ON BONNESEN-STYLE ALEKSANDROV-FENCHEL INEQUALITIES IN $\mathbb{R}^n$

Chunna Zeng

ABSTRACT. In this paper, we investigate the Bonnesen-style Aleksandrov-Fenchel inequalities in $\mathbb{R}^n$, which are the generalization of known Bonnesen-style inequalities. We first define the i-th symmetric mixed homothetic deficit $\Delta_i(K, L)$ and its special case, the i-th Aleksandrov-Fenchel isoperimetric deficit $\Delta_i(K)$. Secondly, we obtain some lower bounds of $(n-1)$-th Aleksandrov Fenchel isoperimetric deficit $\Delta_{n-1}(K)$. Theorem 4 strengthens Groemer's result. As direct consequences, the stronger isoperimetric inequalities are established when $n = 2$ and $n = 3$. Finally, the reverse Bonnesen-style Aleksandrov-Fenchel inequalities are obtained. As a consequence, the new reverse Bonnesen-style inequality is obtained.

1. Introduction

The geometric inequality describes the relation among the invariants of a geometric object in space. Perhaps the classical isoperimetric inequality is the best known geometric inequality. It states that: for a simple closed curve Γ of length L in the Euclidean plane $\mathbb{R}^2$, the area A enclosed by Γ satisfies

$$(1.1) \qquad L^2 - 4\pi A \geq 0.$$

The equality holds if and only if Γ is a circle. It follows that the circle encloses the maximum area among all curves of the same length.

One can find many simplified and beautiful proofs that lead to generalizations of higher dimensions (see [1, 2, 3, 6, 8, 24]), and applications to other branches of mathematics (see [12, 17, 21, 23, 33, 42, 46]).

During the 1920's, Bonnesen initiated a series of inequalities of the following type:

$$(1.2) \qquad \Delta(K) = L^2 - 4\pi A \geq B_K,$$

where the quantity B_K is an invariant of geometric significance with the following basic properties:

  1. B_K is non-negative;
  2. B_K vanishes only when K is a disc.

An equality of the type (1.2) is called the Bonnesen-style inequality. The quantity $\Delta(K) = L^2 - 4\pi A$ is called the isoperimetric deficit of K, and it measures the deficit between a domain K and a disc. Bonnesen proved several inequalities of the form (1.2) in the Euclidean plane (see [4, 5]), but he did not obtain direct generalizations of his two-dimensional results. This was done much later, first by Hadwiger [21] for $n = 3$, and then by Dinghas [7] for arbitrary dimension. Although it is a hard work to obtain some Bonnesen-style inequalities in higher dimensional space, mathematicians are still working on finding unknown invariants of geometric significance (see [9, 10, 18, 20, 22, 25, 31, 32, 35, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48]).

The following Bonnesen-style isoperimetric inequalities are known: Let K be a plane domain with rectifiable boundary ∂K of area A and bounded by a simple closed curve of length L. Denote by r_i and r_e, respectively, the radius of the maximum inscribed circle and the radius of the minimum circumscribed circle of K. Then for $r_i \leq t \leq r_e$,

$$\pi^2 t^2 - Lt + A \leq 0;$$

$$\frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \leq r_i \leq \frac{L}{2\pi} \leq r_e \leq \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi};$$

and

$$L^2 - 4\pi A \geq \pi^2 (r_e - r_i)^2; \qquad L^2 - 4\pi A \geq L^2 \left(\frac{r_e - r_i}{r_e + r_i} \right)^2;$$

$$L^2 - 4\pi A \geq A^2 \left(\frac{1}{r_i} - \frac{1}{r_e} \right)^2; \quad L^2 - 4\pi A \geq L^2 \left(\frac{r - r_i}{r + r_i} \right)^2;$$

$$L^2 - 4\pi A \geq A^2 \left(\frac{1}{r} - \frac{1}{r_e} \right)^2; \quad L^2 - 4\pi A \geq L^2 \left(\frac{r_e - r}{r_e + r} \right)^2.$$

Each equality holds if and only if K is a disc (see [36, 37, 45, 46, 48]). In [39], Zhang established some different forms of Bonnesen-style inequalities associated with mean width of convex body K in $\mathbb{R}^n$,

$$\left(\frac{\bar{\omega}(K)}{2} \right)^{n/(n-1)} - \left(\frac{V(K)}{\kappa_n} \right)^{1/(n-1)}$$

$$\geq \left(\frac{V(K)}{\kappa_n} \right)^{n/(n-1)} \left(\left(\frac{V(K)}{\kappa_n} \right)^{-1/n} - R^{-1} \right),$$

where $\bar{\omega}(K)$ and R are, respectively, the mean width and outradius of K.

The generalization of (1.1) in $\mathbb{R}^n$ shows that, of all domains K with given surface area S, the maximum volume V is attained by the sphere translates

into the isoperimetric inequality (see [26, 27])

$$(1.3) \qquad S^n - n^n \omega_n V^{n-1} \geq 0,$$

where the equality holds if and only if K is a ball, and ω_n is the volume of the unit ball, that is

$$\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)},$$

where $\Gamma(\cdot)$ is the Gamma function.

The classical Aleksandrov-Fenchel inequality is more general than the isoperimetric inequality (1.3) in $\mathbb{R}^n$ (see [14, 15, 16, 34]): Let $K_1, K_2, \ldots, K_n$ be compact convex sets in $\mathbb{R}^n$,

$$(1.4)$$
$$V(K_1, K_2, K_3, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, \ldots, K_n)V(K_2, K_2, K_3, \ldots, K_n).$$

The complete equality condition for the Aleksandrov-Fenchel inequality is unknown. However we have: if $K_3, \ldots, K_n$ are smooth convex bodies, then the equality (1.4) holds if and only if K_1 and K_2 are homothetic.

Since the mixed volume $V(K_1, K_2, \ldots, K_n)$ is symmetric, it is clear that from (1.4) we have

$$(1.5) \qquad V_i^2(K, L) - V_{i-1}(K, L)V_{i+1}(K, L) \geq 0, \quad 1 \leq i \leq n-1.$$

If K, L are smooth convex bodies, then (1.5) holds if and only if K and L are homothetic, and

$$V_i(K, L) = V(\underbrace{K, \ldots, K}_{n-i}, \underbrace{L, \ldots, L}_{i}),$$

where K appears $n-i$ times and L appears i times.

Let L be the unit ball B in (1.5), then we have

$$(1.6) \qquad W_i^2(K) - W_{i-1}(K)W_{i+1}(K) \geq 0, \quad 1 \leq i \leq n-1.$$

If K is a smooth convex body, the equality holds if and only if K is a ball. Let $i = 1$ in the Euclidean plane $\mathbb{R}^2$, then (1.6) is reduced to the classical isoperimetric inequality (1.1).

In this paper, motivated by Bonnesen's work, we define the *i-th symmetric mixed homothetic deficit* of smooth convex bodies K and L as

$$(1.7) \qquad \Delta_i(K, L) = V_i^2(K, L) - V_{i-1}(K, L)V_{i+1}(K, L), \quad 1 \leq i \leq n-1.$$

The symmetric mixed homothetic deficit $\Delta_i(K, L)$ measures the homothetic between K and L. Then one would ask if there is a non-negative invariant $B_{K,L}$ of geometric significance such that

$$(1.8) \quad \Delta_i(K, L) = V_i^2(K, L) - V_{i-1}(K, L)V_{i+1}(K, L) \geq B_{K,L}, \quad 1 \leq i \leq n-1,$$

and the quantity $B_{K,L}$ vanishes only when K and L are homothetic. An inequality of the form (1.8) can be called *the Bonnesen-style symmetric Aleksandrov-Fenchel inequality.*

In particular, the quantity

(1.9) $\Delta_i(K) = W_i^2(K) - W_{i-1}(K)W_{i+1}(K), \quad 1 \le i \le n-1,$

is defined as the *i-th Aleksandrov-Fenchel isoperimetric deficit* of smooth convex body K. Then a *Bonnesen-style Aleksandrov-Fenchel inequality* can be of the form

(1.10) $\Delta_i(K) = W_i^2(K) - W_{i-1}(K)W_{i+1}(K) \ge B_K, \quad 1 \le i \le n-1,$

where B_K is a nonnegative invariant and vanishes if K is a ball in $\mathbb{R}^n$.

Note that the Aleksandrov-Fenchel isoperimetric deficit $\Delta_1(K)$ and the isoperimetric deficit $\Delta(K)$ in the Euclidean plane $\mathbb{R}^2$ have no essential difference except for a constant, then the Bonnesen-style Aleksandrov-Fenchel inequality (1.10) is more general than Bonnesen-style isoperimetric inequality (1.2). Furthermore, many lower bounds of $\Delta_1(K, L)$ in $\mathbb{R}^2$ were found by Blaschke, Flanders and Zhou (see [2, 11]).

Groemer [18] obtained a lower bound of $\Delta_{n-1}(K)$ by Fourier series and spherical harmonics. Let K be a smooth convex body in $\mathbb{R}^n$ and $W_i(K)$ the i-th quermassintegral of K. Denote by K_u the orthogonal projection of K onto the linear subspace $\langle u \rangle^\perp$. Denote by $\bar{w}_{n-1}(K_u)$ the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ the maximum and minimum of $\bar{w}_{n-1}(K_u)$ over all $u \in S^{n-1}$, respectively. Then

(1.11)
$$\Delta_{n-1}(K) = W_{n-1}^2(K) - \kappa_n W_{n-2}(K)$$
$$\ge W_{n-2}^2(K) \left[\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}_{\max}(K)} \right]^2,$$

where the equality holds if and only if K is a ball. For higher dimensional case, the lower bounds of $\Delta_1(K, L)$ are still unknown except for a few inequalities (see [19]). Groemer [19] gave a lower bound of $\Delta_1(K, L)$ of K and L as follows:

(1.12)
$$\Delta_1(K, L) = V_1^2(K, L) - V_0(K, L)V_2(K, L)$$
$$\ge \frac{V(L)^2}{4} \Big(R(K, L) - r(K, L) \Big)^2,$$

where $r(K, L)$ and $R(K, L)$ are, respectively, the relative inradius and outradius of K with respect to L, defined by

$$r(K, L) = \sup\{\lambda : x \in \mathbb{R}^n \text{ and } x + \lambda L \subset K\},$$
$$R(K, L) = \inf\{\lambda : x \in \mathbb{R}^n \text{ and } K \subset x + \lambda L\},$$

and $r(K, L)R(L, K) = 1$.

Bonnesen-style inequalities are obtained through various approaches. Bonnesen, Ren and Zhou obtained Bonnesen-style inequalities by kinematic formulas and the containment measure in integral geometry (see [30, 33, 36, 37, 43, 44, 45, 46, 47, 48]). Other researchers obtained Bonnesen-style inequalities by using the approaches in differential geometry and analysis. In this paper, motivated by Bonnesen's work, we mainly investigate Bonnesen-style

Aleksandrov-Fenchel inequality by means of convex geometry analysis. Some B_K's are obtained. The obtained Bonnesen-style Aleksandrov-Fenchel inequalities are the generalization of Bonnesen-style isoperimetric inequalities. When $n = 2$ and $n = 3$, the stronger isoperimetric inequalities are established. Finally, the reverse Bonnesen-style Aleksandrov-Fenchel inequalities are obtained.

2. Preliminaries

A subset K in the Euclidean space $\mathbb{R}^n$ is *convex* if for any $x, y \in K$ and $0 \le \lambda \le 1$, $\lambda x + (1 - \lambda)y \in K$. A *domain* is a set with nonempty interiors. A *convex body* is a compact convex domain. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$ and $\mathcal{K}_o^n$ if the convex body contains the origin in their interiors. The *Minkowski sum* of convex sets $K_1, K_2, \ldots, K_m$ in $\mathbb{R}^n$, and the *Minkowski scalar product* of convex set K in $\mathbb{R}^n$ for $\lambda \ge 0$, are, respectively, defined by

(2.1) $$K_1 + \cdots + K_m = \{x_1 + \cdots + x_m : x_1 \in K_1, \ldots, \ x_m \in K_m\},$$

and

(2.2) $$\lambda K = \{\lambda x : x \in K\}.$$

A *homothety* of a convex set K is of the form $y + \lambda K$ for $y \in \mathbb{R}^n$, $\lambda > 0$. Let K_1 and K_2 be two convex domains in $\mathbb{R}^n$. If there exist $y \in \mathbb{R}^n$ and $t > 0$, such that $K_1 = y + tK_2$ or $K_2 = y + tK_1$, then K_1 and K_2 are homothetic.

The *support function* of a convex domain K in $\mathbb{R}^n$ is defined by

(2.3) $$h_K(x) = \max\{\langle x, y \rangle : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ is the ordinary standard inner product in $\mathbb{R}^n$ (see [34]). A convex domain in $\mathbb{R}^n$ is uniquely determined by its support function.

Let $K_1, \ldots, K_m$ be convex domains in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m \ge 0$. Then the volume of $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_m$:

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \ldots, i_n = 1}^{m} V(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $V(K_{i_1}, \ldots, K_{i_n})$ are nonnegative, symmetric in the indices, and dependent only on $K_{i_1}, \ldots, K_{i_n}$. $V(K_{i_1}, \ldots, K_{i_n})$ is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$.

If K is a compact convex set and B is the unit ball in $\mathbb{R}^n$, there is the Steiner formula,

$$V(K + \lambda B) = \sum_{i=0}^{n} \binom{n}{i} W_i(K) \lambda^i,$$

where

$$W_i(K) = V_i(K, B), i = 0, 1, \ldots, n$$

is called the *i-th quermassintegral* of K, and $W_0(K) = V(K)$, $W_n(K) = O_{n-1}/n = \kappa_n$, O_{n-1} denotes the surface of unit n-ball.

It is clear that from the Aleksandrov-Fenchel inequality many other inequalities can be deduced by repeated application. Schneider derived an improved version from the Aleksandrov-Fenchel inequality as follows (see [34, Chapter 7]):

Lemma 1. *Let $K, L, M, K_3, \ldots, K_n \in \mathcal{K}^n$ and $\mathcal{L} := (K_3, \ldots, K_n)$, and suppose that*

$$(2.4) \qquad V(K, M, \mathcal{L}) > 0, \ V(L, M, \mathcal{L}) > 0, \ V(M, M, \mathcal{L}) > 0.$$

Then

$$(2.5) \qquad V(K, L, \mathcal{L})^2 \geq V(K, K, \mathcal{L}) V(L, L, \mathcal{L}),$$

and

$$(2.6) \qquad \frac{V(K, K, \mathcal{L})}{V(K, M, \mathcal{L})^2} - \frac{2V(K, L, \mathcal{L})}{V(K, M, \mathcal{L}) V(L, M, \mathcal{L})} + \frac{V(L, L, \mathcal{L})}{V(L, M, \mathcal{L})^2} \leq 0.$$

The following assertions are equivalent:
  (1) *equality in* (2.5);
  (2) *equality in* (2.6).

3. The Bonnesen-style Aleksandrov-Fenchel inequalities

Let $K_3 = \cdots = K_n = B$ in (2.6), and we write $V(K, L) = V(K, L, B, \ldots, B)$. So (2.6) is rewritten in the following form:
$$(3.1)$$
$$2V(K, M)V(L, M)V(K, L) - V(L, M)^2 W_{n-2}(K) - V(K, M)^2 W_{n-2}(L) \geq 0.$$

When $L = B$ and M is a line segment of unit length, it is easily established ([5], p. 49 and [18]) that in this case

$$(3.2) \qquad V(K, M) = W'_{n-2}(K_u),$$

where $u \in S^{n-1}$ has the same direction as M, K_u denotes the orthogonal projection of K onto the linear subspace $\langle u \rangle^{\perp}$, and $W'_{n-2}(K_u)$ is the quermassintegral of K_u in $\mathbb{R}^{n-1}$.

If a convex body $K \in \mathcal{K}^n$ is contained in some hyperplane H, then there are two possibilities for defining its mean width. One can either disregard the fact that $K \subset H$ and define $\bar{w}(K)$ in the same way as it is defined for all convex bodies of $\mathcal{K}^n$, or one can consider H as the underlying Euclidean space. In the latter case K is viewed as a convex body in $\mathbb{R}^{n-1}$ and its mean width is defined accordingly. In this case, to avoid any confusion, the mean width of K is denoted by $\bar{w}_{n-1}(K)$. Then for any $K \subset H$,

$$\frac{O_{n-1}}{\kappa_{n-2}} \bar{w}_{n-1}(K) = \frac{O_n}{\kappa_{n-1}} \bar{w}(K).$$

By (3.2), it follows that

$$(3.3) \qquad W'_{n-2}(K_u) = \frac{\kappa_{n-1}}{2} \bar{w}_{n-1}(K_u).$$

Combining (3.2), (3.3) and (3.1), we come to:

Theorem 1. *Let K be a smooth convex body in $\mathbb{R}^n$ and $W_i(K)$ the i-th quermassintegral of K. Then*

$$(3.4) \qquad \frac{\kappa_n}{4}\bar{w}_{n-1}(K_u)^2 - W_{n-1}(K)\bar{w}_{n-1}(K_u) + W_{n-2}(K) \leq 0,$$

where K_u is the orthogonal projection of K onto the linear subspace $\langle u \rangle^{\perp}$, $\bar{w}_{n-1}(K_u)$ denotes the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ denote, respectively, the maximum and minimum of $\bar{w}_{n-1}(K_u)$ over all $u \in S^{n-1}$. The equality holds if and if K is a ball.

The above Bonnesens inequality can be rewritten in several equivalent forms:

$$(3.5) \qquad W_{n-1}(K)\bar{w}_{n-1}(K_u) \geq W_{n-2}(K) + \frac{\kappa_n}{4}\bar{w}_{n-1}(K_u)^2;$$

$$(3.6) \qquad \Delta_{n-1}(K) = W_{n-1}^2(K) - \kappa_n W_{n-2}(K)$$

$$\geq \left(W_{n-1}(K) - \frac{\kappa_n}{2}\bar{w}_{n-1}(K_u)\right)^2;$$

$$(3.7) \qquad \Delta_{n-1}(K) = W_{n-1}^2(K) - \kappa_n W_{n-2}(K)$$

$$\geq \left(W_{n-1}(K) - \frac{2W_{n-2}(K)}{\bar{w}_{n-1}(K_u)}\right)^2;$$

$$(3.8) \qquad \Delta_{n-1}(K) = W_{n-1}^2(K) - \kappa_n W_{n-2}(K)$$

$$\geq \left(\frac{W_{n-2}(K)}{\bar{w}_{n-1}(K_u)} - \frac{\kappa_n \bar{w}_{n-1}(K_u)}{4}\right)^2.$$

We have the following Bonnesen-style Aleksandrov-Fenchel inequalities:

Theorem 2. *Let K be a smooth convex body in $\mathbb{R}^n$ and $W_i(K)$ the i-th quermassintegral of K. Then*

$$\Delta_{n-1}(K) \geq W_{n-2}^2(K)\left(\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}_{\max}(K)}\right)^2,$$

$$\Delta_{n-1}(K) \geq W_{n-1}^2(K)\left(\frac{\bar{w}_{\max}(K) - \bar{w}_{\min}(K)}{\bar{w}_{\max}(K) + \bar{w}_{\min}(K)}\right)^2,$$

where K_u is the orthogonal projection of K onto the linear subspace $\langle u \rangle^{\perp}$, $\bar{w}_{n-1}(K_u)$ denotes the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ denote, respectively, the maximum and minimum of $\bar{w}_{n-1}(K_u)$ over all $u \in S^{n-1}$. Each equality holds if and only if K is a ball.

Proof. By (3.7), it follows that

$$(3.9) \qquad \sqrt{\Delta_{n-1}(K)} \geq W_{n-1}(K) - \frac{2W_{n-2}(K)}{\bar{w}_{\max}(K)},$$

$$(3.10) \qquad \sqrt{\Delta_{n-1}(K)} \geq \frac{2W_{n-2}(K)}{\bar{w}_{\min}(K)} - W_{n-1}(K).$$

By $x^2 + y^2 \geq \frac{(x+y)^2}{2}$, adding (3.9) and (3.10) yields

$$(3.11) \qquad \Delta_{n-1}(K) \geq W_{n-2}^2(K)\Big(\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}_{\max}(K)}\Big)^2.$$

Inequalities (3.9) and (3.10) can be rewritten as

$$(3.12) \qquad \bar{w}_{\max}(K)\sqrt{\Delta_{n-1}(K)} \geq \bar{w}_{\max}(K)W_{n-1}(K) - 2W_{n-2}(K),$$

$$(3.13) \qquad \bar{w}_{\min}(K)\sqrt{\Delta_{n-1}(K)} \geq 2W_{n-2}(K) - \bar{w}_{\min}(K)W_{n-1}(K).$$

Then adding (3.12) and (3.13) side by side leads to

$$(3.14) \quad \big(\bar{w}_{\max}(K) + \bar{w}_{\min}(K)\big)\sqrt{\Delta_{n-1}(K)} \geq W_{n-1}(K)\big(\bar{w}_{\max}(K) - \bar{w}_{\min}(K)\big).$$

Squaring the inequality, we obtain that

$$(3.15) \qquad \Delta_{n-1}(K) \geq W_{n-1}^2(K)\Big(\frac{\bar{w}_{\max}(K) - \bar{w}_{\min}(K)}{\bar{w}_{\max}(K) + \bar{w}_{\min}(K)}\Big)^2. \qquad \square$$

Similarly, we also obtain the following theorem:

Theorem 3. *Let K be a smooth convex body in $\mathbb{R}^n$ and $W_i(K)$ the i-th quermassintegral of K. Then*

$$\Delta_{n-1}(K) \geq W_{n-2}^2(K)\Big(\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}_{n-1}(K_u)}\Big)^2;$$

$$\Delta_{n-1}(K) \geq W_{n-2}^2(K)\Big(\frac{1}{\bar{w}_{\max}(K)} - \frac{1}{\bar{w}_{n-1}(K_u)}\Big)^2;$$

$$\Delta_{n-1}(K) \geq W_{n-1}^2(K)\Big(\frac{\bar{w}_{n-1}(K_u) - \bar{w}_{\min}(K)}{\bar{w}_{n-1}(K_u) + \bar{w}_{\min}(K)}\Big)^2;$$

$$\Delta_{n-1}(K) \geq W_{n-1}^2(K)\Big(\frac{\bar{w}_{\max}(K) - \bar{w}_{n-1}(K_u)}{\bar{w}_{\max}(K) + \bar{w}_{n-1}(K_u)}\Big)^2;$$

where K_u is the orthogonal projection of K onto the linear subspace $\langle u \rangle^\perp$, $\bar{w}_{n-1}(K_u)$ denotes the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ denote, respectively, the maximum and minimum of $\bar{w}_{n-1}(K_u)$ over all $u \in S^{n-1}$. Each equality holds if and only if K is a ball.

Proof. By (3.7), it follows that

$$(3.16) \qquad \sqrt{\Delta_{n-1}(K)} \geq W_{n-1}(K) - \frac{2W_{n-2}(K)}{\bar{w}_{n-1}(K_u)}.$$

By $x^2 + y^2 \geq \frac{(x+y)^2}{2}$, adding (3.16) and (3.10) yields

$$(3.17) \qquad \Delta_{n-1}(K) \geq W_{n-2}^2(K)\Big(\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}_{n-1}(K_u)}\Big)^2.$$

(3.16) can be rewritten as

$$(3.18) \qquad \bar{w}_{n-1}(K_u)\sqrt{\Delta_{n-1}(K)} \geq \bar{w}_{n-1}(K_u)W_{n-1}(K) - 2W_{n-2}(K).$$

Then adding (3.18) and (3.13) side by side leads to
(3.19)
$$\big(\bar{w}_{n-1}(K_u) + \bar{w}_{\min}(K)\big)\sqrt{\Delta_{n-1}(K)} \geq W_{n-1}(K)\big(\bar{w}_{n-1}(K_u) - \bar{w}_{\min}(K)\big).$$

Squaring the inequality, we obtain that

(3.20)
$$\Delta_{n-1}(K) \geq W_{n-1}^2(K)\Big(\frac{\bar{w}_{n-1}(K_u) - \bar{w}_{\min}(K)}{\bar{w}_{n-1}(K_u) + \bar{w}_{\min}(K)}\Big)^2.$$

Similar with (3.17) and (3.20), we have

$$\Delta_{n-1}(K) \geq W_{n-2}^2(K)\Big(\frac{1}{\bar{w}_{\max}(K)} - \frac{1}{\bar{w}_{n-1}(K_u)}\Big)^2,$$

and

$$\Delta_{n-1}(K) \geq W_{n-1}^2(K)\Big(\frac{\bar{w}_{\max}(K) - \bar{w}_{n-1}(K_u)}{\bar{w}_{\max}(K) + \bar{w}_{n-1}(K_u)}\Big)^2. \qquad \square$$

Next, we consider two corollaries in the case that $n = 2$ and $n = 3$. Let K be a smooth convex body of length L and area A in the Euclidean plane $\mathbb{R}^2$, then $W_0(K) = A, W_1(K) = \frac{L}{2}, W_2(K) = \pi$. Hence

(3.21)
$$\Delta_1(K) = W_1^2(K) - \kappa_2 W_0(K) = \frac{L^2 - 4\pi A}{4}.$$

So the following Bonnesen style inequalities are direct consequences of Theorem 2 and Theorem 3.

Corollary 1. *Let K be a convex body and the boundary ∂K be C^2 in $\mathbb{R}^2$. Denote L and A the length and the area of K, respectively. Then*

$$L^2 - 4\pi A \geq L^2 \Big(\frac{\bar{w}_{\max}(K) - \bar{w}(K_u)}{\bar{w}_{\max}(K) + \bar{w}(K_u)}\Big)^2;$$

$$L^2 - 4\pi A \geq 4A^2 \Big(\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}_{\max}(K)}\Big)^2;$$

$$L^2 - 4\pi A \geq 4A^2 \Big(\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}(K_u)}\Big)^2;$$

$$L^2 - 4\pi A \geq 4A^2 \Big(\frac{1}{\bar{w}_{\max}(K)} - \frac{1}{\bar{w}(K_u)}\Big)^2;$$

$$L^2 - 4\pi A \geq L^2 \Big(\frac{\bar{w}_{\max}(K) - \bar{w}_{\min}(K)}{\bar{w}_{\max}(K) + \bar{w}_{\min}(K)}\Big)^2;$$

$$L^2 - 4\pi A \geq L^2 \Big(\frac{\bar{w}(K_u) - \bar{w}_{\min}(K)}{\bar{w}(K_u) + \bar{w}_{\min}(K)}\Big)^2;$$

where K_u is the orthogonal projection of K onto the linear subspace $\langle u \rangle^{\perp}$, $\bar{w}(K_u)$ denotes the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ denote, respectively, the maximum and minimum of $\bar{w}(K_u)$ over all $u \in S^{n-1}$. Each equality holds if and only if K is a disc.

Let K be a smooth convex body of surface area A and volume K in the Euclidean space $\mathbb{R}^3$, then $W_0(K) = V, W_1(K) = \frac{A}{3}, W_3(K) = \frac{4}{3}\pi$.

Following (1.6), we have

$$\Delta_1(K) = W_1^2(K) - W_0(K)W_2(K) \geq 0 \tag{3.22}$$

and

$$\Delta_2(K) = W_2^2(K) - W_1(K)W_3(K) \geq 0, \tag{3.23}$$

and (3.22) is equivalent to

$$W_2(K) \leq \frac{A^2}{9V}, \tag{3.24}$$

that is

$$W_2^2(K) \leq \frac{A^4}{81V^2}. \tag{3.25}$$

Hence,

$$\Delta_2(K) = W_2^2(K) - \frac{4\pi A}{9} \leq \frac{A^4}{81V^2} - \frac{4\pi A}{9}. \tag{3.26}$$

Combining (3.6) and (3.26) together gives

$$\frac{A^4}{81V^2} - \frac{4\pi A}{9} \geq \Delta_2(K) = W_2^2(K) - \frac{4\pi A}{9} \geq \left(W_2(K) - \frac{2\pi}{3}\bar{w}_2(K_u)\right)^2, \tag{3.27}$$

that is

$$A^3 - 36\pi V^2 \geq \frac{81V^2}{A}\left(W_2(K) - \frac{2\pi}{3}\bar{w}_2(K_u)\right)^2. \tag{3.28}$$

Similarly, combining the equalities in Theorem 2 and Theorem 3 with (3.26), respectively, we have the following Bonnesen-style inequalities in the Euclidean space $\mathbb{R}^3$ (Corollary 2 and Corollary 3).

Corollary 2. *Let K be a convex body and the boundary ∂K be C^2 in Euclidean space $\mathbb{R}^3$. Denote A and V the surface area and the volume of K, respectively. Then*

$$A^3 - 36\pi V^2 \geq \frac{81V^2}{A}\left(W_2(K) - \frac{2\pi}{3}\bar{w}_2(K_u)\right)^2;$$

$$A^3 - 36\pi V^2 \geq \frac{81V^2}{A}\left(\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}_{\max}(K)}\right)^2;$$

$$A^3 - 36\pi V^2 \geq \frac{81V^2}{A}W_2(K)^2\left(\frac{\bar{w}_{\max}(K) - \bar{w}_{\min}(K)}{\bar{w}_{\max}(K) + \bar{w}_{\min}(K)}\right)^2;$$

where $W_2(K)$ is the 2-th quermassintegral of K and K_u is the orthogonal projection of K onto the linear subspace $\langle u \rangle^\perp$, $\bar{w}(K_u)$ denotes the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ denote, respectively, the maximum and minimum of $\bar{w}(K_u)$ over all $u \in S^{n-1}$. Each equality holds if and only if K is a ball.

Corollary 3. *Let K be a convex body and the boundary ∂K be C^2 in Euclidean space $\mathbb{R}^3$. Denote A and V the surface area and the volume of K, respectively. Then*

$$A^3 - 36\pi V^2 \geq 9AV^2 \Big(\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}(K_u)} \Big)^2;$$

$$A^3 - 36\pi V^2 \geq 9AV^2 \Big(\frac{1}{\bar{w}_{\max}(K)} - \frac{1}{\bar{w}(K_u)} \Big)^2;$$

$$A^3 - 36\pi V^2 \geq \frac{81V^2}{A} W_2(K)^2 \Big(\frac{\bar{w}(K_u) - \bar{w}_{\min}(K)}{\bar{w}(K_u) + \bar{w}_{\min}(K)} \Big)^2;$$

$$A^3 - 36\pi V^2 \geq \frac{81V^2}{A} W_2(K)^2 \Big(\frac{\bar{w}_{\max}(K) - \bar{w}(K_u)}{\bar{w}_{\max}(K) + \bar{w}(K_u)} \Big)^2;$$

where $W_2(K)$ is the 2-th quermassintegral of K and K_u is the orthogonal projection of K onto the linear subspace $\langle u \rangle^\perp$, $\bar{w}(K_u)$ denotes the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ denote, respectively, the maximum and minimum of $\bar{w}(K_u)$ over all $u \in S^{n-1}$. Each equality holds if and only if K is a ball.

It is a remarkable fact that only a few Bonnesen-style inequalities for convex bodies in $\mathbb{R}^3$ are known. Higher-dimensional cases are more complicated. In the above we have obtained many lower bounds of Aleksandrov-Fenchel isoperimetric deficit. It is interesting and difficult to compare those lower bounds and to determine which is the best. The following theorem strengthens Gromer's result (the inequation (1.11)).

Theorem 4. *Let K be a smooth convex body in $\mathbb{R}^n$ and $W_i(K)$ the i-th quermassintegral of K. Then*
(3.29)
$$\Delta_{n-1}(K) \geq \frac{\kappa_n^2}{16} \big[\bar{w}_{\max}(K) - \bar{w}_{\min}(K) \big]^2 + \Big[\frac{\kappa_n}{4} \big(\bar{w}_{\max}(K) + \bar{w}_{\min}(K) \big) - W_{n-1}(K) \Big]^2,$$

(3.30)
$$\Delta_{n-1}(K) \geq W_{n-2}^2(K) \left[\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}_{\max}(K)} \right]^2 + \left[\frac{W_{n-2}(K)}{\bar{w}_{\min}(K)} + \frac{W_{n-2}(K)}{\bar{w}_{\max}(K)} \right.$$

$$\left. - W_{n-1}(K) \right]^2,$$

where K_u is the orthogonal projection of K onto the linear subspace $\langle u \rangle^\perp$, $\bar{w}_{n-1}(K_u)$ denotes the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ denote, respectively, the maximum and minimum of $\bar{w}_{n-1}(K_u)$ over all $u \in S^{n-1}$. Each equality holds if and only if K is a ball.

Proof. By (3.4), we have

$$-W_{n-2}(K) \geq \frac{\kappa_n}{4} \bar{w}_{\min}(K)^2 - W_{n-1}(K) \bar{w}_{\min}(K)$$

and

$$-W_{n-2}(K) \geq \frac{\kappa_n}{4}\bar{w}_{\max}(K)^2 - W_{n-1}(K)\bar{w}_{\max}(K).$$

That is,

$$W_{n-1}^2(K) - \kappa_n^2 W_{n-2}(K) \geq \frac{\kappa_n}{4}\bar{w}_{\min}(K)^2 - \kappa_n W_{n-1}(K)\bar{w}_{\min}(K) + W_{n-1}^2(K),$$

$$W_{n-1}^2(K) - \kappa_n^2 W_{n-2}(K) \geq \frac{\kappa_n}{4}\bar{w}_{\max}(K)^2 - \kappa_n W_{n-1}(K)\bar{w}_{\max}(K) + W_{n-1}^2(K).$$

Adding the above inequalities side by side, we obtain

$$
\begin{aligned}
(3.31) \quad & W_{n-1}^2(K) - \kappa_n^2 W_{n-2}(K) \\
& \geq \frac{\kappa_n^2}{8}\big[\bar{w}_{\max}(K)^2 + \bar{w}_{\min}(K)^2\big] - \frac{\kappa_n}{2}W_{n-1}(K)\big[\bar{w}_{\min}(K) + \bar{w}_{\max}(K)\big] \\
& \quad + W_{n-1}^2(K) \\
& = \frac{\kappa_n^2}{8}\big[\bar{w}_{\max}(K)^2 - 2\bar{w}_{\min}(K)\bar{w}_{\max}(K) + \bar{w}_{\min}(K)^2\big] \\
& \quad + \frac{\kappa_n^2}{2}\bar{w}_{\max}(K)\bar{w}_{\min}(K) - \frac{\kappa_n}{2}W_{n-1}(K)\big[\bar{w}_{\min}(K) + \bar{w}_{\max}(K)\big] \\
& \quad + W_{n-1}^2(K),
\end{aligned}
$$

that is,

$$
\begin{aligned}
& W_{n-1}^2(K) - \kappa_n^2 W_{n-2}(K) \\
\geq & \frac{\kappa_n^2}{8}\big[\bar{w}_{\max}(K) - \bar{w}_{\min}(K)\big]^2 + \frac{\kappa_n^2}{4}\bar{w}_{\min}(K)\bar{w}_{\max}(K) - \frac{\kappa_n}{2}W_{n-1}(K)\cdot \\
& \big[\bar{w}_{\min}(K) + \bar{w}_{\max}(K)\big] + W_{n-1}^2(K) \\
= & \frac{\kappa_n^2}{16}\big[\bar{w}_{\max}(K) - \bar{w}_{\min}(K)\big]^2 + \frac{\kappa_n^2}{16}\big[\bar{w}_{\max}(K) - \bar{w}_{\min}(K)\big]^2 + \frac{\kappa_n^2}{4}\bar{w}_{\min}(K)\cdot \\
& \bar{w}_{\max}(K) - \frac{\kappa_n}{2}W_{n-1}(K)\big[\bar{w}_{\min}(K) + \bar{w}_{\max}(K)\big] + W_{n-1}^2(K) \\
= & \frac{\kappa_n^2}{16}\big[\bar{w}_{\max}(K) - \bar{w}_{\min}(K)\big]^2 + \frac{\kappa_n^2}{16}\big[\bar{w}_{\max}(K) + \bar{w}_{\min}(K)\big]^2 - \frac{\kappa_n}{2}W_{n-1}(K)\cdot \\
& \big[\bar{w}_{\min}(K) + \bar{w}_{\max}(K)\big] + W_{n-1}^2(K) \\
= & \frac{\kappa_n^2}{16}\big[\bar{w}_{\max}(K) - \bar{w}_{\min}(K)\big]^2 + \big[\frac{\kappa_n}{4}\big(\bar{w}_{\max}(K) + \bar{w}_{\min}(K)\big) \\
& - W_{n-1}(K)\big]^2.
\end{aligned}
$$

Thus we complete the proof of (3.29).

On the other hand, let $t = \bar{w}_{n-1}(K_u), t_m = \bar{w}_{\min}(K), t_M = \bar{w}_{\max}(K)$ and $r = \frac{1}{t}$, then (3.4) can be rewritten as

$$(3.32) \qquad W_{n-2}(K)r^2 - W_{n-1}(K)r + \frac{\kappa_n}{4} \leq 0.$$

By (3.32), and similarly as (3.29), we have

$$\Delta_{n-1}(K) \geq W_{n-2}^2(K)\Big[\frac{1}{t_m} - \frac{1}{t_M}\Big]^2 + \Big[\frac{W_{n-2}(K)}{t_m} + \frac{W_{n-2}(K)}{t_M} - W_{n-1}(K)\Big]^2.$$

So we complete the proof of Theorem 4. $\qquad\qquad\qquad\qquad\qquad\square$

In particular, when $n = 2$, $\Delta_1(K) = W_1^2(K) - \kappa_2 W_0(K) = \frac{L^2-4\pi A}{4} = \frac{\Delta(K)}{4}$, we obtain the direct consequence of Theorem 4.

Corollary 4. *Let K be a convex body and the boundary ∂K be C^2 in $\mathbb{R}^2$. Denote L and A the length and the area of K, respectively. Then*

$$L^2 - 4\pi A \geq \frac{\pi^2}{4}\Big[\bar{w}_{\max}(K) - \bar{w}_{\min}(K)\Big]^2 + \Big[\frac{\pi}{2}\bar{w}_{\max}(K) + \frac{\kappa_n}{2}\bar{w}_{\min}(K) - L\Big]^2,$$

$$L^2 - 4\pi A \geq A^2 \left[\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}_{\max}(K)}\right]^2 + \left[\frac{A}{\bar{w}_{\min}(K)} + \frac{A}{\bar{w}_{\max}(K)} - \frac{L}{2}\right]^2,$$

where K_u is the orthogonal projection of K onto the linear subspace $\langle u\rangle^\perp$, $\bar{w}(K_u)$ denotes the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ denote, respectively, the maximum and minimum of $\bar{w}(K_u)$ over all $u \in S^{n-1}$. Each equality holds if and only if K is a disc.

When $n = 3$, combining (3.29) and (3.26), (3.30) and (3.26), respectively, it follows that:

Corollary 5. *Let K be a convex body and the boundary ∂K be C^2 in Euclidean space $\mathbb{R}^3$. Denote A and V the surface area and the volume of K, respectively. Then*

$$A^3 - 36\pi V^2 \geq \frac{81V^2}{A}\Big[\frac{\pi^2}{9}\Big(\bar{w}_{\max}(K) - \bar{w}_{\min}(K)\Big)^2$$
$$+ \Big(\frac{\pi}{3}\bar{w}_{\max}(K) + \frac{\pi}{3}\bar{w}_{\min}(K) - W_2(K)\Big)^2\Big],$$

$$A^3 - 36\pi V^2 \geq \frac{81V^2}{A}\Big[W_1^2(K)\Big(\frac{1}{\bar{w}_{\min}(K)} - \frac{1}{\bar{w}_{\max}(K)}\Big)^2$$
$$+ \Big(\frac{W_1(K)}{\bar{w}_{\min}(K)} + \frac{W_1(K)}{\bar{w}_{\max}(K)} - W_2(K)\Big)^2\Big],$$

where $W_2(K)$ is the 2-th quermassintegral of K and K_u is the orthogonal projection of K onto the linear subspace $\langle u\rangle^\perp$, $\bar{w}(K_u)$ denotes the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ denote, respectively, the maximum and minimum of $\bar{w}(K_u)$ over all $u \in S^{n-1}$. Each equality holds if and only if K is a ball.

Theorem 5. *Let K be a smooth convex body in $\mathbb{R}^n$ and $W_i(K)$ the i-th quermassintegral of K. Then*

$$\frac{2W_{n-1}(K) - 2\sqrt{\Delta_{n-1}(K)}}{\kappa_n} \leq \bar{w}_{\min}(K) \leq \bar{w}_{\max}(K)$$

$$\leq \frac{2W_{n-1}(K) + 2\sqrt{\Delta_{n-1}(K)}}{\kappa_n},$$

where K_u is the orthogonal projection of K onto the linear subspace $\langle u \rangle^{\perp}$, $\bar{w}_{n-1}(K_u)$ denotes the mean width of K_u, and $\bar{w}_{\max}(K), \bar{w}_{\min}(K)$ denote, respectively, the maximum and minimum of $\bar{w}_{n-1}(K_u)$ over all $u \in S^{n-1}$. Each equality holds if and only if K is a ball.

Proof. Notice that the equality

$$\frac{\kappa_n}{4}\bar{w}_{n-1}(K_u)^2 - W_{n-1}(K)\bar{w}_{n-1}(K_u) + W_{n-2}(K) = 0$$

has two difference roots

$$\frac{2W_{n-1}(K) + 2\sqrt{\Delta_{n-1}(K)}}{\kappa_n}, \frac{2W_{n-1}(K) - 2\sqrt{\Delta_{n-1}(K)}}{\kappa_n}$$

if $\Delta_{n-1}(K) = W_{n-1}^2(K) - \kappa_n W_{n-2}(K) > 0$. The inequality $\frac{\kappa_n}{4}\bar{w}_{n-1}(K_u)^2 - W_{n-1}(K)\bar{w}_{n-1}(K_u) + W_{n-2}(K) \leq 0$ holds for any $\bar{w}_{n-1}(K_u)$ in the closed interval

$$\left[\frac{2W_{n-1}(K) + 2\sqrt{\Delta_{n-1}(K)}}{\kappa_n}, \frac{2W_{n-1}(K) - 2\sqrt{\Delta_{n-1}(K)}}{\kappa_n}\right]. \qquad \square$$

4. Reverse Bonnesen-style Aleksandrov-Fenchel inequalities

Mathematicians are also interested in reverse Bonnesen-style inequalities. Only a few upper bounds for convex domains are known (see [13, 29, 28, 35, 48]) and an upper bound for an oval domain in $\mathbb{R}^2$ was given by Bottema. The higher-dimensional cases are more complicated.

In the Euclidean plane $\mathbb{R}^2$, if the boundary of the convex set K is strictly convex and with the C^2 smooth boundary ∂K, Bottema gave an upper bound of the isoperimetric deficit (see [30, 33]):

$$\Delta(K) = L^2 - 4\pi A \leq \pi^2 (\rho_M - \rho_m)^2,$$

where ρ_M and ρ_m are the maximum and minimum of the continues curvature radius of ∂K. The equality holds if and only if $\rho_M = \rho_m$, that is, ∂K is a circle.

In contrast to the Bonnesen-style Alesandrov-Fenchel inequality, one may wish to consider the following type reverse Bonnesen-style Alesandrov-Fenchel inequality, that is, for a smooth convex body K in $\mathbb{R}^n$, is there a geometric invariant U_K, such that

$$\Delta_{n-1}(K) = W_{n-1}^2(K) - \kappa_n W_{n-2}(K) \leq U_K?$$

Here U_K is non-negative and vanishes when K is a ball.

In order to obtain the reverse Bonnesen-style Aleksandrov-Fenchel inequalities, we first introduce the following lemma:

Lemma 2. *Let K, L be two smooth convex bodies in $\mathbb{R}^n$. Denote by $r(K, L)$ and $R(K, L)$ the relative inradius and the relative circumradius of K with respect to L, respectively. Then*

$$(4.1) \quad r(K, L)V_{i+1}(K, L) \leq V_i(K, L) \leq R(K, L)V_{i+1}(K, L), \quad 0 \leq i \leq n-1.$$

Proof. By $r(K, L)L \subseteq K$ and $K \subseteq R(K, L)L$, and the monotonicity of mixed volumes, the following inequalities

$$(4.2) \qquad\qquad r(K, L) \leq \frac{V_0(K, L)}{V_1(K, L)}$$

and

$$(4.3) \qquad\qquad \frac{V_{n-1}(K, L)}{V_n(K, L)} \leq R(K, L)$$

are established. Following the Aleksandrov-Fenchel inequality, we have

$$(4.4) \qquad V_i^2(K, L) - V_{i-1}(K, L)V_{i+1}(K, L) \geq 0, \quad 1 \leq i \leq n-1.$$

By (4.2), (4.3) and (4.4), we have

$$(4.5) \qquad r(K, L) \leq \frac{V_0(K, L)}{V_1(K, L)} \leq \frac{V_1(K, L)}{V_2(K, L)} \leq \cdots$$
$$\leq \frac{V_i(K, L)}{V_{i+1}(K, L)} \leq \cdots \leq \frac{V_{n-1}(K, L)}{V_n(K, L)} \leq R(K, L).$$

Thus we complete the proof of Lemma 2. $\qquad\qquad\qquad\qquad\square$

Let $L = B$ in (4.5), we have

$$(4.6)$$
$$r(K, B) \leq \frac{W_0(K)}{W_1(K)} \cdots \leq \frac{W_{i-2}(K)}{W_{i-1}(K)} \leq \frac{W_{i-1}(K)}{W_i(K)} \cdots \leq \frac{W_{n-1}(K)}{W_n(K)} \leq R(K, B),$$

then

$$(4.7) \qquad\qquad \frac{W_{i-1}(K)}{W_i(K)} - \frac{W_{i-2}(K)}{W_{i-1}(K)} \leq R(K, B) - r(K, B),$$

that is

$$(4.8) \quad W_{i-1}^2(K) - W_i(K)W_{i-2}(K) \leq W_i(K)W_{i-1}(K)\Big(R(K, B) - r(K, B)\Big).$$

So we have the following theorem:

Theorem 6. *Let K be a smooth convex body in $\mathbb{R}^n$ and $W_i(K)$ the i-th quermassintegral of K. If $1 \leq i \leq n-1$, then we have*

$$(4.9)$$
$$\Delta_i(K) = W_i^2(K) - W_{i+1}(K)W_{i-1}(K) \leq W_{i+1}(K)W_i(K)\Big(R(K, B) - r(K, B)\Big),$$

where $r(K, B)$ and $R(K, B)$ are, respectively, the relative inradius and the relative circumradius of K with respect to B.

Especially, when $i = n-1$, we obtain that:

Theorem 7. *Let K be a smooth convex body in $\mathbb{R}^n$ and $W_i(K)$ the i-th quermassintegral of K. Then*

(4.10)
$$\Delta_{n-1}(K) = W_{n-1}^2(K) - \kappa_n W_{n-2}(K) \leq \kappa_n W_{n-1}(K)\big(R(K, B) - r(K, B)\big),$$

where $r(K, B)$ and $R(K, B)$ are, respectively, the relative inradius and the relative circumradius of K with respect to B.

Let $n = 2$, then $\Delta_1(K) = \frac{\Delta(K)}{4}$. The following reverse Bonnesen-style inequality is a direct consequence of Theorem 7:

Corollary 6. *Let K be a convex body and the boundary ∂K be C^2 in $\mathbb{R}^2$. Denote L and A the length and the area of K, respectively. Then*

$$\Delta(K) = L^2 - 4\pi A \leq 2\pi L\big(R(K, B) - r(K, B)\big),$$

where $r(K, B)$ and $R(K, B)$ are, respectively, the relative inradius and the relative circumradius of K with respect to B.

References

[1] T. F. Banchoff and W. F. Pohl, *A generalization of the isoperimetric inequality*, J. Differential Geom. **6** (1971), 175–213.

[2] W. Blaschke, *Vorlesungen über Intergralgeometrie*, 3rd ed. Deutsch. Verlag Wiss., Berlin, 1955.

[3] J. Bokowski and E. Heil, *Integral representation of quermassintegrals and Bonnesen-style inequalities*, Arch. Math. **47** (1986), no. 1, 79–89.

[4] T. Bonnesen, *Les probléms des isopérimétres et des isépiphanes*, Paris, 1929.

[5] T. Bonnesen and W. Fenchel, *Theorie der konvexen Köeper*, 2nd ed., Berlin-Heidelberg-New York, 1974.

[6] Yu. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Springer-Verlag Berlin Heidelberg, 1988.

[7] A. Dinghas, *Bemerkung zu einer Verschärfung der isoperimetrischen Ungleichung durch H. Hadwiger*, Math. Nachr. **1** (1948), 284–286.

[8] V. Diskant, *Bounds for the discrepancy between convex bodies in terms of the isoperimetric difference*, Sibirskii Mat. Zh. **13** (1972), 767–772. English translation: Siberian Math. J. **13** (1973), 529–532.

[9] _____, *Strengthening of an isoperimetric inequality*, Sibirskii Mat. Zh. **14** (1973), 873–877. English translation: Sib. Math. J. **14** (1973), 608–611.

[10] _____, *A generalization of Bonnesen's inequalities*, Dokl. Math. **14** (1973), 1728–1731.

[11] H. Flanders, *A proof of Minkowski's inequality for convex curves*, Amer. Math. Monthly **75** (1968), no. 6, 581–593.

[12] M. Gage, *An isoperimetric inequality with applications to curve shortening*, Duke Math. J. **50** (1983), no. 4, 1225–1229.

[13] X. Gao, *A new reverse isoperimetric inequality and its stability*, Math. Inequal. Appl. **15** (2012), no. 3, 733–743.

[14] R. Gardner, *Geometric Tomography*, Cambridge Univ. Press, New York, 1995.

[15] _____, *The Brunn-Minkowski inequality, Minkowski's first inequality, and their duals*, J. Math. Anal. Appl. **245** (2000), no. 2, 502–512.

[16] _____, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. **39** (2002), no. 3, 355–405.

[17] M. Green and S. Osher, *Steiner polynomials, Wulff flows, and some new isoperimetric inequalities for convex plane curves*, Asian J. Math. **3** (1999), no. 3, 659–676.

[18] H. Groemer, *Geometric applications of Fourier series and spherical harmonics*, Encyclopedia of Mathematics and its Applications, 61. Cambridge University Press, Cambridge, 1996.

[19] H. Groemer and R. Schneider, *Stability estimates for some geometric inequalities*, Bull. Lond. Math. Soc. **23** (1991), no. 1, 67–74.

[20] L. Gysin, *The isoperimetric inequality for nonsimple closed curves*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 197–203.

[21] H. Hadwiger, *Die isoperimetrische Ungleichung in Raum*, Elem. Math. **3** (1948), 25–38.

[22] _____, *Kurze Herleitung einer verschärften isoperimetrischen Ungleichung für konvexe Körper*, Rev. Fac. Sci. Univ. Istanbul, Sér. A **14** (1949), 1–6.

[23] _____, *Vorlesungen über Inhalt, Oberflache und Isoperimetrie*, Springer, Berlin, 1957.

[24] W. Y. Hsiang, *An elementary proof of the isoperimetric problem*, Chinese Ann. Math. **23** (2002), no. 1, 7–12.

[25] D. Klain, *Bonnesen-type inequalities for surfaces of constant curvature*, Adv. in Appl. Math. **39** (2007), no. 2, 143–154.

[26] R. Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 1182–1238.

[27] _____, *Bonnesen-style isoperimetric inequality*, Amer. Math. Monthly **86** (1979), no. 1, 1–29.

[28] S. Pan, X. Tang, and X. Wang, *A refined reverse isoperimetric inequality in the plane*, Math. Inequal. Appl. **13** (2010), no. 2, 329–338.

[29] S. Pan and H. Xu, *Stability of a reverse isoperimetric inequality*, J. Math. Anal. Appl. **350** (2009), no. 1, 348–353.

[30] D. Ren, *Topics in Integral Geometry*, World Scientific, Sigapore, 1994.

[31] J. R. Sangwine-Yager, *Bonnesen-style inequalities for Minkowski relative geometry*, Trans. Amer. Math. Soc. **307** (1988), no. 1, 373–382.

[32] _____, *Mixe Volumes*, Handbook of Covex Geometry, Vol. A, 43–71, Edited by P. Gruber & J. Wills, North-Holland, 1993.

[33] L. A. Santaló, *Integral Geometry and Geometric Probability*, Reading, MA: Addison-Wesley, 1976.

[34] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge Univ. Press, Cambridge, 1993.

[35] Y. Xia, *On reverse isoperimetric inequalities in two-dimensional space forms and related results*, Math. Inequal. Appl. **18** (2015), no. 3, 1025–1032.

[36] W. Xu, J. Zhou, and B. Zhu, *On containment measure and the mixed isoperimetric inequality*, J. Inequal. Appl. **2013** (2103), 540, 11 pp.

[37] C. Zeng, L. Ma, J. Zhou, and F. Chen, *The Bonnesen isoperimetric inequality in a surface of constant curvature*, Sci. China Math. **55** (2012), no. 9, 1913–1919.

[38] C. Zeng, J. Zhou, and S. Yue, *A symmetric mixed isoperimetric inequality for two planar convex domains*, Acta Math. Sinica **55** (2012), no. 2, 355–362.

[39] G. Zhang, *Geometric inequalities and inclusion measures of convex bodies*, Mathematika **41** (1994), no. 1, 95–116.

[40] _____, *The affine Sobolev inequality*, J. Differential Geom. **53** (1999), no. 1, 183–202.

[41] X.-M. Zhang, *Bonnesen-style inequalities and pseudo-perimeters for polygons*, J. Geom. **60** (1997), no. 1-2, 188–201.

[42] _____, *Schur-convex functions and isoperimetric inequalities*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 461–470.

[43] J. Zhou, *Bonnesen-type inequalities*, Acta Math. Sin. (Chin. Ser.) **50** (2007), no. 6, 1397–1402.

[44] J. Zhou and F. Chen, *The Bonnesen-type inequality in a plane of constant cuvature*, J. Korean Math. Soc. **44** (2007), no. 6, 1363–1372.

[45] J. Zhou, Y. Du, and F. Cheng, *Some Bonnesen-style inequalities for higher dimensions*, Acta Math. Sin. (Engl. Ser.) **28** (2012), no. 12, 2561–2568.

[46] J. Zhou and D. Ren, *Geometric inequalities from the viewpoint of integral geometry*, Acta Math. Sci. Ser. A Chin. Ed. **30** (2010), no. 5, 1322–1339.

[47] J. Zhou, Y. Xia, and C. Zeng, *Some new Bonnesen-style inequalities*, J. Korean Math. Soc. **48** (2011), no. 2, 421–430.

[48] J. Zhou, C. Zhou, and F. Ma, *Isoperimetric deficit upper limit of a planar convex set*, Rend. Circ. Mat. Palermo (2) **81** (2009), 363–367.

CHUNNA ZENG
COLLEGE OF MATHEMATICS SCIENCE
CHONGQING NORMAL UNIVERSITY
CHONGQING 401331, P. R. CHINA
AND
INSTITUTE OF DISCRETE MATHEMATICS AND GEOMETRY
VIENNA UNIVERSITY OF TECHNOLOGY
WIEN 1040, AUSTRIA
E-mail address: zengchn@163.com