, \min\{1, \|A\|^{-1}\}) \). Since \( \|T x_t - v\| \leq \|x_t - v\| \), \( \{T x_t\} \) is bounded and so are \( \{ATx_t\} \) and \( \{Ax_t\} \). Moreover, since \( h \) is a bounded mapping, \( \{h(x_t)\} \) is bounded. This implies that
\[ \|x_t - T x_t\| = t\|\gamma h(x_t) - AT x_t\| \to 0 \quad \text{as} \quad t \to 0. \]

Using Proposition 3.1, we establish strong convergence of \( \{x_t\} \).

**Theorem 3.2.** Let \( E \) be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Assume that every weakly compact convex subset of \( E \) has the FPP for nonexpansive mappings. Let \( \{x_t\} \) be defined via (3.1). Then, as \( t \to 0 \), \( \{x_t\} \) converges strongly to a fixed point \( p \) of \( T \), which is the unique solution in \( Fix(T) \) of the variational inequality
\[ (A - \gamma h)p, J(p - q) \leq 0, \quad \forall q \in Fix(T). \]

**Proof.** First, we show the uniqueness of the solution of the variational inequality (3.2). Suppose both \( p_1 \in Fix(T) \) and \( p_2 \in Fix(T) \) are solutions of the variational inequality (3.2). We have
\[ \langle (A - \gamma h)p_1, J(p_1 - p_2) \rangle \leq 0 \]
and
\[ \langle (A - \gamma h)p_2, J(p_2 - p_1) \rangle \leq 0. \]
Adding up the above two inequalities, we obtain
\[ \langle (A - \gamma h)p_1 - (A - \gamma h)p_2, J(p_1 - p_2) \rangle \leq 0. \]

Note that
\[ \langle (A - \gamma h)p_1 - (A - \gamma h)p_2, J(p_1 - p_2) \rangle = \langle A(p_1 - p_2) J(p_1 - p_2) \rangle \]
\[ \quad - \gamma \langle h(p_1) - h(p_2), J(p_1 - p_2) \rangle \]
\[ \quad \geq \gamma \|p_1 - p_2\|^2 - \gamma \|p_1 - p_2\|^2 \]
\[ \quad = (\gamma - \gamma k)\|p_1 - p_2\|^2 \geq 0. \]

Consequently, we have \( p_1 = p_2 \) and the uniqueness is proved. We use \( \tilde{p} \) to the unique solution of the variational inequality (3.2).

Now, we may assume, without loss of generality, that \( t \leq \|A\|^{-1} \). From Proposition 3.1(c), we have that \( \{x_t\} \) is bounded.

Assume that \( t_n \to 0 \) as \( n \to \infty \). Set \( x_n := x_{t_n} \). We use the so-called optimization method. Define \( \phi : C \to \mathbb{R} \) by \( \phi(z) = LIM_n(\|x_n - z\|^2) \), \( z \in C \), where \( LIM \) is a Banach limit on \( t^\infty \). Then \( \phi \) is continuous and convex, \( \phi(z) \to \infty \) as \( \|z\| \to \infty \). Since \( E \) is reflexive, \( \phi \) attains its infimum over \( C \) ([2, p. 79]). Let
\[ K = \{u \in C : \phi(u) = \min_{z \in C} \phi(z)\}. \]
We see easily that $K$ is a nonempty closed bounded convex subset of $E$. Note that $\|x_n - T x_n\| \to 0$ as $n \to \infty$ by Proposition 3.1(c). Thus, it follows that for each $u \in K$,
\[
\phi(Tu) = \text{LIM}_n(\|x_n - Tu\|^2) = \text{LIM}_n(\|Tx_n - Tu\|^2) \leq \text{LIM}_n(\|x_n - u\|^2) = \phi(u),
\]
which implies that $T(K) \subset K$, that is, $K$ is invariant under $T$. So, by the hypothesis, $T$ has a fixed point $p \in K$. For $x - Ap \in C$ and $t$ with $0 < t < \min\{1, \|A\|^{-1}\}$, by Lemma 2.6, we get
\[
\|x_n - p - t(x - Ap)\|^2 \leq \|x_n - p\|^2 - 2t(x - Ap, J(x_n - p - t(x - Ap)).
\]
Let $\varepsilon > 0$ be given. Since the norm of $E$ is uniformly Gâteaux differentiable, the duality mapping $J$ is norm-to-weak* uniformly continuous on bounded subsets of $E$. Therefore
\[
\langle x - Ap, J(x_n - p - t(x - Ap) - J(x_n - p) \rangle < \varepsilon
\]
for $t$ is close enough to 0. Consequently, we have
\[
\langle x - Ap, J(x_n - p) \rangle < \varepsilon + \langle x - Ap, J(x_n - p - t(x - Ap)) \rangle \leq \varepsilon + \frac{1}{2t}(\|x_n - p\|^2 - \|x_n - p - t(x - Ap)\|^2).
\]
Since $p$ is a minimizer of $\phi$ over $C$, we have
\[
\text{LIM}_n(\langle x - Ap, J(x_n - p) \rangle) \leq \varepsilon + \frac{1}{2t}(\text{LIM}_n(\|x_n - p\|^2) - \text{LIM}_n(\|x_n - p - t(x - Ap)\|^2)) \leq \varepsilon.
\]
Thus, we obtain
\begin{equation}
\text{LIM}_n(\langle x - Ap, J(x_n - p) \rangle) \leq 0, \quad \forall x \in C.
\end{equation}

On the other hand, since $x_n - p = t_n(\gamma h(x_n) - Ap) + (I - t_n A)(Tx_n - p)$, it follows that
\[
\|x_n - p\|^2 = t_n(\gamma h(x_n) - Ap, J(x_n - p) + (I - t_n A)(Tx_n - p), J(x_n - p)) \leq t_n(\gamma h(x_n) - Ap, J(x_n - p)) + (1 - t_n)\|x_n - p\|^2,
\]
which implies that for $x \in C$,
\begin{equation}
\|x_n - p\|^2 \leq \frac{1}{\gamma}(\gamma h(x_n) - Ap, J(x_n - p)) = \frac{1}{\gamma}(\gamma h(x_n) - x, J(x_n - p)) + \frac{1}{\gamma}(x - Ap, J(x_n - p)).
\end{equation}
Combining (3.3) and (3.4), we obtain
\[
\lim_{n} \left( \|x_n - p\|^2 \right) \leq \frac{1}{\gamma} \lim_{n} \left( \langle \gamma h(x_n) - x, J(x_n - p) \rangle \right) + \frac{1}{\gamma} \lim_{n} \left( \langle x - Ap, J(x_n - p) \rangle \right)
\]
\[
\leq \frac{1}{\gamma} \lim_{n} \left( \langle \gamma h(x_n) - x, J(x_n - p) \rangle \right).
\]
In particular,
\[
\gamma \lim_{n} \left( \|x_n - p\|^2 \right) \leq \lim_{n} \left( \langle \gamma h(x_n) - \gamma h(p), J(x_n - p) \rangle \right) \leq \gamma k \lim_{n} \left( \|x_n - p\|^2 \right).
\]
Hence, \(\gamma \lim_{n} \left( \|x_n - p\|^2 \right) \leq 0\).
Since \(\gamma > \gamma k\), we have
\[
\lim_{n} \left( \|x_n - p\|^2 \right) = 0,
\]
and hence there exists a subsequence which is still denoted \(\{x_n\}\) such that \(x_n \to p\).
Next, we prove that \(p\) solves the variational inequality (3.2). Indeed, from Proposition 3.1(b), we have for \(q \in \text{Fix}(T)\),
\[
\langle (A - \gamma h)x_t, J(x_t - q) \rangle \leq \langle (I - T)x_t, J(x_t - q) \rangle.
\]
Replacing \(t\) with \(t_n\), letting \(n \to \infty\) and noting that \((I - T)x_{tn} \to (I - T)p = 0\), we obtain
\[
\langle (A - \gamma h)p, J(p - q) \rangle \leq 0.
\]
That is, \(p \in \text{Fix}(T)\) is a solution of the variational inequality (3.2). Then \(p = \tilde{p}\). In summary, we have that each cluster point of \(\{x_n\}\) converges strongly to \(p\) as \(t_n \to 0\). This complete the proof. \(\square\)

Next, we substitute the fixed point property assumption, mentioned in Theorem 3.2, by assuming that the space \(E\) is strict convex.

**Theorem 3.3.** Let \(E\) be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let \(\{x_t\}\) be defined via (3.1). Then, as \(t \to 0\), \(\{x_t\}\) converges strongly to a fixed point \(p\) of \(T\), which is the unique solution in \(\text{Fix}(T)\) of the variational inequality (3.2).

**Proof.** Let \(w \in \text{Fix}(T)\). As in the proof of Theorem 3.2, we define \(\phi : C \to \mathbb{R}\) by \(\phi(z) = \lim_{n} \left( \|x_n - z\|^2 \right)\), \(z \in C\), where \(\lim\) is a Banach limit on \(l^\infty\). Let
\[
K = \{u \in C : \phi(u) = \min_{z \in C} \phi(z)\}.
\]
Then, by the proof of Theorem 3.2, \(K\) is invariant under \(T\), Moreover \(K\) contains a fixed point of \(T\). To this end, define the function \(g : K \to \mathbb{R}\) by \(g(u) = \|u - w\|\). Then, by Theorem 1.2 of [2] (or Theorem 2.5.7 of [1]) we conclude that the set
\[
K^* = \{v \in K : g(v) = \min \{g(u) : u \in K\}\}
\]
is nonempty, and by Lemma 2.5, \( K^o \) is singleton. Denote such a singleton by \( p \in K \). Then we also know that \( Tw = w \) and 
\[
\|Tp - w\| = \|Tp - Tw\| \leq \|p - w\|.
\]
Therefore \( Tp = p \). We now follows the proof of Theorem 3.2.

Now, we propose the following general iterative algorithm which generates a sequence in an explicit way:

\[
\begin{aligned}
x_1 &= x \in C \\
x_{n+1} &= \alpha_n \gamma h(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 1,
\end{aligned}
\]

where \( \{\alpha_n\} \) is a sequence in \((0,1)\).

Using Theorem 3.2 and Theorem 3.3, we obtain strong convergence of the sequence \( \{x_n\} \) generated by (3.5).

**Theorem 3.4.** Let \( \{x_n\} \) be a sequence generated by the explicit algorithm (3.5). Let \( \{\alpha_n\} \) satisfy the following conditions:

(C1) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^\infty \alpha_n = \infty \);

(C2) \( |\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n, \quad \sum_{n=1}^\infty \sigma_n < \infty \).

If one of the following assumptions holds:

(H1) \( E \) is a reflexive Banach space with a uniformly Gâteaux differentiable norm, and every weak compact convex subset of \( E \) has the FPP for nonexpansive mappings;

(H2) \( E \) is a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm,

then \( \{x_n\} \) converges strongly to a fixed point \( p \) of \( T \), which is the unique solution in \( \text{Fix}(T) \) of the variational inequality (3.2).

**Proof.** By condition (C1), we may assume, without loss of generality, that \( \alpha_n < \|A\|^{-1} \) for all \( n \geq 1 \). By Lemma 2.2, we have \( \|I - \alpha_n A\| \leq (1 - \alpha_n \bar{\gamma}) \).

Now we divide the proof into five steps.

**Step 1.** We show that \( \{x_n\} \) is bounded. Indeed, pick any \( p \in \text{Fix}(T) \) to obtain

\[
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \gamma h(x_n) + (I - \alpha_n A)Tx_n - p\| \\
&= \|\alpha_n(\gamma h(x_n) - \gamma h(p)) + \alpha_n(\gamma h(p) - Ap) + (I - \alpha_n A)(Tx_n - p)\| \\
&\leq \alpha_n \gamma k \|x_n - p\| + \alpha_n \|\gamma h(p) - Ap\| + (1 - \alpha_n \bar{\gamma}) \|x_n - p\| \\
&\leq (1 - \alpha_n(\bar{\gamma} - \gamma k)) \|x_n - p\| + \alpha_n(\bar{\gamma} - \gamma k) \frac{\|\gamma h(p) - Ap\|}{\bar{\gamma} - \gamma k}.
\end{aligned}
\]

It follows from induction that

\[
\|x_n - p\| \leq \max\left\{ \|x_1 - p\|, \frac{\|\gamma h(p) - Ap\|}{\bar{\gamma} - \gamma k} \right\}, \quad \forall n \geq 1.
\]
Hence \( \{x_n\} \) is bounded. Moreover, since \( h \) is a bounded mapping, \( \{h(x_n)\} \) is bounded. Also, \( \{Tx_n\} \) and \( \{ATx_n\} \) are bounded.

As a direct consequence, from condition (C1) we get

\[
\|x_{n+1} - Tx_n\| = \alpha_n\|\gamma h(x_n) - ATx_n\| \to 0 \quad \text{as} \quad n \to \infty.
\]

**Step 2.** We show that \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \) Indeed, from (3.5), it is easily seen that

\[
\|x_{n+1} - x_n\| = \|(I - \alpha_{n+1}A)\gamma h(x_n) + \alpha_n\gamma h(x_n) - \gamma h(x_n)\| + \alpha_n\gamma h(x_n) - \gamma h(x_n)\|
\]

for \( \forall n \geq 1. \) So, from the condition (C2), we obtain

\[
\|x_{n+1} - x_n\| \leq (1 - \alpha_{n+1}(\gamma - k))\|x_{n+1} - x_n\| + o(\alpha_{n+1} + \sigma_n)M
\]

for \( \forall n \geq 1, \) where \( M = \sup_{n \geq 1} \|ATx_n - \gamma h(x_n)\|. \) Put \( s_n = \|x_{n+1} - x_n\|, \)

\( \lambda_n = \alpha_{n+1}(\gamma - k), \lambda_n \delta_n = o(\alpha_{n+1})M \) and \( \omega_n = \sigma_nM. \) Then, from the conditions (C1) and (C2), it follows that \( \lambda_n \to 0 \) as \( n \to \infty, \sum_{n=1}^{\infty} \lambda_n = \infty \) and \( \sum_{n=1}^{\infty} \omega_n = M \sum_{n=1}^{\infty} \sigma_n < \infty. \) Since (3.7) reduces

\[
s_{n+1} = (1 - \lambda_n)s_n + \lambda_n \delta_n + \omega_n,
\]

it follows from Lemma 2.3 that

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]

**Step 3.** We show that \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0. \) In fact, from (3.6) and Step 2 it follows that

\[
\|Tx_n - x_n\| \leq \|Tx_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty.
\]

**Step 4.** We show that \( \lim_{n \to 0} \gamma h(p) - Ap, J(x_n - p) \) \( \leq 0, \) where \( p = \lim_{n \to 0} x_n \) and \( x_n \) is defined by (3.1). In fact, let \( x_t = t\gamma h(x_t) + (I - tA)Tx_t. \) Then, it follows from Theorem 3.2 or Theorem 3.3 that \( \{x_t\} \) converges strongly to \( p \in Fix(T) \) which is the unique solution of the variational inequality (3.2). Noting that

\[
x_t - x_n = t\gamma h(x_t) + Tx_t - tATx_t - x_n
\]

\[
= t\gamma h(x_t) - Ax_t + (Tx_t - x_n) - t(AX_t - Ax_t)
\]

\[
= t\gamma h(x_t) - Ax_t + (Tx_t - x_n) + (Tx_t - x_n) + t^2A(\gamma h(x_t) - ATx_t),
\]
we have
\[
\|x_t - x_n\|^2 = t(\gamma h(x_t) - Ax_t, J(x_t - x_n)) + \langle Tx_t - Tx_n, J(x_t - x_n) \rangle \\
+ \langle Tx_n - x_n, J(x_t - x_n) \rangle + t^2 \langle A(\gamma h(x_t) - ATx_t), J(x_t - x_n) \rangle \\
\leq t(\gamma h(x_t) - Ax_t, J(x_t - x_n)) + \|x_t - x_n\|^2 \\
+ \|Tx_n - x_n\|\|x_t - x_n\| + t^2 \|A(\gamma h(x_t) - ATx_t)\|\|x_t - x_n\|,
\]
which implies that
\[
\gamma h(x_t) - Ax_t, J(x_n - x_t) \leq \frac{\|Tx_n - x_n\|}{t} + t\|A(\gamma h(x_t) - ATx_t)\|\|x_t - x_n\| \\
\leq \frac{\|Tx_n - x_n\|}{t} + tL,
\]
where \( L > 0 \) is a constant such that \( L = \sup\{\|A(\gamma h(x_t) - ATx_t)\|\|x_t - x_n\| : n \geq 1 \text{ and } t \in (0, \min \{1, \|A\|^{-1}\})\}. \) Since \( x_n - Tx_n \to 0 \) by Step 3, taking the upper limit as \( n \to \infty \) in (3.8), we derive
\[
\limsup_{n \to \infty} (\gamma h(x_t) - Ax_t, J(x_n - x_t)) \leq tM.
\]
Taking the lim sup as \( t \to 0 \) in (3.9) and noticing that the fact that the two limits are interchangeable due to the fact that \( J \) is uniformly continuous on bounded subsets of \( E \) from the strong topology of \( E \) to the weak* topology of \( E^* \), we obtain
\[
\limsup_{n \to \infty} (\gamma h(p) - Ap, J(x_n - p)) \leq 0.
\]
\textbf{Step 5.} We show that \( \lim_{n \to \infty} x_n = p \), where \( p = \lim_{t \to 0} x_t \in \text{Fix}(T) \), \( x_t \) being defined by (3.1), which is the unique solution of the variational inequality (3.2).

Indeed, from (3.5), Lemma 2.2 and Lemma 2.6, we derive
\[
\|x_{n+1} - p\|^2 = \|\alpha_n(\gamma h(x_n)) - Ap\|^2 + (I - \alpha_nA)Tx_n - (I - \alpha_nA)p\|^2 \\
\leq \|(I - \alpha_nA)(Tx_n - p)\|^2 + 2\alpha_n(\gamma h(x_n) - Ap, J(x_{n+1} - p)) \\
\leq (1 - \alpha_n\gamma)\|x_n - p\|^2 + 2\alpha_n(\gamma h(x_n) - \gamma h(p), J(x_{n+1} - p)) \\
+ 2\alpha_n(\gamma h(p) - Ap, J(x_{n+1} - p)) \\
\leq (1 - \alpha_n\gamma)\|x_n - p\|^2 + 2\alpha_n\gamma \|x_n - p\|\|x_{n+1} - p\| \\
+ 2\alpha_n(\gamma h(p) - Ap, J(x_{n+1} - p)) \\
\leq (1 - \alpha_n\gamma)\|x_n - p\|^2 + \alpha_n\gamma \|x_n - p\|^2 + \|x_{n+1} - p\|^2 \\
+ 2\alpha_n(\gamma h(p) - Ap, J(x_{n+1} - p)).
\]
This implies that
\begin{equation}
||x_{n+1} - p||^2 \\
\leq \left( 1 - \frac{2\alpha_n(\tau - \gamma k)}{1 - \alpha_n\gamma k} \right) ||x_n - p||^2 + \frac{2\alpha_n(\gamma h(p) - Ap, J(x_{n+1} - p))}{1 - \alpha_n\gamma k} \\
+ \frac{2\alpha_n^2(\tau - \gamma k)^2}{(\tau - \gamma k)^2 - 1} \cdot \frac{2\alpha_n(\gamma h(p) - Ap, J(x_{n+1} - p))}{1 - \alpha_n\gamma k} L \\
\leq \left( 1 - \frac{2\alpha_n(\tau - \gamma k)}{1 - \alpha_n\gamma k} \right) ||x_n - p||^2 + \frac{2\alpha_n(\gamma h(p) - Ap, J(x_{n+1} - p))}{1 - \alpha_n\gamma k} \\
+ \frac{2\alpha_n^2(\tau - \gamma k)^2}{(\tau - \gamma k)^2 - 1} \cdot \frac{1}{\gamma - \gamma k}(\gamma h(p) - Ap, J(x_{n+1} - p)),
\end{equation}
where $L = \sup \{ ||x - p|| : n \geq 1 \}$. Put $\lambda_n = \frac{2\alpha_n(\tau - \gamma k)}{1 - \alpha_n\gamma k}$ and
\[ \delta_n = \frac{\alpha_n^2(\tau - \gamma k)^2}{2(\tau - \gamma k)^2 - 1} \cdot \frac{1}{\gamma - \gamma k}(\gamma h(p) - Ap, J(x_{n+1} - p)). \]
Then it follows from the condition (C1) and Step 4 that $\lim_{n \to \infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\limsup_{n \to \infty} \delta_n \leq 0$. (3.10) reduces to
\begin{equation}
||x_{n+1} - p||^2 \leq (1 - \lambda_n)||x_n - p||^2 + \lambda_n \delta_n.
\end{equation}
Thus, applying Lemma 2.3 together with $\omega_n = 0$ to (3.11), we conclude that $\lim_{n \to \infty} x_n = p$. This completes the proof. \qed

**Corollary 3.5.** Let $E$ be a uniformly smooth Banach space. Let $\{x_n\}$ be a sequence generated by the explicit algorithm (3.5). Let $\{\alpha_n\}$ satisfy the conditions (C1) and (C2) in Theorem 3.4. Then $\{x_n\}$ converges strongly to a fixed point $p$ of $T$, which is the unique solution in $\text{Fix}(T)$ of the variational inequality (3.2).

Removing the condition $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=1}^{\infty} \sigma_n < \infty$ on the sequence $\{\alpha_n\}$ in Theorem 3.4, we have the following result.

**Theorem 3.6.** Let $\{x_n\}$ be a sequence generated by the following explicit algorithm:
\begin{equation}
\begin{cases}
x_1 = x \in C \\
x_{n+1} = \alpha_n\gamma h(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)Tx_n, \quad n \geq 1,
\end{cases}
\end{equation}
where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$, which satisfy the following conditions:
\begin{enumerate}
\item[(C1)] $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
\item[(C2)] $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.
\end{enumerate}
If one of the following assumptions holds:

- "new para"
(H1) \( E \) is a reflexive Banach space with a uniformly Gâteaux differentiable norm, and every weakly compact convex subset of \( E \) has the FPP for nonexpansive mappings;

(H2) \( E \) is a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm,

then \( \{x_n\} \) converges strongly to a fixed point \( p \) of \( T \), which is the unique solution in \( \text{Fix}(T) \) of the variational inequality (3.2).

Proof. We only include the difference from the proof of Theorem 3.5. By conditions (C1) and (C2), we may assume, without loss of generality, that \( \frac{\alpha_n}{1 - \beta_n} < \|A\|^{-1} \) for all \( n \geq 1 \). By Lemma 2.2, we have \( \|(1 - \beta_n)I - \alpha_n A\| \leq (1 - \beta_n - \alpha_n \gamma) \).

**Step 1.** We show that \( \{x_n\} \), \( \{h(x_n)\} \), \( \{Tx_n\} \) and \( \{ATx_n\} \) are bounded. Indeed, pick any \( p \in \text{Fix}(T) \) to obtain

\[
\|x_{n+1} - p\| = \|\alpha_n \gamma h(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)T x_n - p\|
\]

\[
= \|\alpha_n (\gamma h(x_n) - \gamma h(p)) + \alpha_n (\gamma h(p) - Ap) + \beta_n (x_n - p) + ((1 - \beta_n)I - \alpha_n A)(T x_n - p)\|
\]

\[
\leq \alpha_n \gamma k \|x_n - p\| + \alpha_n \|\gamma h(p) - Ap\| + \beta_n \|x_n - p\| + (1 - \beta_n - \alpha_n \gamma) \|x_n - p\|
\]

\[
= (1 - \alpha_n(\gamma - \gamma) - \alpha_n(\gamma - \gamma) \|\gamma h(p) - Ap\| / (\gamma - \gamma)).
\]

The rest follows from Step 1 of the proof of Theorem 3.4.

**Step 2.** We show that \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \). To this end, define a sequence \( \{z_n\} \) by \( z_n = (x_{n+1} - \beta_n x_n) / (1 - \beta_n) \) so that

\[
(3.13) \quad x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n.
\]

We now observe that

\[
(3.14) \quad \frac{z_{n+1} - z_n}{1 - \beta_{n+1}} = \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}
\]

\[
= \frac{\alpha_n \gamma h(x_{n+1}) + \beta_{n+1} x_{n+1} + ((1 - \beta_{n+1})I - \alpha_{n+1} A)T x_{n+1} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}}
\]

\[
- \frac{\alpha_n \gamma h(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)T x_n - \beta_n x_n}{1 - \beta_n}
\]

\[
= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\gamma h(x_{n+1}) - AT x_{n+1}) + T x_{n+1} - T x_n + \frac{\alpha_n}{1 - \beta_n} (AT x_n - \gamma h(x_n)).
\]
It follows from (3.14) that
\begin{equation}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\|\gamma h(x_{n+1})\| + \|AT x_{n+1}\|) + \frac{\alpha_n}{1 - \beta_n}(\|\gamma h(x_n)\| + \|AT x_n\|).
\end{equation}

By conditions (C1), (C2) and (3.15), we obtain that
\[
\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]

Hence by Lemma 2.4, we have
\begin{equation}
\lim_{n \to \infty} \|z_n - x_n\| = 0.
\end{equation}

It then follows from condition (C2), (3.13) and (3.16) that
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n)\|z_n - x_n\| = 0.
\]

**Step 3.** We show that \(\lim_{n \to \infty} \|x_n - Tx_n\| = 0\). In fact, from (3.12) it follows that
\[
\|Tx_n - x_n\| \leq \|Tx_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\
\leq \alpha_n \gamma h(x_n) - \alpha_n AT x_n + \beta_n \|x_n - Tx_n\| + \|x_{n+1} - x_n\|.
\]

This implies that
\[
(1 - \beta_n)\|Tx_n - x_n\| \leq \alpha_n (\|\gamma h(x_n)\| + \|AT x_n\|) + \|x_{n+1} - x_n\|.
\]

Thus, by conditions (C1) and (C2) and Step 2, we have
\[
\lim_{n \to \infty} \|Tx_n - x_n\| = 0.
\]

**Step 4.** We show that \(\limsup_{n \to \infty} \langle \gamma h(p) - Ap, J(x_n - p) \rangle \leq 0\), where \(p = \lim_{t \to 0} x_t\) and \(x_t\) is defined by (3.1). The result follows from Step 4 in the proof of Theorem 3.4.

**Step 5.** We show that \(\lim_{n \to \infty} x_n = p\), where \(p = \lim_{t \to 0} x_t \in Fix(T)\), \(x_t\) being defined by (3.1), which is the unique solution of the variational inequality (3.2).

Indeed, from (3.12), observe that
\[
x_{n+1} - p = \alpha_n (\gamma h(x_n) - Ap) + \beta_n (x_n - p) + ((1 - \beta_n)I - \alpha_n A)(Tx_n - p).
\]
By Lemma 2.2 and Lemma 2.6, we derive
\[
\|x_{n+1} - p\|^2 \leq (\beta_n\|x_n - p\|^2 + \|(1 - \beta_n)I - \alpha_n A(Tx_n - p)\|)^2 \\
+ 2\alpha_n\langle \gamma h(x_n) - Ap, J(x_{n+1} - p) \rangle \\
\leq (\beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\gamma\|x_n - p\|)^2 \\
+ 2\alpha_n\langle \gamma h(x_n) - Ap, J(x_{n+1} - p) \rangle \\
= (1 - \alpha_n\gamma\|x_n - p\|^2 + 2\alpha_n\langle \gamma h(x_n) - \gamma h(p), J(x_{n+1} - p) \rangle \\
+ 2\alpha_n\langle \gamma h(p) - Ap, J(x_{n+1} - p) \rangle \\
\leq (1 - \alpha_n\gamma\|x_n - p\|^2 + 2\alpha_n\gamma k\|x_n - p\|\|x_{n+1} - p\| \\
+ 2\alpha_n\langle \gamma h(p) - Ap, J(x_{n+1} - p) \rangle \\
\leq (1 - \alpha_n\gamma\|x_n - p\|^2 + \alpha_n\gamma k(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\
+ 2\alpha_n\langle \gamma h(p) - Ap, J(x_{n+1} - p) \rangle.
\]
The remainder follows from the proof of Theorem 3.4. □

Remark 3.7. Our results in this paper extend, improve and develop the corresponding results in [8, 9, 10, 13] and the references therein.

References


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