J. Appl. Math. & Informatics Vol. **35***(2017), No. 3 - 4, pp. 261 - 265*
https://doi.org/10.14317/jami.2017.261

ON CHARACTERIZATIONS OF THE NORMAL DISTRIBUTION BY INDEPENDENCE PROPERTY

MIN-YOUNG LEE

ABSTRACT. Let X and Y be independent identically distributed nondegenerate random variables with common absolutely continuous probability distribution function $F(x)$ and the corresponding probability density function $f(x)$ and $E(X^2) < \infty$. Put $Z = \max(X, Y)$ and $W = \min(X, Y)$. In this paper, it is proved that $Z - W$ and $Z + W$ or $(X - Y)^2$ and $X + Y$ are independent if and only if X and Y have normal distribution.

AMS Mathematics Subject Classification : 60E10, 62E10.
Key words and phrases : independent identically distributed, normal distribution, transformation invariant statistics, independence property.

1. Introduction

Let X and Y be independent identically distributed(i.i.d.) random variables with common absolutely continuous probability distribution function $F(x)$ and the corresponding probability density function $f(x)$.

By definition, the random variable X has a normal distribution with parameters μ and σ^2 if it has density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}, \ -\infty < x < \infty,$$

where $-\infty < \mu < \infty$ and $\sigma^2 > 0$.

It is known that its characteristic function is

$$\varphi(t) = \exp\left\{ i\mu t - \frac{1}{2}\sigma^2 t^2 \right\}.$$

Lukacs (1955) proved the following theorem : let X and Y be two nondegenerate and positive random variables, and suppose that they are independently distributed. X/Y and $X + Y$ are independently distributed if and only if both X and Y have gamma distributions with the same scale parameter. Bansal

Received January 10, 2017. Revised February 21, 2017. Accepted February 23, 2017

et.al.(1999) showed the following theorem : let X and Y be i.i.d. and $\frac{2XY}{\sqrt{X^2+Y^2}}$ be standard normal distributed. Then X has a standard normal distribution. Also, Lee and Galambos (2012) obtained that $\frac{Z}{Z+W}$ and $Z+W$ are independent if and only if X and Y have gamma distribution where $Z = \max(X, Y)$ and $W = \min(X, Y)$.

It is worth to consider such characterizations by independence property of random variables.

Hence, in this paper, we are interested in the extended characterization that $Z-W$ and $Z+W$ or $(X-Y)^2$ and $X+Y$ are independent if and only if X and Y have normal distribution.

2. Main results

Theorem 2.1. Let X and Y be i.i.d. random variables with common absolutely continuous probability distribution function $F(x)$ and the corresponding probability density function $f(x)$ and $E(X^2) < \infty$. Then $Z-W$ and $Z+W$ are independent if and only if X and Y have normal distribution.

Proof. Write $Z = \max(X, Y)$ and $W = \min(X, Y)$. Since $Z-W$ and $Z+W$ are transformation invariant statistics, by Lukacs and Laha (1964), $Z-W$ is independent of $Z+W = X+Y$ for normal variables. So, we have to prove the converse.

We denote the characteristic functions of $Z-W$, $Z+W$ and $(Z-W, Z+W)$ by $\phi_1(t)$, $\phi_2(s)$ and $\phi(t,s)$, respectively.

The independence of $Z-W$ and $Z+W$ is equivalent to

$$\phi(t,s) = \phi_1(t) \cdot \phi_2(s). \tag{2.1}$$

The left hand side of (2.1) becomes

$$\begin{aligned}\phi(t,s) = &\int\int_{-\infty<x\le y<\infty} \exp\{it(y-x)+is(x+y)\}\, f(x)\, f(y)\, dx\ dy \\ &+ \int\int_{-\infty<y<x<\infty} \exp\{it(x-y)+is(x+y)\}\, f(x)\, f(y)\, dx\ dy.\end{aligned}$$

Also, the right hand side of (2.1) becomes

$$\begin{aligned}\phi_1(t)\phi_2(s) = &\Big(\int\int_{-\infty<x\le y<\infty} \exp\{it(y-x)\} f(x)\, f(y)\, dx\ dy \\ &+ \int\int_{-\infty<y<x<\infty} \exp\{it(x-y)\} f(x)\, f(y)\, dx\ dy\Big) \\ &\cdot \Big(\int\int_{-\infty<x\le y<\infty} \exp\{is(x+y)\}\, f(x)\, f(y)\, dx\ dy \\ &+ \int\int_{-\infty<y<x<\infty} \exp\{is(x+y)\}\, f(x)\, f(y)\, dx\ dy\Big).\end{aligned}$$

Then (2.1) gives

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\exp\{it(|x-y|)+is(x+y)\}\, f(x)\, f(y)\, dx\ dy\\ &= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{it(|x-y|)\} f(x)\, f(y)\, dx\ dy\right) \\ &\quad\cdot\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{is(x+y)\}\, f(x)\, f(y)\, dx\ dy\right).\end{aligned} \tag{2.2}$$

The integrals in (2.2) exist not only for reals t and s but also for complex values $t = u + iv$, $s = u^* + iv^*$, where u and u^* are reals, for which $v = Im(t) \geq 0$, $v^* = Im(s) \geq 0$ and they are analytic for all t, s for $v > 0$, $v^* > 0$ [see Lukacs (1955)].

Differentiating (2.2) twice with respect to t and setting $t = 0$, we get

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &(x-y)^2 \exp\{is(x+y)\}\, f(x)\, f(y)\, dx\ dy\\ &= \theta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{is(x+y)\}\, f(x)\, f(y)\, dx\ dy.\end{aligned} \tag{2.3}$$

where $\theta = E\left[(X-Y)^2\right]$.

Note that, by the assumed continuity of $F(x)$, $P(x = y) = 0$, so $\theta > 0$.

Denote the characteristic function of X by

$$\varphi(s) = \int_{-\infty}^{\infty} e^{isx}\, f(x)\, dx. \tag{2.4}$$

Then we know that

$$\varphi'(s) = i\int_{-\infty}^{\infty} x\, e^{isx}\, f(x)\, dx \ \text{ and } \ \varphi''(s) = -\int_{-\infty}^{\infty} x^2\, e^{isx}\, f(x)\, dx. \tag{2.5}$$

By using (2.4) and (2.5), we can express (2.3) as a differential equation with respect to the characteristic function $\varphi(t)$ and get

$$-\varphi''(s)\varphi(s) + (\varphi'(s))^2 = \frac{\theta}{2}\{\varphi(s)\}^2$$

that is,

$$\left\{\frac{\varphi(s)}{\varphi'(s)}\right\}' = \frac{\theta}{2}\left\{\frac{\varphi(s)}{\varphi'(s)}\right\}^2,\ \theta > 0.$$

After integrating and taking the initial conditions $\varphi(0) = 1$, $\varphi'(0) = iE(X)$, we obtain

$$\frac{\varphi'(s)}{\varphi(s)} = iE(X) - \frac{\theta}{2}s,\ \theta > 0. \tag{2.6}$$

Hence, from (2.6), by uniqueness theorem of the differential equation for $\theta > 0$, there exists a unique solution

$$\varphi(s) = \exp\{iE(X)s - \frac{\theta}{4}s^2\}.$$

Consequently, $F(x)$ is a normal distribution. □

Theorem 2.2. Let X and Y be i.i.d. random variables with common absolutely continuous probability distribution function $F(x)$ and the corresponding probability density function $f(x)$ and $E(X^2) < \infty$. Then $(X-Y)^2$ and $X+Y$ are independent if and only if X and Y have normal distribution.

Proof. Since $(X-Y)^2$ and $X+Y$ are transformation invariant statistics, by Lukacs and Laha (1964), $(X-Y)^2$ is independent of $Z+W = X+Y$ for normal variables. So, we have to prove the converse.

We denote the characteristic functions of $(X-Y)^2$, $X+Y$ and $\big((X-Y)^2, X+Y\big)$ by $\varphi_1(t)$, $\varphi_2(s)$ and $\phi(t,s)$, respectively.

The independence of $(X-Y)^2$ and $X+Y$ is equivalent to

$$\phi(t,s) = \phi_1(t) \cdot \phi_2(s). \tag{2.7}$$

Then (2.7) gives

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\big\{it(x-y)^2 + is(x+y)\big\} f(x)\, f(y)\, dx\ dy \\ &\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{it(x-y)^2\} f(x)\, f(y)\, dx\ dy \\ &\qquad \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{is(x+y)\}\, f(x)\, f(y)\, dx\ dy. \end{aligned} \tag{2.8}$$

The integrals in (2.8) exist not only for reals t and s but also for complex values $t = u + iv$, $s = u^* + iv^*$, where u and u^* are reals, for which $v = Im(t) \geq 0$, $v^* = Im(s) \geq 0$ and they are analytic for all t, s for $v > 0$, $v^* > 0$ [see Lukacs (1955)].

Differentiating (2.8) one time with respect to t and setting $t = 0$, we get

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^2 \exp\{is(x+y)\}\, f(x)\, f(y)\, dx\ dy \\ &\quad = \theta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{is(x+y)\}\, f(x)\, f(y)\, dx\ dy. \end{aligned} \tag{2.9}$$

where $\theta = E\left[(X-Y)^2\right]$.

Note that, by the assumed continuity of $F(x)$, $P(x=y) = 0$, so $\theta > 0$.

Denote the characteristic function of X by

$$\varphi(s) = \int_{-\infty}^{\infty} e^{isx}\, f(x)\, dx. \tag{2.10}$$

Then we know that

$$\varphi'(s) = i\int_{-\infty}^{\infty} x\, e^{isx}\, f(x)\, dx \quad \text{and} \quad \varphi''(s) = -\int_{-\infty}^{\infty} x^2\, e^{isx}\, f(x)\, dx. \tag{2.11}$$

By using (2.10) and (2.11), we can express (2.9) as a differential equation with respect to the characteristic function $\varphi(t)$ and get

$$2\varphi''(s)\varphi(s) - 2\{\varphi'(s)\}^2 = -\theta\{\varphi(s)\}^2$$

that is,

$$\frac{\varphi''(s)\varphi(s) - \{\varphi'(s)\}^2}{\{\varphi(s)\}^2} = -\frac{\theta}{2}, \theta > 0.$$

After integrating and taking the initial conditions $\varphi(0) = 1$, $\varphi'(0) = iE(X)$, we obtain

$$\frac{\varphi'(s)}{\varphi(s)} = -\frac{\theta}{2}s + iE[X], \theta > 0. \tag{2.12}$$

Hence, from (2.12), by uniqueness theorem of the differential equation for $\theta > 0$, there exists a unique solution

$$\varphi(s) = \exp\{iE[X]s - \frac{\theta}{4}s^2\}.$$

Consequently, $F(x)$ is a normal distribution. □

References

1. M. Ahsanuallah, *Record Statistics*, Nova Science publishers, Inc, Commack NY, 1995.
2. N. Bansal, G.G. Hamedani, Key, S. Eric, Volkmer, Hans, Zhang, Hao, J. Behboodian, *Some characterizations of the normal distribution*, Statistics and Probability Letters **42** (1999), 393-400.
3. M-Y. Lee, J. Galamos, *A Kotz and Steutel type of characterization of the gamma distribution*, Aequat.Math. Statist. **84** (2012), 121-124.
4. E. Lukacs, *A characterization of the gamma distribution*, Ann. Math. Statist. **26** (1955), 319-324.
5. E. Lukacs, R. G. Laha, *Applications of Characteristic Functions*, Charles Griffin, London, 1964

Min-Young Lee received M.S. and Ph.D. from Temple University. Since 1991 he has been at Dankook University. His research interests include characterizations of distribution, order and record Statistics.

Department of Mathematics, Dankook University, Cheonan 330-714, Korea
e-mail: leemy@dankook.ac.kr