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ON THE (p, q)-ANALOGUE OF EULER ZETA FUNCTION

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Abstract. In this paper we define (p, q)-analogue of Euler zeta function. In order to define (p, q)-analogue of Euler zeta function, we introduce the (p, q)-analogue of Euler numbers and polynomials by generalizing the Euler numbers and polynomials, Carlitz's type q-Euler numbers and polynomials. We also give some interesting properties, explicit formulas, a connection with (p, q)-analogue of Euler numbers and polynomials. Finally, we investigate the zeros of the (p, q)-analogue of Euler polynomials by using computer.

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1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-13]). In this paper, we define (p, q)-analogue of Euler polynomials and numbers and study some properties of the (p, q)-analogue of Euler polynomials and numbers.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -2, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

We remember that the classical Euler numbers E_n and Euler polynomials $T_n(x)$ are defined by the following generating functions(see [1, 2, 3, 4, 5])

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.1}$$

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and

$$\left(\frac{2}{e^t+1}\right)e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad (|t|<\pi). \tag{1.2}$$

respectively.

The (p,q)-number is defined by

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}.$$

It is clear that (p,q)-number contains symmetric property, and this number is q-number when $p=1$. In particular, we can see $\lim_{q\to 1}[n]_{p,q}=n$ with $p=1$.

By using (p,q)-number, we define the (p,q)-analogue of Euler polynomials and numbers, which generalized the previously known numbers and polynomials, including the Carlitz's type q-Euler numbers and polynomials. We begin by recalling here the Carlitz's type q-Euler numbers and polynomials(see 1, 2, 3, 4, 5, 13]).

Definition 1.1. The Carlitz's type q-Euler polynomials $E_{n,q}(x)$ are defined by means of the generating function

$$F_q(t,x)=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_q t}. \tag{1.3}$$

and their values at $x=0$ are called the Carlitz's type q-Euler numbers and denoted $E_{n,q}$(see [12]).

Many kinds of of generalizations of these polynomials and numbers have been presented in the literature(see [1-13]). Based on this idea, we generalize the Carlitz's type q-Euler number $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$. It follows that we define the following (p,q)-analogues of the the Carlitz's type q-Euler number $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$.

In the following section, we define (p,q)-analogue of Euler zeta function. We introduce the (p,q)-analogue of Euler polynomials and numbers. After that we will investigate some their properties. Finally, we investigate the zeros of the (p,q)-analogue of Euler polynomials by using computer.

2. (p,q)-analogue of Euler numbers and polynomials

In this section, we define (p,q)-analogue of Euler numbers and polynomials and provide some of their relevant properties.

Definition 2.1. For $0<q<p\le 1$, the Carlitz's type (p,q)-Euler numbers $E_{n,p,q}$ and polynomials $E_{n,p,q}(x)$ are defined by means of the generating functions

$$F_{p,q}(t)=\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m]_{p,q}t}. \tag{2.1}$$

and

$$F_{p,q}(t,x)=\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_{p,q}t}, \tag{2.2}$$

respectively.

Setting $p=1$ in (2.1) and (2.2), we can obtain the corresponding definitions for the Carlitz's type q-Euler number $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$ respectively. Obviously, if we put $p=1$, then we have

$$E_{n,p,q}(x)=E_{n,q}(x), \quad E_{n,p,q}=E_{n,q}.$$

Putting $p=1$, we have

$$\lim_{q\to 1}E_{n,p,q}(x)=E_n(x), \quad \lim_{q\to 1}E_{n,p,q}=E_n.$$

By using above equation (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,p,q}\frac{t^n}{n!}&=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m]_{p,q}t}\\&=\sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{l+1}p^{n-l}}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 2.2. *For $n\in\mathbb{Z}_+$, we have*

$$E_{n,p,q}=[2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{l+1}p^{n-l}}.$$

If we put $p=1$ in the above theorem we obtain(cf. [12, Theorem 1])

$$E_{n,p,q}=[2]_q\left(\frac{1}{1-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{l+1}}.$$

By (2.2), we obtain

$$E_{n,p,q}(x)=[2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\frac{1}{1+q^{l+1}p^{n-l}}. \tag{2.4}$$

By using (2.2) and (2.4), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\frac{1}{1+q^{l+1}p^{n-l}}\right)\frac{t^n}{n!}\\&=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_{p,q}t}.\end{aligned} \tag{2.5}$$

Since $[x+y]_{p,q} = p^y[x]_{p,q} + q^x[y]_{p,q}$, we see that

$$
\begin{aligned}
&E_{n,p,q}(x) \\
&= [2]_q \sum_{l=0}^{n} \binom{n}{l} [x]_{p,q}^{n-l} q^{xl} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \left(\frac{1}{p-q}\right)^l \frac{1}{1+q^{k+1}p^{n-k}}.
\end{aligned} \tag{2.6}
$$

Next, we introduce Carlitz's type (h,p,q)-Euler polynomials $E_{n,p,q}^{(h)}(x)$.

Definition 2.3. The Carlitz's type (h,p,q)-Euler polynomials $E_{n,p,q}^{(h)}(x)$ are defined by

$$
E_{n,p,q}^{(h)}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m p^{hm} [m+x]_{p,q}^n. \tag{2.7}
$$

By using (2.7) and (p,q)-number, we have the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, we have*

$$
E_{n,p,q}^{(h)}(x) = [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \frac{1}{1+q^{l+1}p^{n-l+h}}.
$$

By (2.6) and Theorem 2.4, we have

$$
E_{n,p,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{p,q}^{n-l} q^{xl} E_{l,p,q}^{(n-l)}
$$

The following elementary properties of the (p,q)-analogue of Euler numbers $E_{n,p,q}$ and polynomials $E_{n,p,q}(x)$ are readily derived form (2.1) and (2.2). We, therefore, choose to omit details involved.

Theorem 2.5. *(Distribution relation) For any positive integer m(=odd), we have*

$$
E_{n,p,q}(x) = \frac{[2]_q}{[2]_{q^m}} [m]_{p,q}^n \sum_{a=0}^{m-1} (-1)^a q^a E_{n,p^m,q^m}\left(\frac{a+x}{m}\right), n \in \mathbb{N}_0.
$$

Theorem 2.6. *(Property of complement)*

$$
E_{n,p^{-1},q^{-1}}(1-x) = (-1)^n p^n q^n E_{n,p,q}(x).
$$

Theorem 2.7. *For $n \in \mathbb{Z}_+$, we have*

$$
qE_{n,p,q}(1) + E_{n,p,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}
$$

By (2.1) and (2.2), we get

$$
\begin{aligned}
&- [2]_q \sum_{l=0}^{\infty} (-1)^{l+n} q^{l+n} e^{[l+n]_{p,q}t} + [2]_q \sum_{l=0}^{\infty} (-1)^l q^l e^{[l]_{p,q}t} \\
&= [2]_q \sum_{l=0}^{n-1} (-1)^l q^l e^{[l]_{p,q}t}.
\end{aligned} \tag{2.8}
$$

Hence we have

$$\begin{aligned}&(-1)^{n+1}q^n\sum_{m=0}^{\infty}E_{m,p,q}(n)\frac{t^m}{m!}+\sum_{m=0}^{\infty}E_{m,p,q}\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left([2]_q\sum_{l=0}^{n-1}(-1)^l q^l[l]_{p,q}^m\right)\frac{t^m}{m!}.\end{aligned}\tag{2.9}$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of (2.9), we have the following theorem.

Theorem 2.8. *For $n \in \mathbb{Z}_+$, we have*

$$\sum_{l=0}^{n-1}(-1)^l q^l[l]_{p,q}^m=\frac{(-1)^{n+1}q^nE_{m,p,q}(n)+E_{m,p,q}}{[2]_q}.$$

3. (p, q)-analogue of Euler zeta function

By using (p, q)-analogue of Euler numbers and polynomials, (p, q)-Euler zeta function and Hurwitz (p, q)-Euler zeta functions are defined. These functions interpolate the (p, q)-analogue of Euler numbers $E_{n,p,q}$, and polynomials $E_{n,p,q}(x)$, respectively. From (2.1), we note that

$$\begin{aligned}\frac{d^k}{dt^k}F_{p,q}(t)\bigg|_{t=0}&=[2]_q\sum_{m=0}^{\infty}(-1)^n q^m[m]_{p,q}^k\\&=E_{k,p,q},(k\in\mathbb{N}).\end{aligned}$$

By using the above equation, we are now ready to define (p, q)-Euler zeta functions.

Definition 3.1. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

$$\zeta_{p,q}(s)=[2]_q\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{[n]_{p,q}^s}.\tag{3.1}$$

Note that $\zeta_{p,q}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $p = 1, q \to 1$, then $\zeta_{p,q}(s) = \zeta_E(s)$ which is the Euler zeta functions(see [4]). Relation between $\zeta_{p,q}(s)$ and $E_{k,p,q}$ is given by the following theorem.

Theorem 3.2. *For $k \in \mathbb{N}$, we have*

$$\zeta_{p,q}(-k)=E_{k,p,q}.$$

Observe that $\zeta_{p,q}(s)$ function interpolates $E_{k,p,q}$ numbers at non-negative integers. By using (2.2), we note that

$$\frac{d^k}{dt^k}F_{p,q}(t,x)\bigg|_{t=0}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m[m+x]_{p,q}^k\tag{3.2}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,p,q}(x)\frac{t^n}{n!}\right)\Bigg|_{t=0} = E_{k,p,q}(x), \text{ for } k \in \mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the Hurwitz (p,q)-Euler zeta functions.

Definition 3.3. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $x \notin \mathbb{Z}_0^-$.

$$\zeta_{p,q}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[n+x]_{p,q}^s}. \tag{3.4}$$

Note that $\zeta_{p,q}(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $p = 1$ and $q \to 1$, then $\zeta_{p,q}(s,x) = \zeta_E(s,x)$ which is the Hurwitz Euler zeta functions(see [4, 5]). Relation between $\zeta_{p,q}(s,x)$ and $E_{k,p,q}(x)$ is given by the following theorem.

Theorem 3.4. *For $k \in \mathbb{N}$, we have*

$$\zeta_{p,q}(-k,x) = E_{k,p,q}(x).$$

Observe that $\zeta_{p,q}(-k,x)$ function interpolates $E_{k,p,q}(x)$ numbers at non-negative integers.

4. Zeros of the (p,q)-analogue of Euler polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the (p,q)-analogue of Euler polynomials $E_{n,p,q}(x)$. The (p,q)-analogue of Euler polynomials $E_{n,p,q}(x)$ can be determined explicitly. A few of them are

$$\begin{aligned}
E_{0,p,q}(x) &= 1,\\
E_{1,p,q}(x) &= -\frac{(1+q)(-p^x - p^x q^2 + q^x + pq^{1+x})}{(p-q)(1+pq)(1+q^2)},\\
E_{2,p,q}(x) &= \frac{p^{2x} + p^{1+2x}q^2 + p^{2x}q^3 + p^{1+2x}q^5 - 2p^x q^x + q^{2x} - 2p^{2+x}q^{1+x}}{(p-q)^2(1+p^2q)(1-q+q^2)(1+pq^2)}\\
&\quad - \frac{2p^x q^{3+x} - 2p^{2+x}q^{4+x} + p^2 q^{1+2x} + pq^{2+2x} + p^3 q^{3+2x}}{(p-q)^2(1+p^2q)(1-q+q^2)(1+pq^2)}.
\end{aligned}$$

Our numerical results for approximate solutions of real zeros of $E_{n,p,q}(x)$ are displayed(Tables 1, 2).

Table 1. Numbers of real and complex zeros of $E_{n,p,q}(x)$

degree n	real zeros	complex zeros
1	1	0
2	2	0
3	1	2
4	2	2
5	1	4
6	2	4
7	1	6
10	2	8
15	1	14
20	2	18
25	1	24
30	2	28
35	1	34
40	2	38
45	1	44

In Table 1, we choose $p = 1/2$ and $q = 1/10$.

Next, we calculated an approximate solution satisfying (p, q)-analogue of Euler polynomials $E_{n,p,q}(x) = 0$ for $x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $E_{n,p,q}(x) = 0, p = 1/2, q = 1/10$

degree n	x
1	0.0241325
2	−0.0706366, 0.085358
3	0.133545
4	−0.119556, 0.168612
5	0.194723
6	−0.141066, 0.21479

We investigate the beautiful zeros of the (p, q)-analogue of Euler polynomials $E_{n,p,q}(x)$ by using a computer. We plot the zeros of the (p, q)-analogue of Euler polynomials $E_{n,p,q}(x)$ for $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 40, p = 1/2$ and $q = 1/4$. In Figure 1(top-right), we choose $n = 40, p = 1/2$ and $q = 1/6$. In Figure 1(bottom-left), we choose $n = 40, p = 1/2$ and $q = 1/8$. In Figure 1(bottom-right), we choose $n = 40, p = 1/2$ and $q = 1/10$.

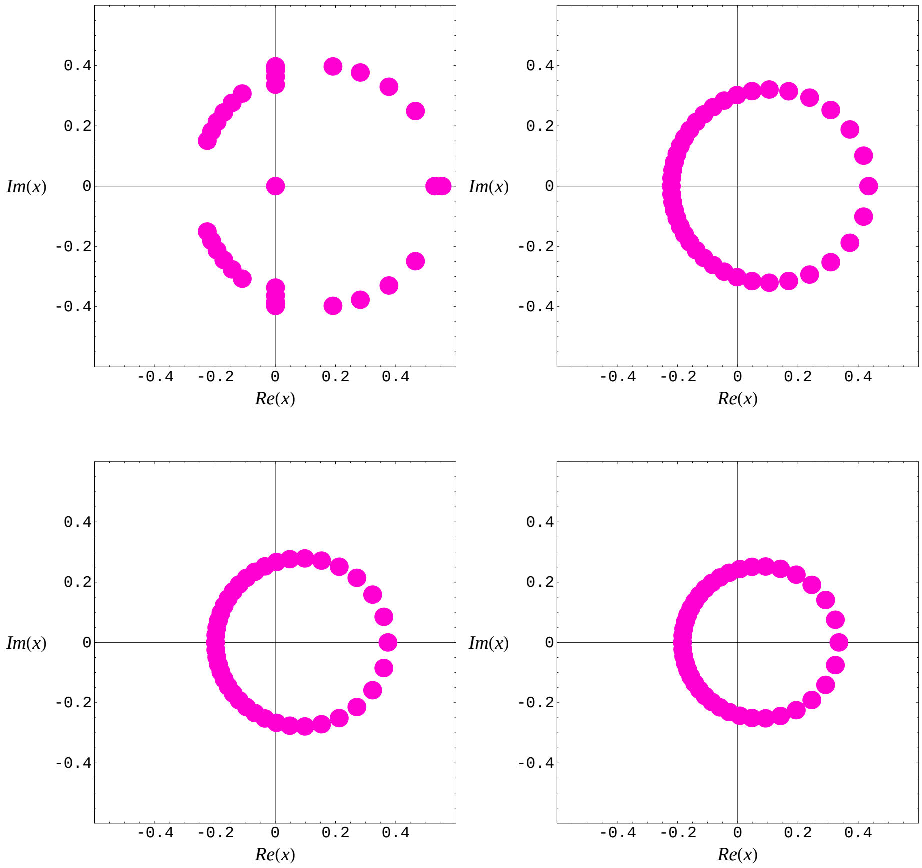

FIGURE 1. Zeros of $T_{n,p,q}^{(k)}(x)$

We observe a remarkable regular structure of the real roots of the (p,q)-analogue of Euler polynomials $E_{n,p,q}(x)$. We also hope to verify a remarkable regular structure of the real roots of the (p,q)-analogue of Euler polynomials $E_{n,p,q}(x)$(Table 1). By numerical computations, we will make a series of the following conjectures:

Conjecture 4.1. Prove that $E_{n,p,q}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions. However, $E_{n,p,q}(x)$ has not $Re(x) = a$ reflection symmetry for $a \in \mathbb{R}$.

Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n(see Figure 1). We are able to decide if $E_{n,p,q}(x)) = 0$ has n distinct solutions(see Tables 1, 2).

Conjecture 4.2. Prove that $E_{n,p,q}(x) = 0$ has n distinct solutions.

Since n is the degree of the polynomial $E_{n,p,q}(x)$, the number of real zeros $R_{E_{n,p,q}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{E_{n,p,q}(x)} = n - C_{E_{n,p,q}(x)}$, where $C_{E_{n,p,q}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,p,q}(x)}$ and $C_{E_{n,p,q}(x)}$. The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the (p, q)-analogue of Euler polynomials $E_{n,p,q}(x)$ which appear in mathematics and physics. The reader may refer to [6, 7, 8, 9, 10, 12] for the details.

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