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AN ERDŐS-KO-RADO THEOREM FOR MINIMAL COVERS

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Abstract. Let $[n] = \{1, 2, \ldots, n\}$. A set $\mathbf{A} = \{A_1, A_2, \ldots, A_l\}$ is a minimal cover of $[n]$ if $\bigcup_{1\le i\le l} A_i = [n]$ and

$$\bigcup_{\substack{1\le i\le l,\\ i\ne j_0}} A_i \ne [n] \quad \text{for all } j_0 \in [l].$$

Let $\mathcal{C}(n)$ denote the collection of all minimal covers of $[n]$, and write $C_n = |\mathcal{C}(n)|$. Let $\mathbf{A} \in \mathcal{C}(n)$. An element $u \in [n]$ is critical in $\mathbf{A}$ if it appears exactly once in $\mathbf{A}$. Two minimal covers $\mathbf{A}$, $\mathbf{B} \in \mathcal{C}(n)$ are said to be restricted t-intersecting if they share at least t sets each containing an element which is critical in both $\mathbf{A}$ and $\mathbf{B}$.

A family $\mathcal{A} \subseteq \mathcal{C}(n)$ is said to be restricted t-intersecting if every pair of distinct elements in $\mathcal{A}$ are restricted t-intersecting. In this paper, we prove that there exists a constant $n_0 = n_0(t)$ depending on t, such that for all $n \ge n_0$, if $\mathcal{A} \subseteq \mathcal{C}(n)$ is restricted t-intersecting, then $|\mathcal{A}| \le C_{n-t}$. Moreover, the bound is attained if and only if $\mathcal{A}$ is isomorphic to the family $\mathcal{D}_0(t)$ consisting of all minimal covers which contain the singleton parts $\{1\}$, …, $\{t\}$. A similar result also holds for restricted r-cross intersecting families of minimal covers.

1. Introduction

Let $[n] = \{1, \ldots, n\}$, and let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$. A family $\mathcal{A}$ of subsets of $[n]$ is *t-intersecting* if $|A \cap B| \ge t$ for all $A, B \in \mathcal{A}$. One of the most beautiful results in extremal combinatorics is the Erdős-Ko-Rado theorem.

Theorem 1.1 (Erdős, Ko, and Rado [11], Frankl [13], Wilson [38])**.** *Suppose $\mathcal{A} \subseteq \binom{[n]}{k}$ is t-intersecting and $n > 2k - t$. Then for $n \ge (k - t + 1)(t + 1)$, we have*

$$|\mathcal{A}| \le \binom{n-t}{k-t}.$$

Moreover, if $n > (k - t + 1)(t + 1)$, then equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{k} : T \subseteq A\}$ for some t-set T.

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Let $\mathcal{A}_i \subseteq \binom{[n]}{k_i}$ for $i = 1, 2, \ldots, r$. We say that the families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$ are *r-cross t-intersecting* if $|A_1 \cap A_2 \cap \cdots \cap A_r| \geq t$ holds for all $A_i \in \mathcal{A}_i$. When $t = 1$, we will just say *r-cross intersecting* instead of r-cross 1-intersecting. When $r = 2$ and $t = 1$, we will just say *cross-intersecting* instead of 2-cross intersecting.

Theorem 1.2 (Bey [3], Matsumoto and Tokushige [32], Pyber [34])**.** *Let $\mathcal{A}_1 \subseteq \binom{[n]}{k_1}$ and $\mathcal{A}_2 \subseteq \binom{[n]}{k_2}$ be cross-intersecting. If $k_1, k_2 \leq n/2$, then*

$$|\mathcal{A}_1||\mathcal{A}_2| \leq \binom{n-1}{k_1-1}\binom{n-1}{k_2-1}.$$

Equality holds for $k_1 + k_2 < n$ if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ consist of all k_1-element resp. k_2-element sets containing a fixed element.

In the celebrated paper [1], Ahlswede and Khachatrian extended the Erdős-Ko-Rado theorem by determining the structure of all t-intersecting set systems of maximum size for all possible n (see also [12, 14, 16, 24, 26, 35, 36] for some related results). There have been many recent results showing that a version of the Erdős-Ko-Rado theorem holds for combinatorial objects other than set systems. For example, an analogue of the Erdős-Ko-Rado theorem for the Hamming scheme is proved in [33]. A complete solution for the t-intersection problem in the Hamming space is given in [2]. Some recent work done on this problem and its variants can be found in [4, 5, 6, 8, 9, 10, 15, 18, 19, 25, 30, 31, 37]. The Erdős-Ko-Rado type results also appear in vector spaces [7, 17], set partitions [20, 22, 21, 29] and weak compositions [23, 27, 28].

In this paper, we consider Erdős-Ko-Rado type results for minimal covers. Let $\mathcal{P}(n)$ be the set of all subsets of $[n]$, and let $\mathcal{P}^2(n)$ be the set of all subsets of $\mathcal{P}(n)$. Let $Z \subseteq [n]$. A set $\mathbf{A} = \{A_1, A_2, \ldots, A_l\} \subseteq \mathcal{P}(n)$ is a *cover* of Z if $\bigcup_{1 \leq i \leq l} A_i = Z$. It is a *minimal cover* of Z if it is a cover of Z and

$$\bigcup_{\substack{1 \leq i \leq l, \\ i \neq j_0}} A_i \neq Z \quad \text{for all } j_0 \in [l].$$

Let $\mathcal{C}(Z)$ denote the collection of all minimal covers of Z. Note that $\mathcal{C}(Z) \subseteq \mathcal{P}^2(n)$. When $Z = [n]$, we shall write $\mathcal{C}(n)$ instead of $\mathcal{C}([n])$. Let $C_n = |\mathcal{C}(n)|$. For $1 \leq n \leq 3$, we have

$$\begin{aligned}
\mathcal{C}(1) &= \{\{\{1\}\}\}, \\
\mathcal{C}(2) &= \{\{\{1\},\{2\}\},\ \{\{1,2\}\}\}, \\
\mathcal{C}(3) &= \{\{\{1\},\{2\},\{3\}\},\ \{\{1,2\},\{3\}\},\ \{\{1,3\},\{2\}\},\ \{\{2,3\},\{1\}\}, \\
&\qquad \{\{1,2\},\{1,3\}\},\ \{\{1,2\},\{2,3\}\},\ \{\{1,3\},\{2,3\}\},\ \{\{1,2,3\}\}\},
\end{aligned}$$

and thus $C_1 = 1$, $C_2 = 2$ and $C_3 = 8$.

Let σ be a permutation on $[n]$. For each $A \subseteq [n]$, we define $\sigma(A) = \{\sigma(a) : a \in A\}$. For each $\mathbf{A} \subseteq \mathcal{P}(n)$, we define $\sigma(\mathbf{A}) = \{\sigma(A) : A \in \mathbf{A}\}$, and for each $\mathcal{A} \subseteq \mathcal{P}^2(n)$, we define $\sigma(\mathcal{A}) = \{\sigma(\mathbf{A}) : \mathbf{A} \in \mathcal{A}\}$. Two families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}^2(n)$ are

said to be *isomorphic*, denoted by $\mathcal{A} \cong \mathcal{B}$, if they are the same up to relabelling of the underlying elements, i.e., $\sigma(\mathcal{A}) = \mathcal{B}$.

Let

$$\begin{aligned}
\mathcal{Q}_0(t) &= \{\mathbf{A} : \mathbf{A} \text{ is a minimal cover of } [n] \setminus [t]\},\\
\mathcal{Q}_1(t) &= \{\mathbf{A} \in \mathcal{Q}_0(t) : \{t+1\} \notin \mathbf{A}\},\\
\mathcal{Q}_2(t) &= \{\mathbf{A} : \mathbf{A} \text{ is a minimal cover of } [n] \setminus [t+1]\},\\
\mathcal{D}_0(t) &= \{\{\{1\}, \{2\}, \ldots, \{t\}\} \cup \mathbf{A}\} : \mathbf{A} \in \mathcal{Q}_0(t)\}.
\end{aligned}$$

For $1 \le l \le t$, let

$$\begin{aligned}
\mathcal{D}_l(t) = &\{\{\{1, t+1\}, \{2, t+1\}, \ldots, \{l, t+1\}, \{l+1\}, \ldots, \{t\}\} \cup \mathbf{A} : \mathbf{A} \in \mathcal{Q}_1(t)\}\\
&\cup \{\{\{1, t+1\}, \{2, t+1\}, \ldots, \{l, t+1\}, \{l+1\}, \ldots, \{t\}\} \cup \mathbf{A} : \mathbf{A} \in \mathcal{Q}_2(t)\}.
\end{aligned}$$

Notice that when $l = t$, we have

$$\begin{aligned}
\mathcal{D}_l(l) = &\{\{\{1, l+1\}, \{2, l+1\}, \ldots, \{l, l+1\}\} \cup \mathbf{A} : \mathbf{A} \in \mathcal{Q}_1(t)\}\\
&\cup \{\{\{1, l+1\}, \{2, l+1\}, \ldots, \{l, l+1\}\} \cup \mathbf{A} : \mathbf{A} \in \mathcal{Q}_2(t)\}.
\end{aligned}$$

Clearly $\mathcal{D}_0(t) \subseteq \mathcal{C}(n)$, and $|\mathcal{D}_0(t)| = C_{n-t}$. For each $\mathbf{A} \in \mathcal{Q}_1(t)$, the mapping defined by

$$\begin{aligned}
&\{\{1, t+1\}, \{2, t+1\}, \ldots, \{l, t+1\}, \{l+1\}, \ldots, \{t\}\} \cup \mathbf{A}\\
&\qquad \mapsto \{\{1\}, \{2\}, \ldots, \{l\}, \{l+1\}, \ldots, \{t\}\} \cup \mathbf{A},
\end{aligned}$$

is one-to-one. For each $\mathbf{A} \in \mathcal{Q}_2(t)$, the mapping defined by

$$\begin{aligned}
&\{\{1, t+1\}, \{2, t+1\}, \ldots, \{l, t+1\}, \{l+1\}, \ldots, \{t\}\} \cup \mathbf{A}\\
&\qquad \mapsto \{\{1\}, \{2\}, \ldots, \{l\}, \{l+1\}, \ldots, \{t\}, \{t+1\}\} \cup \mathbf{A},
\end{aligned}$$

is also one-to-one. Hence

$$|\mathcal{D}_l(t)| = |\mathcal{D}_0(t)| = C_{n-t} \tag{1}$$

for $1 \le l \le t$. However, some of the elements in $\mathcal{D}_l(t)$ do not lie in $\mathcal{C}(n)$. For example, if $n \ge t+3$, then the set

$$\begin{aligned}
\mathbf{A}' = &\{\{1, t+1\}, \{2, t+1\}, \ldots, \{l, t+1\}, \{l+1\}, \ldots, \{t\}, \{t+1, t+2\},\\
&\{t+2, t+3, \ldots, n\}\}
\end{aligned}$$

is in $\mathcal{D}_l(t)$, but it is not in $\mathcal{C}(n)$ since removing $\{t+1, t+2\}$ from $\mathbf{A}'$ results in a collection of sets which is still a cover of $[n]$. Therefore, for $n \ge t+3$,

$$|\mathcal{D}_l(t) \cap \mathcal{C}(n)| < |\mathcal{D}_l(t)| = C_{n-t}. \tag{2}$$

A family $\mathcal{A} \subseteq \mathcal{C}(n)$ is said to be *t-intersecting* if $|\mathbf{A} \cap \mathbf{B}| \ge t$ for all $\mathbf{A}, \mathbf{B} \in \mathcal{A}$. We suggest the following conjecture on the characterisation of t-intersecting families of maximum size.

Conjecture 1.3. *There exists a constant $n_0 = n_0(t)$ depending on t, such that for all $n \geq n_0$, if $\mathcal{A} \subseteq \mathcal{C}(n)$ is t-intersecting, then*

$$|\mathcal{A}| \leq C_{n-t}.$$

Moreover, equality holds if and only if $\mathcal{A} \cong \mathcal{D}_0(t)$.

In this paper, we prove a weaker version of Conjecture 1.3 (see Theorem 1.4 below). To this end, we require a stronger notion of intersection. For a fixed $j \in [n]$, $\mathbf{A} \in \mathcal{P}^2(n)$, we define

$$N_j(\mathbf{A}) = |\{A \in \mathbf{A} : j \in A\}|$$

to be the number of times j appears in $\mathbf{A}$. If $N_j(\mathbf{A}) = 1$, then j is said to be *critical* in $\mathbf{A}$. For example, if $\mathbf{A} = \{\{1,2,3\},\{1,2,4\},\{1,5,6\}\} \in \mathcal{C}(6)$, then $N_2(\mathbf{A}) = 2$ since 2 appears twice in $\mathbf{A}$. Also, 5 is critical in $\mathbf{A}$ since $N_5(\mathbf{A}) = 1$.

Given any $\mathbf{A}, \mathbf{B} \in \mathcal{C}(n)$, we write $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t$ if there exist t distinct elements $A_1, \ldots, A_t \in \mathbf{A} \cap \mathbf{B}$ each containing an element which is critical in both $\mathbf{A}$ and $\mathbf{B}$, i.e., for all $1 \leq i \leq t$, there exists $a_i \in A_i$ such that $N_{a_i}(\mathbf{A}) = 1 = N_{a_i}(\mathbf{B})$. For example, if $\mathbf{A} = \{\{1,2,3\},\{1,2,4\},\{1,5,6\}\}$ and $\mathbf{B} = \{\{1,2,3\},\{2,4,6\},\{2,3,5\}\}$, then $|\mathbf{A} \cap \mathbf{B}| = 1$, but $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) = 0$ because $\mathbf{A} \cap \mathbf{B} = \{\{1,2,3\}\}$ and none of the elements in $\{1,2,3\}$ is critical in both $\mathbf{A}$ and $\mathbf{B}$. On the other hand, if $\mathbf{C} = \{\{1,2,3\},\{1,4,5\},\{1,2,6\}\}$, then $\mathrm{Inter}(\mathbf{A}, \mathbf{C}) \geq 1$ since $\{1,2,3\} \in \mathbf{A} \cap \mathbf{C}$ and 3 is critical in both $\mathbf{A}$ and $\mathbf{C}$. In general, if $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t+1$, then $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t$. Also, $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t$ implies that $|\mathbf{A} \cap \mathbf{B}| \geq t$.

A family $\mathcal{A} \subseteq \mathcal{C}(n)$ is said to be *restricted t-intersecting* if $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t$ for any $\mathbf{A}, \mathbf{B} \in \mathcal{A}$.

Theorem 1.4. *There exists a constant $n_0 = n_0(t)$ depending on t, such that for all $n \geq n_0$, if $\mathcal{A} \subseteq \mathcal{C}(n)$ is restricted t-intersecting, then*

$$|\mathcal{A}| \leq C_{n-t}.$$

Moreover, equality holds if and only if $\mathcal{A} \cong \mathcal{D}_0(t)$.

Families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \subseteq \mathcal{C}(n)$ are said to be *r-cross t-intersecting* if $|\mathbf{A}_1 \cap \mathbf{A}_2 \cap \cdots \cap \mathbf{A}_r| \geq t$ for all $\mathbf{A}_i \in \mathcal{A}_i$. As in the case for sets, we will just say r-cross intersecting to mean r-cross 1-intersecting and cross t-intersecting to mean 2-cross t-intersecting.

Conjecture 1.5. *There exists a constant $n_0 = n_0(r)$ depending on r, such that for all $n \geq n_0$, if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \subseteq \mathcal{C}(n)$ are r-cross intersecting, then*

$$\prod_{i=1}^{r} |\mathcal{A}_i| \leq C_{n-1}^r.$$

Moreover, equality holds if and only if $\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_r$ and $\mathcal{A}_1 \cong \mathcal{D}_0(1)$.

We will prove a weaker version of Conjecture 1.5 (Theorem 1.6). Given any $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r \in \mathcal{C}(n)$, we write $\mathrm{Inter}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r) \geq t$ if there exist t distinct elements $A_1, A_2, \ldots, A_t \in \mathbf{A}_1 \cap \mathbf{A}_2 \cap \cdots \cap \mathbf{A}_r$ each containing a critical element in all of the $\mathbf{A}_j$. Families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \subseteq \mathcal{C}(n)$ are said to be *restricted r-cross t-intersecting* if $\mathrm{Inter}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r) \geq t$ for all $\mathbf{A}_i \in \mathcal{A}_i$. As before, we will just say restricted r-cross intersecting to mean restricted r-cross 1-intersecting and restricted cross t-intersecting to mean restricted 2-cross t-intersecting.

Theorem 1.6. *There exists a constant $n_0 = n_0(r)$ depending on r, such that for all $n \geq n_0$, if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \subseteq \mathcal{C}(n)$ are restricted r-cross intersecting, then*

$$\prod_{i=1}^{r} |\mathcal{A}_i| \leq C_{n-1}^{r}.$$

Moreover, equality holds if and only if $\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_r$ and $\mathcal{A}_1 \cong \mathcal{D}_0(1)$.

Theorem 1.4 and Theorem 1.6 are proved in Sections 3 and 4 respectively.

2. Splitting operation

Lemma 2.1. *Every set in a minimal cover of $[n]$ contains a critical element. In particular, if $\mathbf{A} \in \mathcal{C}(n)$ and $B = \{j\}$ is a singleton in $\mathbf{A}$, then j is critical in $\mathbf{A}$.*

Proof. Let $\mathbf{A} \in \mathcal{C}(n)$, and $A \in \mathbf{A}$. By definition, removing A from $\mathbf{A}$ results in an element of $\mathcal{P}^2(n)$ which is no longer a cover of $[n]$. So there must be an element in A which does not appear elsewhere in $\mathbf{A}$. Thus, this element must be critical in $\mathbf{A}$. □

Let $T \subseteq [n]$ and $|T| \geq 2$. For each $\mathbf{A} \in \mathcal{C}(n)$ with $T \in \mathbf{A}$, we define

$$P(T, \mathbf{A}) = \{\{q\} : q \in T \text{ and } q \text{ is critical in } \mathbf{A}\}.$$

By Lemma 2.1, $P(T, \mathbf{A}) \neq \varnothing$. The T-*split* of $\mathbf{A}$, denoted by $s_T(\mathbf{A})$, is defined as follow: If T is not a set in $\mathbf{A}$, then the T-split is just $\mathbf{A}$ itself. Otherwise, we replace T by all the singleton sets each consisting of a critical element found in T. Formally,

(O1) $s_T(\mathbf{A}) = \mathbf{A}$, if $T \notin \mathbf{A}$;
(O2) $s_T(\mathbf{A}) = (\mathbf{A} \setminus \{T\}) \cup P(T, \mathbf{A})$, if $T \in \mathbf{A}$.

Lemma 2.2. *$s_T(\mathbf{A}) \in \mathcal{C}(n)$ for all $\mathbf{A} \in \mathcal{C}(n)$.*

Proof. We can assume that $T \in \mathbf{A}$. By removing T from $\mathbf{A}$ and adding the singleton set $\{v\}$ for every critical element $v \in T$, we clearly still have that $s_T(\mathbf{A})$ covers $[n]$. Furthermore, as we have only reduced the number of occurrences of non-critical elements, every set in $s_T(\mathbf{A})$ still has a critical element, and so it must be a minimal cover of $[n]$. □

For a family $\mathcal{A} \subseteq \mathcal{C}(n)$, let $s_T(\mathcal{A}) = \{s_T(\mathbf{A}) : \mathbf{A} \in \mathcal{A}\}$. By Lemma 2.2, $s_T(\mathcal{A}) \subseteq \mathcal{C}(n)$. Any family $\mathcal{A} \subseteq \mathcal{C}(n)$ can be decomposed with respect to a given $T \subseteq [n]$ with $|T| \geq 2$ as follows:

$$\mathcal{A} = (\mathcal{A} \setminus \mathcal{A}_T) \cup \mathcal{A}_T,$$

where $\mathcal{A}_T = \{\mathbf{A} \in \mathcal{A} : s_T(\mathbf{A}) \notin \mathcal{A}\}$. Define the *$T$-splitting* of $\mathcal{A}$ to be the family

$$S_T(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_T) \cup s_T(\mathcal{A}_T).$$

Lemma 2.3. *$|S_T(\mathcal{A})| = |\mathcal{A}|$ for all $\mathcal{A} \subseteq \mathcal{C}(n)$.*

Proof. If $\mathcal{A}_T = \varnothing$, then $S_T(\mathcal{A}) = \mathcal{A}$ and the lemma holds. Suppose $\mathcal{A}_T \neq \varnothing$. Clearly, $s_T(\mathcal{A}_T) \cap (\mathcal{A} \setminus \mathcal{A}_T) = \varnothing$. So, it is sufficient to show that s_T is one-to-one on $\mathcal{A}_T$, i.e., $s_T(\mathbf{A}) = s_T(\mathbf{B})$ implies that $\mathbf{A} = \mathbf{B}$ for $\mathbf{A}, \mathbf{B} \in \mathcal{A}_T$. Note that $T \in \mathbf{A} \cap \mathbf{B}$ and both $s_T(\mathbf{A})$ and $s_T(\mathbf{B})$ are obtained by operation (O2). So,

$$s_T(\mathbf{A}) = \mathbf{A} \setminus \{T\} \cup P(T, \mathbf{A}),$$
$$s_T(\mathbf{B}) = \mathbf{B} \setminus \{T\} \cup P(T, \mathbf{B}).$$

If $P(T, \mathbf{A}) \cap \mathbf{B} \setminus \{T\} \neq \varnothing$, then $\{q\} \in \mathbf{B} \setminus \{T\}$ for some $q \in T$. So, q appears at least 2 times in $\mathbf{B}$ (once in $\{q\}$ and once in T), contradicting Lemma 2.1. Thus, $P(T, \mathbf{A}) \cap \mathbf{B} \setminus \{T\} = \varnothing$. Similarly, $P(T, \mathbf{B}) \cap \mathbf{A} \setminus \{T\} = \varnothing$. Therefore, $P(T, \mathbf{A}) = P(T, \mathbf{B})$ and $\mathbf{A} \setminus \{T\} = \mathbf{B} \setminus \{T\}$. Hence, $\mathbf{A} = \mathbf{B}$. □

Let $I(n, t)$ be the set of all restricted cross t-intersecting families in $\mathcal{C}(n)$, i.e.,

$$I(n, t) = \{(\mathcal{A}_1, \mathcal{A}_2) : \mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{C}(n) \text{ are restricted cross } t\text{-intersecting}\}.$$

Note that $(\mathcal{A}, \mathcal{A}) \in I(n, t)$ if and only if $\mathcal{A}$ is restricted t-intersecting. Given any $(\mathcal{A}_1, \mathcal{A}_2) \in I(n, t)$, the T-splitting of $(\mathcal{A}_1, \mathcal{A}_2)$ is defined to be the set $(S_T(\mathcal{A}_1), S_T(\mathcal{A}_2))$.

For any $(\mathcal{A}_1, \mathcal{A}_2) \in I(n, t)$, splitting operations preserve the size (Lemma 2.3) and the intersecting property (Lemma 2.4).

Lemma 2.4. *Let $T \subseteq [n]$ with $|T| \geq 2$. If $(\mathcal{A}_1, \mathcal{A}_2) \in I(n, t)$, then*

$$(S_T(\mathcal{A}_1), S_T(\mathcal{A}_2)) \in I(n, t).$$

Proof. Note that $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t$ for all $\mathbf{A} \in \mathcal{A}_1 \setminus (\mathcal{A}_1)_T$ and $\mathbf{B} \in \mathcal{A}_2 \setminus (\mathcal{A}_2)_T$, where $(\mathcal{A}_1)_T = \{\mathbf{A} \in \mathcal{A}_1 : s_T(\mathbf{A}) \notin \mathcal{A}_1\}$, $(\mathcal{A}_2)_T = \{\mathbf{A} \in \mathcal{A}_2 : s_T(\mathbf{A}) \notin \mathcal{A}_2\}$. So, it is sufficient to show that $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t$ for any $\mathbf{A} \in S_T(\mathcal{A}_1)$ and $\mathbf{B} \in s_T((\mathcal{A}_2)_T)$ (the case $\mathbf{A} \in S_T(\mathcal{A}_2)$ and $\mathbf{B} \in s_T((\mathcal{A}_1)_T)$ can be proved similarly).

(**Case 1**) Suppose $\mathbf{A} \in \mathcal{A}_1 \setminus (\mathcal{A}_1)_T$ and $\mathbf{B} \in s_T((\mathcal{A}_2)_T)$.

Let $\mathbf{B} = s_T(\mathbf{C})$ for some $\mathbf{C} \in (\mathcal{A}_2)_T$. Then $T \in \mathbf{C}$ and $\mathbf{B} = \mathbf{C} \setminus \{T\} \cup P(T, \mathbf{C})$. Suppose $T \notin \mathbf{A}$. Then $T \notin \mathbf{A} \cap \mathbf{C}$. Since $\mathrm{Inter}(\mathbf{A}, \mathbf{C}) \geq t$, there exist $A_1, \ldots, A_t \in \mathbf{A} \cap \mathbf{C}$ each containing a critical element in both $\mathbf{A}$ and $\mathbf{C}$. Since $T \neq A_i$ for all i, we have $A_1, \ldots, A_t \in \mathbf{A} \cap \mathbf{C} \setminus \{T\} \subseteq \mathbf{A} \cap \mathbf{B}$. So, $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t$.

Suppose $T \in \mathbf{A}$. Then $\mathbf{A} \setminus \{T\} \cup P(T, \mathbf{A}) = s_T(\mathbf{A}) \in \mathcal{A}_1$. Since $C \in \mathcal{A}_2$, we have $\mathrm{Inter}(s_T(\mathbf{A}), \mathbf{C}) \geq t$, and so there exist $B_1, \ldots, B_t \in s_T(\mathbf{A}) \cap \mathbf{C}$ each containing a critical element in both $s_T(\mathbf{A})$ and $\mathbf{C}$. If $B_{i_0} \in P(T, \mathbf{A})$ for some i_0, then $B_{i_0} = \{q_0\}$ for some $q_0 \in T$, and q_0 appears at least 2 times in $\mathbf{C}$ (once in B_{i_0} and once in T), contradicting Lemma 2.1. Thus, $B_i \in \mathbf{A} \setminus \{T\}$ for all i. This implies that $B_1, \ldots, B_t \in \mathbf{A} \cap \mathbf{C} \setminus \{T\} \subseteq \mathbf{A} \cap \mathbf{B}$. Hence, $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t$. (**Case 2**) Suppose $\mathbf{A} \in s_T((\mathcal{A}_1)_T)$ and $\mathbf{B} \in s_T((\mathcal{A}_2)_T)$.

Let $\mathbf{A} = s_T(\mathbf{C})$ and $\mathbf{B} = s_T(\mathbf{D})$ for some $\mathbf{C} \in (\mathcal{A}_1)_T$ and $\mathbf{D} \in (\mathcal{A}_2)_T$. Then

$$\mathbf{A} = \mathbf{C} \setminus \{T\} \cup P(T, \mathbf{C}),$$
$$\mathbf{B} = \mathbf{D} \setminus \{T\} \cup P(T, \mathbf{D}).$$

Since $\mathrm{Inter}(\mathbf{C}, \mathbf{D}) \geq t$, there exist $C_1, \ldots, C_t \in \mathbf{C} \cap \mathbf{D}$ each containing a critical element in $\mathbf{C}$ and $\mathbf{D}$. If $T \neq C_i$ for all i, then $C_1, \ldots, C_t \in (\mathbf{C} \setminus \{T\}) \cap (\mathbf{D} \setminus \{T\}) \subseteq \mathbf{A} \cap \mathbf{B}$. Hence, $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t$. Suppose $T = C_{i_0}$ for some i_0. For convenience, we may assume that $T = C_1$. Since $C_i \neq T$ for all $i \neq 1$, we have $C_2, \ldots, C_t \in (\mathbf{C} \setminus \{T\}) \cap (\mathbf{D} \setminus \{T\}) \subseteq \mathbf{A} \cap \mathbf{B}$. Let $c_1 \in C_1$ be a critical element in C_1. Then $\{c_1\} \in P(T, \mathbf{C}) \cap P(T, \mathbf{D}) \subseteq \mathbf{A} \cap \mathbf{B}$, and c_1 is critical in both $\mathbf{A}$ and $\mathbf{B}$. Since $\{c_1\}, C_2, \ldots, C_t \in \mathbf{A} \cap \mathbf{B}$, we deduce that $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) \geq t$.

This completes the proof of the lemma. □

A family $\mathcal{A} \subseteq \mathcal{C}(n)$ is *compressed* if for any $T \subseteq [n]$ with $|T| \geq 2$, we have $S_T(\mathcal{A}) = \mathcal{A}$. For any $\mathbf{A} \in \mathcal{C}(n)$, define $\beta(\mathbf{A}) = |\{A \in \mathbf{A} : |A| = 1\}|$, i.e., $\beta(\mathbf{A})$ is the number of singletons in $\mathbf{A}$.

Lemma 2.5. *Let $(\mathcal{A}_1, \mathcal{A}_2) \in I(n, t)$. By repeatedly applying the splitting operations on $(\mathcal{A}_1, \mathcal{A}_2)$, we eventually obtain compressed families $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ with $|\mathcal{A}_1^*| = |\mathcal{A}_1|$, $|\mathcal{A}_2^*| = |\mathcal{A}_2|$, and $(\mathcal{A}_1^*, \mathcal{A}_2^*) \in I(n, t)$.*

Proof. For any $(\mathcal{A}_1, \mathcal{A}_2) \in I(n, t)$, let $w((\mathcal{A}_1, \mathcal{A}_2)) = \sum_{\mathbf{A} \in \mathcal{A}_1} \beta(\mathbf{A}) + \sum_{\mathbf{A} \in \mathcal{A}_2} \beta(\mathbf{A})$. Note that if $S_T(\mathcal{A}_i) \neq \mathcal{A}_i$ for some $i \in \{1, 2\}$, then $\sum_{\mathbf{A} \in S_T(\mathcal{A}_i)} \beta(\mathbf{A}) > \sum_{\mathbf{A} \in \mathcal{A}_i} \beta(\mathbf{A})$. This implies that $w((\mathcal{A}_1, \mathcal{A}_2)) < w((S_T(\mathcal{A}_1), S_T(\mathcal{A}_2)))$. So, the splitting operations cannot go on forever. Eventually, we will obtain $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ with $|\mathcal{A}_1^*| = |\mathcal{A}_1|$, $|\mathcal{A}_2^*| = |\mathcal{A}_2|$, and $(\mathcal{A}_1^*, \mathcal{A}_2^*) \in I(n, t)$ (Lemmas 2.3 and 2.4). □

Let $\mathcal{A} \in \mathcal{C}(n)$. For each $\mathbf{A} \in \mathcal{A}$, let

$$\gamma(\mathbf{A}) = \{x : \{x\} \in \mathbf{A}\},$$

i.e., $\gamma(\mathbf{A})$ is the union of all the singletons in $\mathbf{A}$. Let $\gamma(\mathcal{A}) = \{\gamma(\mathbf{A}) : \mathbf{A} \in \mathcal{A}\}$.

Lemma 2.6. *If $\mathcal{A}_1$ is compressed and $(\mathcal{A}_1, \mathcal{A}_2) \in I(n, t)$, then $\gamma(\mathcal{A}_1), \gamma(\mathcal{A}_2)$ are cross t-intersecting families of subsets.*

Proof. Suppose $\gamma(\mathcal{A}_1), \gamma(\mathcal{A}_2)$ are not cross t-intersecting. Then there exist $\mathbf{A} \in \mathcal{A}_1$ and $\mathbf{B} \in \mathcal{A}_2$ with $|\gamma(\mathbf{A}) \cap \gamma(\mathbf{B})| \leq t - 1$. Let $\mathrm{Inter}(\mathbf{A}, \mathbf{B}) = s$. Note that $s \geq t$ and there exist $A_1, \ldots, A_s \in \mathbf{A} \cap \mathbf{B}$ each containing a critical element in both $\mathbf{A}$ and $\mathbf{B}$. The sets A_1, …, A_s cannot be all singletons since

$|\gamma(\mathbf{A}) \cap \gamma(\mathbf{B})| \leq t-1$. By relabeling if necessary, we may assume that $|A_i| = 1$ for $1 \leq i \leq l$ and $|A_i| \geq 2$ for $l+1 \leq i \leq s$. If none of the A_i's are singletons, then we may assume that $l = 0$ and $|A_i| \geq 2$ for $1 \leq i \leq s$. Note that $l \leq t-1$ since $|\gamma(\mathbf{A}) \cap \gamma(\mathbf{B})| \leq t-1$.

Since $\mathcal{A}_1$ is compressed, we have $\mathbf{A}_1 = s_{A_{l+1}}(\mathbf{A}) \in \mathcal{A}_1$. Note that $\mathrm{Inter}(\mathbf{A}_1, \mathbf{B}) = s-1$. Let $\mathbf{A}_2 = s_{A_{l+2}}(\mathbf{A}_1)$. Then $\mathbf{A}_2 \in \mathcal{A}_1$ and $\mathrm{Inter}(\mathbf{A}_2, \mathbf{B}) = s-2$. By applying the splitting operations for $A_{l+1}, A_{l+2}, \ldots, A_s$, we will obtain

$$\mathbf{C} = s_{A_s}\left(s_{A_{s-1}}\left(\cdots s_{A_{l+1}}(\mathbf{A})\right)\right) \in \mathcal{A}_1.$$

Furthermore, $\mathrm{Inter}(\mathbf{C}, \mathbf{B}) = l \leq t-1$. This contradicts the fact that $\{\mathcal{A}_1, \mathcal{A}_2\} \in I(n,t)$. Hence, $\gamma(\mathcal{A}_1), \gamma(\mathcal{A}_2)$ are cross t-intersecting. □

The proof of the following lemma is straightforward and hence omitted.

Lemma 2.7. *If σ is a permutation of $[n]$, then for any $\mathcal{A} \subseteq \mathcal{C}(n)$, $T \subseteq [n]$ with $|T| \geq 2$, we have $\sigma(S_T(\mathcal{A})) = S_{\sigma(T)}(\sigma(\mathcal{A}))$.*

Lemma 2.8. *Let $\mathbf{A} \in \mathcal{C}(n)$, $T \subseteq [n]$ and $|T| \geq 2$. If $A \in s_T(\mathbf{A})$ and $|A| \geq 2$, then $A \in \mathbf{A}$ and $A \nsubseteq T$.*

Proof. Note that $s_T(\mathbf{A}) = \mathbf{A} \setminus \{T\} \cup P(T, \mathbf{A})$ and $|B| = 1$ for all $B \in P(T, \mathbf{A})$. Therefore, $A \in \mathbf{A} \setminus \{T\} \subseteq \mathbf{A}$. If $A \subseteq T$, then every element in A will appear at least 2 times in $\mathbf{A}$ (once in A and once T), contradicting Lemma 2.1. Hence, $A \nsubseteq T$. □

Lemma 2.9. *Let $n \geq t+3$. Suppose $(\mathcal{A}_1, \mathcal{A}_2) \in I(n,t)$ and $|\mathcal{A}_1|, |\mathcal{A}_2| \geq |\mathcal{D}_0(t)| > 1$. Let $T \subseteq [n]$ and $|T| \geq 2$. If $S_T(\mathcal{A}_1) = S_T(\mathcal{A}_2) \cong \mathcal{D}_0(t)$, then $\mathcal{A}_1 = \mathcal{A}_2$, and $\mathcal{A}_1 \cong \mathcal{D}_0(t)$.*

Proof. There is a permutation σ of $[n]$ with $\sigma(S_T(\mathcal{A}_i)) = \mathcal{D}_0(t)$ for $1 \leq i \leq 2$. By Lemma 2.7, $S_{\sigma(T)}(\sigma(\mathcal{A}_i))) = \mathcal{D}_0(t)$. If $\sigma(\mathcal{A}_1) = \sigma(\mathcal{A}_2)$, and $\sigma(\mathcal{A}_1) \cong \mathcal{D}_0(t)$, then $\mathcal{A}_1 = \mathcal{A}_2$, and $\mathcal{A}_1 \cong \mathcal{D}_0(t)$. Furthermore, $(\sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2)) \in I(n,t)$ and $|\sigma(\mathcal{A}_1)|, |\sigma(\mathcal{A}_2)| \geq |\mathcal{D}_0(t)| > 1$. So, without loss of generality, we may assume that $S_T(\mathcal{A}_i) = \mathcal{D}_0(t)$ for $i = 1, 2$.

Recall that

$$\mathcal{D}_0(t) = \{\{\{1\}, \{2\}, \ldots, \{t\}\} \cup \mathbf{A}\} : \mathbf{A} \in \mathcal{Q}_0(t)\},$$

where $\mathcal{Q}_0(t) = \{\mathbf{A} : \mathbf{A}$ is a minimal cover of $[n] \setminus [t]\}$. Let

$$\mathbf{B} = \{\{1\}, \{2\}, \ldots, \{t\}, \{t+1, t+2, \ldots, n\}\} \in \mathcal{D}_0(t).$$

We first prove the following claim.

(**Claim 1.**) If $\mathbf{B} \in \mathcal{A}_1 \cup \mathcal{A}_2$, then $\mathbf{B} \in \mathcal{A}_1 \cap \mathcal{A}_2$, and $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{D}_0(t)$.

Without loss of generality, we may assume that $\mathbf{B} \in \mathcal{A}_1$.

We first prove that $\mathcal{A}_2 = \mathcal{D}_0(t)$. Assume, for a contradiction, that $\mathcal{A}_2 \neq \mathcal{D}_0(t)$. Then there exists a $\mathbf{C} \in \mathcal{A}_2$ such that $\{i\} \notin \mathbf{C}$ for some $1 \leq i \leq t$. Now, the condition $\mathrm{Inter}(\mathbf{B}, \mathbf{C}) \geq t$ implies that

$$\mathbf{C} = \{\{1\}, \{2\}, \ldots, \{i-1\}, V, \{i+1\}, \ldots, \{t\}, \{t+1, t+2, \ldots, n\}\},$$

where $i \in V$ and $|V| \geq 2$. Note that $V \cap \{1, 2, \ldots, i-1, i+1, \ldots, t\} = \varnothing$ for otherwise there exists $j \in \{1, 2, \ldots, i-1, i+1, \ldots, t\}$ such that j appears at least 2 times in $\mathbf{C}$ (once in $\{j\}$ and once in V), contradicting Lemma 2.1. Similarly, $\{t+1, t+2, \ldots, n\} \nsubseteq V$. Since $s_T(\mathbf{C}) \in \mathcal{D}_0(t)$, we have $\{i\} \in s_T(\mathbf{C})$. Thus, $T = V$.

Let

$$\mathbf{D} = \{\{1\}, \{2\}, \ldots, \{t\}, \{t+1, t+2\}, \{t+1, t+3, \ldots, n\}\} \in \mathcal{D}_0(t).$$

Note that $\mathrm{Inter}(\mathbf{D}, \mathbf{C}) = t-1$ since $\mathbf{D} \cap \mathbf{C} = \{\{1\}, \{2\}, \ldots, \{i-1\}, \{i+1\}, \ldots, \{t\}\}$. Therefore, $\mathbf{D} \notin \mathcal{A}_1$. Since $\mathbf{D} \in S_T(\mathcal{A}_1) = \mathcal{D}_0(t)$, there is $\mathbf{E} \in \mathcal{A}_1$ with $T \in \mathbf{E}$ and $s_T(\mathbf{E}) = \mathbf{D}$. By Lemma 2.8, $\{t+1, t+2\}, \{t+1, t+3, \ldots, n\} \in \mathbf{E}$. From $\mathrm{Inter}(\mathbf{E}, \mathbf{C}) \geq t$, we must have

$$\begin{aligned}\mathbf{E} = \{&\{1\}, \{2\}, \ldots, \{i-1\}, V, \{i+1\}, \ldots, \{t\},\\ &\{t+1, t+2\}, \{t+1, t+3, \ldots, n\}\}.\end{aligned}$$

Let $\mathbf{F} \in \mathcal{A}_2 \setminus \{\mathbf{C}\}$ (such an $\mathbf{F}$ exists because $|\mathcal{A}_2| \geq 2$). The aim is to arrive at a contradiction by showing that such $\mathbf{F}$ could never exist.

Suppose $\{t+1, t+2, \ldots, n\} \in \mathbf{F}$. Then $\{t+1, t+2\}, \{t+1, t+3, \ldots, n\} \notin \mathbf{F}$ (otherwise it contradicts Lemma 2.1). From $\mathrm{Inter}(\mathbf{E}, \mathbf{F}) \geq t$, we have $\{\{1\}, \{2\}, \ldots, \{i-1\}, V, \{i+1\}, \ldots, \{t\}\} \subseteq \mathbf{F}$. Thus, $\mathbf{F} = \mathbf{C}$, a contradiction. So, we may assume that $\{t+1, t+2, \ldots, n\} \notin \mathbf{F}$. Now, from $\mathrm{Inter}(\mathbf{B}, \mathbf{F}) \geq t$, we have $\{\{1\}, \{2\}, \ldots, \{t\}\} \subseteq \mathbf{F}$. This implies that $V \notin \mathbf{F}$ (otherwise both $\{i\}, V \in \mathbf{F}$, contradicting Lemma 2.1). From $\mathrm{Inter}(\mathbf{E}, \mathbf{F}) \geq t$, we have $\{t+1, t+2\} \in \mathbf{F}$ or $\{t+1, t+3, \ldots, n\} \in \mathbf{F}$. In either case, we always have $\{t+1\} \notin \mathbf{F}$.

Next, we claim that $\{j\} \notin \mathbf{F}$ for $t+1 \leq j \leq n$. Since $\{t+1\} \notin \mathbf{F}$ from the preceding paragraph, it remains to show that $\{j\} \notin \mathbf{F}$ for $t+2 \leq j \leq n$. For $t+2 \leq j \leq n$, let

$$\mathbf{G}_j = \{\{1\}, \{2\}, \ldots, \{t\}, \{t+1, j\}, \{t+2, t+3, \ldots, n\}\} \in \mathcal{D}_0(t).$$

Note that $\mathbf{G}_j \notin \mathcal{A}_1$ since $\mathrm{Inter}(\mathbf{G}_j, \mathbf{C}) = t-1$. Since $\mathbf{G}_j \in S_T(\mathcal{A}_1)$, there is $\mathbf{H}_j \in \mathcal{A}_1$ with $T \in \mathbf{H}_j$ such that $s_T(\mathbf{H}_j) = \mathbf{G}_j$. By Lemma 2.8, $\{t+1, j\}, \{t+2, t+3, \ldots, n\} \in \mathbf{H}_j$. From $\mathrm{Inter}(\mathbf{H}_j, \mathbf{C}) \geq t$, we must have

$$\begin{aligned}\mathbf{H}_j = \{&\{1\}, \{2\}, \ldots, \{i-1\}, V, \{i+1\}, \ldots, \{t\},\\ &\{t+1, j\}, \{t+2, t+3, \ldots, n\}\}.\end{aligned}$$

Now, $\mathrm{Inter}(\mathbf{H}_j, \mathbf{F}) \geq t$ implies that either $\{t+1, j\} \in \mathbf{F}$ or $\{t+2, t+3, \ldots, n\} \in \mathbf{F}$. Thus, $\{j\} \notin \mathbf{F}$ for $t+2 \leq j \leq n$; otherwise j would appear twice in $\mathbf{F}$, once in $\{j\}$ and once in either $\{t+1, j\}$ or $\{t+2, t+3, \ldots, n\}$ contradicting Lemma 2.1. Hence $\{j\} \notin \mathbf{F}$ for all $t+1 \leq j \leq n$.

For $t+1 \leq j \leq t+3$, let $Y_j = \{t+1, t+2, \ldots, n\} \setminus \{j\}$ and

$$\mathbf{Y}_j = \{\{1\}, \{2\}, \ldots, \{t\}, \{j\}, Y_j\} \in \mathcal{D}_0(t).$$

Now, $\mathbf{Y}_j \notin \mathcal{A}_1$ since $\mathrm{Inter}(\mathbf{Y}_j, \mathbf{C}) = t-1$. Therefore, there exists $\mathbf{Z}_j \in \mathcal{A}_1$ with $T \in \mathbf{Z}_j$ and $s_T(\mathbf{Z}_j) = \mathbf{Y}_j$. By Lemma 2.8, $Y_j \in \mathbf{Z}_j$. Moreover, $|Y_j| \geq 2$ since $n - t \geq 3$. From $\mathrm{Inter}(\mathbf{Z}_j, \mathbf{C}) \geq t$, we must have

$$\{\{1\}, \{2\}, \ldots, \{i-1\}, V, \{i+1\}, \ldots, \{t\}\} \subseteq \mathbf{Z}_j.$$

Therefore,

$$\mathbf{Z}_j = \begin{cases} \{\{1\}, \{2\}, \ldots, \{i-1\}, V, \{i+1\}, \ldots, \{t\}, Y_j\}, & \text{if } j \in V; \\ \{\{1\}, \{2\}, \ldots, \{i-1\}, V, \{i+1\}, \ldots, \{t\}, \{j\}, Y_j\}, & \text{if } j \notin V. \end{cases}$$

If $j \in V$, then $\mathrm{Inter}(\mathbf{Z}_j, \mathbf{F}) \geq t$ implies that $Y_j \in \mathbf{F}$. If $j \notin V$, then $\mathrm{Inter}(\mathbf{Z}_j, \mathbf{F}) \geq t$ implies that either $\{j\} \in \mathbf{F}$ or $Y_j \in \mathbf{F}$. Since $\{j\} \notin \mathbf{F}$ for $t+1 \leq j \leq n$, we can only have $Y_j \in \mathbf{F}$. Hence, $Y_j \in \mathbf{F}$ in all $t+1 \leq j \leq t+3$. In particular, we have $Y_{t+1} = \{t+2, t+3, \ldots, n\}$, $Y_{t+2} = \{t+1, t+3, \ldots, n\}$, $Y_{t+3} = \{t+1, t+2, t+4 \ldots, n\} \in \mathbf{F}$ and this contradicts Lemma 2.1, because every element in $\{t+2, t+3, \ldots, n\}$ appears at least 2 times in $\mathbf{F}$.

We conclude that no such $\mathbf{F}$ exists. This contradiction shows that $\mathcal{A}_2 = \mathcal{D}_0(t)$. Consequently, $\mathbf{B} \in \mathcal{A}_2$ and thus $\mathbf{B} \in \mathcal{A}_1 \cap \mathcal{A}_2$. By repeating the above argument starting with $\mathbf{B} \in \mathcal{A}_2$, we deduce that $\mathcal{A}_1 = \mathcal{D}_0(t)$. Hence, Claim 1 is proved.

We now proceed to prove the lemma. If $\mathbf{B} \in \mathcal{A}_1$, then the result of the lemma holds by Claim 1. So we may suppose that $\mathbf{B} \notin \mathcal{A}_1$.

Then there exists $\mathbf{Q} \in \mathcal{A}_1$ with $s_T(\mathbf{Q}) = \mathbf{B}$. Note that $T \in \mathbf{Q}$ and

$$\mathbf{B} = \mathbf{Q} \setminus \{T\} \cup P(T, \mathbf{Q}).$$

By Lemma 2.8, $\{t+1, t+2, \ldots, n\} \in \mathbf{Q}$ and $\{t+1, t+2, \ldots, n\} \nsubseteq T$. Note that $P(T, \mathbf{Q}) \subseteq \{\{1\}, \{2\}, \ldots, \{t\}\}$. If $|P(T, \mathbf{Q})| \geq 2$, then $|\mathbf{Q} \setminus \{T\}| \leq t-1$ and $|\mathbf{Q}| \leq t$. Since $(\mathcal{A}_1, \mathcal{A}_2) \in I(n, t)$, we must have $\mathcal{A}_2 = \{\mathbf{Q}\}$, contradicting $|\mathcal{A}_2| > 1$. Thus, $|P(T, \mathbf{Q})| = 1$ and $P(T, \mathbf{Q}) = \{\{j_0\}\}$ for some $1 \leq j_0 \leq t$.

Suppose $|T| \geq 3$. Then $T = \{j_0\} \cup X$ for some $X \subseteq \{t+1, t+2, \ldots, n\}$ with $|X| \geq 2$. Let $Y = \{t+1, t+2, \ldots, n\} \setminus X$ and

$$\mathbf{R} = \{\{1\}, \{2\}, \ldots, \{t\}, X, Y\}.$$

Note that $\mathbf{R} \notin \mathcal{A}_2$ since $\mathrm{Inter}(\mathbf{R}, \mathbf{Q}) = t-1$. In fact $\mathbf{Q} \cap \mathbf{R} = \{\{1\}, \ldots, \{j_0 - 1\}, \{j_0+1\}, \ldots, \{t\}\}$. Since $\mathbf{R} \in S_T(\mathcal{A}_2) = \mathcal{D}_0(t)$, there exists $\mathbf{S} \in \mathcal{A}_2$ with $T \in \mathbf{S}$ and $s_T(\mathbf{S}) = \mathbf{R}$. By Lemma 2.8, $X \in \mathbf{S}$ and $X \nsubseteq T$, a contradiction. Hence, $|T| = 2$ and $T = \{i_0, j_0\}$ or some $i_0 \in \{t+1, t+2, \ldots, n\}$. Subsequently, from $s_T(\mathbf{Q}) = \mathbf{B}$, we deduce that

$$\mathbf{Q} = \{\{1\}, \{2\}, \ldots, \{j_0 - 1\}, T = \{i_0, j_0\}, \{j_0+1\}, \ldots, \{t\}, \{t+1, t+2, \ldots, n\}\},$$

and $|\mathbf{Q}| = t+1$.

Since $\mathbf{B} \notin \mathcal{A}_1$ and $\mathbf{Q} \in \mathcal{A}_1$, it follows from Claim 1 that

$$\begin{aligned} \mathbf{B} =& \{\{1\}, \{2\}, \ldots, \{t\}, \{t+1, t+2, \ldots, n\}\} \notin \mathcal{A}_i, \quad \text{and} \\ \mathbf{Q} =& \{\{1\}, \{2\}, \ldots, \{j_0 - 1\}, T = \{i_0, j_0\}, \{j_0+1\}, \ldots, \{t\}, \end{aligned}$$

$$\{t+1, t+2, \ldots, n\}\} \in \mathcal{A}_i \quad \text{for } i = 1, 2.$$

Let $\mathbf{U} \in \mathcal{A}_2 \setminus \{\mathbf{Q}\}$. If $T \notin \mathbf{U}$, then $s_T(\mathbf{U}) = \mathbf{U} \in \mathcal{D}_0(t)$ and so $\{\{1\}, \{2\}, \ldots, \{t\}\} \subseteq \mathbf{U}$. Next, $\mathrm{Inter}(\mathbf{U}, \mathbf{Q}) \geq t$ implies that $\{t+1, t+2, \ldots, n\} \in \mathbf{U}$. Thus, $\mathbf{U} = \mathbf{B} \in \mathcal{A}_2$, a contradiction. Hence $T \in \mathbf{U}$.

Suppose $\{k\} \notin \mathbf{U}$ for some $k \in \{1, 2, \ldots, t\} \setminus \{j_0\}$. Then there is a set $K \in \mathbf{U}$ with $|K| \geq 2$ and $k \in K$. Since $K \neq T$, we have $K \in s_T(\mathbf{U})$. Also, since $s_T(\mathbf{U}) \in \mathcal{D}_0(t)$, we have $\{k\} \in s_T(\mathbf{U})$. This contradicts Lemma 2.1 because k appears twice in $s_T(\mathbf{U})$ (once in $\{k\}$ and once in K). Hence, $\{k\} \in \mathbf{U}$ for all $k \in \{1, 2, \ldots, t\} \setminus \{j_0\}$. This implies that every element $\mathbf{U} \in \mathcal{A}_2$ is of the form

$$\{\{1\}, \{2\}, \ldots, \{j_0 - 1\}, T = \{i_0, j_0\}, \{j_0 + 1\}, \ldots, \{t\}\} \cup \mathbf{W},$$

where $\mathbf{W}$ is a minimal cover of $[n] \setminus [t]$ with $\{i_0\} \notin \mathbf{W}$. Therefore, $\mathcal{A}_2 \subseteq \mathcal{D}$, where $\mathcal{D} \cong \mathcal{D}_1(t)$. In fact, since not all elements in $\mathcal{D}_1(t)$ are minimal covers, we have $\mathcal{A}_2 \subseteq \mathcal{D} \cap \mathcal{C}(n)$ and so it follows from (2) that

$$|\mathcal{A}_2| \leq |\mathcal{D}_1(t) \cap \mathcal{C}(n)| < C_{n-t},$$

contradicting the assumption that $|\mathcal{A}_2| \geq |\mathcal{D}_0(t)| = C_{n-t}$.

This completes the proof of the lemma. □

3. Proof of Theorem 1.4

For each $Z \subseteq [n]$, let $\widetilde{\mathcal{C}}(Z) = \{\mathbf{A} \in \mathcal{C}(Z) : \mathbf{A}$ does not contain any singleton$\}$. When $Z = [n]$, we shall write $\widetilde{\mathcal{C}}(n)$ instead of $\widetilde{\mathcal{C}}([n])$. Let $\widetilde{C}_n = |\widetilde{\mathcal{C}}(n)|$.

Lemma 3.1. *Let $n \geq 2$. Then*

$$C_n = \sum_{k=0}^{n} \binom{n}{k} \widetilde{C}_{n-k}, \tag{3}$$

$$\widetilde{C}_n \geq \sum_{k=1}^{n-1} \binom{n-1}{k} \widetilde{C}_{n-1-k}, \tag{4}$$

with the conventions $C_0 = \widetilde{C}_0 = 1$.

Proof. Let $T \subseteq [n]$ and $\mathcal{C}(n)(T)$ be the set of all $\mathbf{A} \in \mathcal{C}(n)$ such that the only singletons in $\mathbf{A}$ are those in T, i.e.,

$$\mathcal{C}(n)(T) = \{\mathbf{A} \in \mathcal{C}(n) : \{x\} \in \mathbf{A} \text{ if and only if } x \in T\}.$$

Note that if $\mathbf{A} \in \mathcal{C}(n)(T)$, then every $x \in T$ is critical in $\mathbf{A}$ (Lemma 2.1). Therefore, $\mathbf{A} \setminus \{\{x\} : x \in T\} \in \widetilde{\mathcal{C}}([n] \setminus T)$. Hence, $|\mathcal{C}(n)(T)| = \widetilde{C}_{n-|T|}$.

Note that $\bigcup_{T \subseteq [n]} \mathcal{C}(n)(T) \subseteq \mathcal{C}(n)$. Now, for each $\mathbf{A}_0 \in \mathcal{C}(n)$, there is a $T_0 \subseteq [n]$ such that $\{x\} \in \mathbf{A}$ if and only if $x \in T_0$. Thus, $\mathbf{A}_0 \in \mathcal{C}(n)(T_0)$ and $\bigcup_{T \subseteq [n]} \mathcal{C}(n)(T) = \mathcal{C}(n)$.

Note that $\mathcal{C}(n)(T) \cap \mathcal{C}(n)(T') = \varnothing$ for $T \neq T'$. So,

$$C_n = |\mathcal{C}(n)| = \left| \bigcup_{T \subseteq [n]} \mathcal{C}(n)(T) \right| = \sum_{k=0}^{n} \binom{n}{k} \widetilde{C}_{n-k},$$

proving (3).

Let $T \subseteq [n-1]$, $|T| \geq 1$ and $V(T)$ be the set of all $\mathbf{A} \in \mathcal{C}(n-1)$ such that $T \in \mathbf{A}$ and $\mathbf{A} \setminus \{T\}$ is a minimal cover of $[n-1] \setminus T$ that does not contain any singletons, i.e.,

$$V(T) = \{\mathbf{A} \in \mathcal{C}(n-1) : T \in \mathbf{A} \text{ and } \mathbf{A} \setminus \{T\} \in \widetilde{\mathcal{C}}([n-1] \setminus T)\}.$$

Then $|V(T)| = \widetilde{C}_{n-1-|T|}$. Let

$$\overline{V}(T) = \{(\mathbf{A} \setminus \{T\}) \cup \{\{T \cup \{n\}\}\} : \mathbf{A} \in V(T)\}.$$

Note that $\overline{V}(T) \subseteq \widetilde{\mathcal{C}}(n)$ and $|\overline{V}(T)| = |V(T)| = \widetilde{C}_{n-1-|T|}$. Furthermore, $\overline{V}(T) \cap \overline{V}(T') = \varnothing$ for $T \neq T'$. So, from $\bigcup_{T \subseteq [n-1], |T| \geq 1} \overline{V}(T) \subseteq \widetilde{\mathcal{C}}(n)$, we have

$$\sum_{k=1}^{n-1} \binom{n-1}{k} \widetilde{C}_{n-1-k} = \left| \bigcup_{T \subseteq [n-1], |T| \geq 1} \overline{V}(T) \right| \leq |\widetilde{\mathcal{C}}(n)| = \widetilde{C}_n,$$

proving (4). □

Given a real number x, we shall denote the greatest integer less than or equal to x, by $\lfloor x \rfloor$. Note that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

Lemma 3.2. *Given any positive integers m, c and t with $m \geq 2$, there is a positive integer $n_0 = n_0(m, c, t)$ depending on m, c and t, such that for $n \geq n_0$,*

$$\widetilde{C}_{n-t} > c^n \sum_{\lfloor \frac{n}{m} \rfloor \leq k \leq n} \binom{n}{k} \widetilde{C}_{n-k}.$$

Proof. Since $\widetilde{C}_{n-\lfloor n/m \rfloor + 2} \geq \widetilde{C}_{n-k}$ for all $\lfloor n/m \rfloor \leq k \leq n$, we have

$$\begin{aligned} \sum_{\lfloor \frac{n}{m} \rfloor \leq k \leq n} \binom{n}{k} \widetilde{C}_{n-k} &\leq \widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor + 2} \sum_{\lfloor \frac{n}{m} \rfloor \leq k \leq n} \binom{n}{k} \\ &\leq 2^n \widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor + 2}. \end{aligned}$$

So, it is sufficient to show that $\widetilde{C}_{n-t} / \widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor + 2} > (2c)^n$.

Now, $n - \lfloor \frac{n}{m} \rfloor + 4 > (2c)^{4m} + 1$ provided that $n \geq \frac{m}{m-1}(2c)^{4m}$. So, by (4), $\widetilde{C}_l / \widetilde{C}_{l-2} \geq l - 1 > (2c)^{4m}$ for $l \geq n - \lfloor \frac{n}{m} \rfloor + 4$. Therefore,

$$\begin{aligned} \frac{\widetilde{C}_{n-t}}{\widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor + 2}} &\geq \left(\frac{\widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor + 2u}}{\widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor + 2u - 2}} \right) \cdots \left(\frac{\widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor + 6}}{\widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor + 4}} \right) \left(\frac{\widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor + 4}}{\widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor + 2}} \right) \\ &> ((2c)^{4m})^{u-1}, \end{aligned}$$

where $u = \lfloor \frac{1}{2}(\lfloor \frac{n}{m} \rfloor - t - 2) \rfloor$. Note that $u - 1 \geq \frac{1}{2}(\frac{n}{m} - t - 3) - 2 \geq \frac{n}{4m}$ provided that $n \geq 2m(t+7)$. Hence, for sufficiently large n, $\widetilde{C}_{n-t}/\widetilde{C}_{n-\lfloor \frac{n}{m} \rfloor+2} > (2c)^n$. $\square$

Let $\mathcal{A} \subseteq \mathcal{C}(n)$ be compressed. Recall that $\gamma(\mathcal{A}) = \{\gamma(\mathbf{A}) : \mathbf{A} \in \mathcal{A}\}$, where $\gamma(\mathbf{A})$ is the union of all the singletons in $\mathbf{A}$. We say $\gamma(\mathcal{A})$ is *trivial* if there is a fixed t-set, say T, such that $T \subseteq \gamma(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{A}$.

Lemma 3.3. *There is a positive integer $n_0 = n_0(t)$ depending on t, such that for $n \geq n_0$, if $\mathcal{A} \subseteq \mathcal{C}(n)$ is compressed and is restricted t-intersecting, and $\gamma(\mathcal{A})$ is non-trivial, then*

$$|\mathcal{A}| < C_{n-t}.$$

Proof. Note that $(\mathcal{A}, \mathcal{A}) \in I(n,t)$. By Lemma 2.6, $\gamma(\mathcal{A})$ and $\gamma(\mathcal{A})$ are cross t-intersecting, i.e., $\gamma(\mathcal{A})$ is t-intersecting. For $k \geq t$, let $\mathcal{F}_k = \gamma(\mathcal{A}) \cap \binom{[n]}{k}$. Then $\mathcal{F}_k$ is t-intersecting. If $\mathcal{F}_t \neq \varnothing$, then $\gamma(\mathcal{A})$ is trivial. So, we may assume that $\mathcal{F}_t = \varnothing$. By using Lemma 2.1, it is not hard to see that for each $\mathbf{A} \in \mathcal{A}$,

$$\mathbf{A} \setminus \{\{x\} \ : \ x \in \gamma(\mathbf{A})\} \in \widetilde{\mathcal{C}}([n] \setminus \gamma(\mathbf{A})).$$

Therefore,

$$\begin{aligned} |\mathcal{A}| &\leq \sum_{t+1 \leq k \leq n} |\mathcal{F}_k| \widetilde{C}_{n-k} \\ &= \sum_{t+1 \leq k \leq \lfloor \frac{n}{t+1} + t - 1 \rfloor} |\mathcal{F}_k| \widetilde{C}_{n-k} + \sum_{\lfloor \frac{n}{t+1} + t - 1 \rfloor + 1 \leq k \leq n} |\mathcal{F}_k| \widetilde{C}_{n-k}. \end{aligned}$$

By Theorem 1.1, $|\mathcal{F}_k| \leq \binom{n-t}{k-t}$ for $t+1 \leq k \leq \left\lfloor \frac{n}{t+1} + t - 1 \right\rfloor$. Therefore,

$$\begin{aligned} \sum_{t+1 \leq k \leq \lfloor \frac{n}{t+1} + t - 1 \rfloor} |\mathcal{F}_k| \widetilde{C}_{n-k} &\leq \sum_{t+1 \leq k \leq \lfloor \frac{n}{t+1} + t - 1 \rfloor} \binom{n-t}{k-t} \widetilde{C}_{n-k} \\ &= \sum_{1 \leq k \leq \lfloor \frac{n}{t+1} + t - 1 \rfloor - t} \binom{n-t}{k} \widetilde{C}_{n-t-k} \\ &\leq \sum_{1 \leq k \leq n-t} \binom{n-t}{k} \widetilde{C}_{n-t-k} \\ &= C_{n-t} - \widetilde{C}_{n-t}, \end{aligned}$$

where the last equality follows from equation (3).

On the other hand, $|\mathcal{F}_k| \leq \binom{n}{k}$ for $\left\lfloor \frac{n}{t+1} + t - 1 \right\rfloor + 1 \leq k \leq n$. Therefore,

$$\sum_{\lfloor \frac{n}{t+1} + t - 1 \rfloor + 1 \leq k \leq n} |\mathcal{F}_k| \widetilde{C}_{n-k} \leq \sum_{\lfloor \frac{n}{t+1} + t - 1 \rfloor + 1 \leq k \leq n} \binom{n}{k} \widetilde{C}_{n-k}$$

$$\leq \sum_{\lfloor \frac{n}{t+1} \rfloor \leq k \leq n} \binom{n}{k} \widetilde{C}_{n-k}$$
$$< \widetilde{C}_{n-t},$$

where the last inequality follows from Lemma 3.2 for sufficiently large n in terms of t. Hence, $|\mathcal{A}| < C_{n-t}$. □

Proof of Theorem 1.4. Note that $(\mathcal{D}_0(t), \mathcal{D}_0(t)) \in I(n,t)$ and $|\mathcal{D}_0(t)| = C_{n-t}$ for $0 \leq l \leq t$. Let $\mathcal{A}$ be restricted t-intersecting of maximum size. Then $(\mathcal{A}, \mathcal{A}) \in I(n,t)$ and $|\mathcal{A}| \geq C_{n-t}$. Repeatedly apply the splitting operations until we obtain a compressed $\mathcal{A}^*$ with $|\mathcal{A}^*| = |\mathcal{A}|$ and $(\mathcal{A}^*, \mathcal{A}^*) \in I(n,t)$ (Lemma 2.5). By Lemma 2.6, $\gamma(\mathcal{A}^*)$ is t-intersecting. If $\gamma(\mathcal{A}^*)$ is non-trivial, then by Lemma 3.3, $|\mathcal{A}| = |\mathcal{A}^*| < C_{n-t}$, a contradiction. Hence, $\gamma(\mathcal{A}^*)$ is trivial.

Let $T = \{x_1, \ldots, x_t\}$ be the t-set such that $T \subseteq \gamma(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{A}^*$. Let σ be a permutation of $[n]$ with $\sigma(x_i) = i$ for all i. Then $\sigma(\mathcal{A}^*) \subseteq \mathcal{D}_0(t)$. Since $|\sigma(\mathcal{A}^*)| = |\mathcal{A}^*| \geq |\mathcal{D}_0(t)|$, we deduce that $\sigma(\mathcal{A}^*) = \mathcal{D}_0(t)$. By Lemma 2.9, we conclude that $\mathcal{A} \cong \mathcal{D}_0(t)$.

This completes the proof of Theorem 1.4. □

4. Proof of Theorem 1.6

Let

$$\mathcal{C}_k(n) = \{\mathbf{A} \in \mathcal{C}(n) : |\mathbf{A}| = k\},$$

and $C_{n,k} = |\mathcal{C}_k(n)|$. Clearly,

$$C_n = \sum_{k=1}^{n} C_{n,k}. \tag{5}$$

Lemma 4.1. *For $n \geq 1$, $\widetilde{C}_{n+1} \leq 2^{n+1} C_n$.*

Proof. We first define a function $f : \widetilde{\mathcal{C}}(n+1) \to \mathcal{C}(n)$. Let $\mathbf{A} = \{A_1, A_2, \ldots, A_k\} \in \widetilde{\mathcal{C}}(n+1)$. If $\{A_1 \setminus \{n+1\}, A_2 \setminus \{n+1\}, \ldots, A_k \setminus \{n+1\}\} \in \mathcal{C}(n)$, then we say $\mathbf{A}$ is of Type I, otherwise, we say $\mathbf{A}$ is of Type II.

If $\mathbf{A}$ is of Type I, then we set

$$f(\mathbf{A}) = \{A_1 \setminus \{n+1\}, A_2 \setminus \{n+1\}, \ldots, A_k \setminus \{n+1\}\}.$$

By Lemma 2.1, every set A_i contains a critical element in $\mathbf{A}$. Furthermore, if every A_i contains a critical element different from $n+1$, then $\{A_1 \setminus \{n+1\}, A_2 \setminus \{n+1\}, \ldots, A_k \setminus \{n+1\}\} \in \mathcal{C}(n)$. So, $\{A_1 \setminus \{n+1\}, A_2 \setminus \{n+1\}, \ldots, A_k \setminus \{n+1\}\} \notin \mathcal{C}(n)$ if and only if there exists a unique $i_0 \in \{1, \ldots, k\}$ such that $n+1$ is the only critical element contained in A_{i_0}, in which case we have $A_{i_0} \setminus \{n+1\} \subseteq A_1 \cup \cdots \cup A_{i_0-1} \cup A_{i_0+1} \cup \cdots \cup A_k$ and $\{A_1, \ldots, A_{i_0-1}, A_{i_0+1}, \ldots, A_k\} \in \mathcal{C}(n)$. Therefore, if $\mathbf{A}$ is of Type II, then we set

$$f(\mathbf{A}) = \{A_1, \ldots, A_{i_0-1}, A_{i_0+1}, \ldots, A_k\},$$

which is well-defined by the uniqueness of i_0.

Let $\mathbf{B} = \{B_1, B_2, \ldots, B_k\} \in \mathcal{C}_k(n)$. Consider $\overline{\mathbf{B}} = \{\overline{B}_1, \overline{B}_2, \ldots, \overline{B}_k\}$ where $\overline{B}_i = B_i \cup \{n+1\}$ if $|B_i| = 1$, and $\overline{B}_i = B_i \cup \{n+1\}$ or B_i if $|B_i| \neq 1$. Note that $\overline{\mathbf{B}} = \{\overline{B}_1, \overline{B}_2, \ldots, \overline{B}_k\} \in \widetilde{\mathcal{C}}(n+1)$ and $f(\overline{\mathbf{B}}) = \mathbf{B}$. Therefore, the number of Type I minimal covers in $f^{-1}(\mathbf{B})$ is at most $2^k \leq 2^n$.

Let $\mathbf{C} \in f^{-1}(\mathbf{B})$ be of Type II. Then $|B_i| \geq 2$ for $1 \leq i \leq k$ and $\mathbf{C} = \{B_0, B_1, B_2, \ldots, B_k\}$ where $B_0 = A \cup \{n+1\}$, $A \subseteq [n]$ and $A \neq \varnothing$. So, the number of Type II minimal covers in $f^{-1}(\mathbf{B})$ is at most 2^n. Hence, $|f^{-1}(\mathbf{B})| \leq 2^n + 2^n = 2^{n+1}$.

Note that

$$\begin{aligned}
\widetilde{C}_{n+1} &= \sum_{k=1}^{n} \sum_{\mathbf{B} \in \mathcal{C}_k(n)} f^{-1}(\mathbf{B}) \\
&\leq 2^{n+1} \sum_{k=1}^{n} \sum_{\mathbf{B} \in \mathcal{C}_k(n)} 1 \\
&= 2^{n+1} \sum_{k=1}^{n} C_{n,k} = 2^{n+1} C_n \qquad \text{(by equation (5))}.
\end{aligned}$$

□

Let $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{C}(n)$ be compressed. We say that $(\gamma(\mathcal{A}_1), \gamma(\mathcal{A}_2))$ is *trivial* if there exists $x \in [n]$, such that $x \in \gamma(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{A}_1$ and $\mathbf{A} \in \mathcal{A}_2$.

Lemma 4.2. *There is a positive integer n_0, such that for $n \geq n_0$, if $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{C}(n)$ are compressed, $(\mathcal{A}_1, \mathcal{A}_2) \in I(n,1)$, and $(\gamma(\mathcal{A}_1), \gamma(\mathcal{A}_2))$ is non-trivial, then*

$$|\mathcal{A}_1||\mathcal{A}_2| < C_{n-1}^2.$$

Proof. For $1 \leq i \leq 2$ and $k \geq 1$, let $\mathcal{F}_{ik} = \gamma(\mathcal{A}_i) \cap \binom{[n]}{k}$. By Lemma 2.6, $\gamma(\mathcal{A}_1), \gamma(\mathcal{A}_2)$ are cross intersecting. Therefore, if $\mathcal{F}_{i1} \neq \varnothing$ for $i = 1, 2$, then $(\gamma(\mathcal{A}_1), \gamma(\mathcal{A}_2))$ is trivial. So, we may assume that $\mathcal{F}_{21} = \varnothing$. By using Lemma 2.1, it is not hard to see that for each $\mathbf{A} \in \mathcal{A}_i$,

$$\mathbf{A} \setminus \{\{x\} \ : \ x \in \gamma(\mathbf{A})\} \in \widetilde{\mathcal{C}}([n] \setminus \gamma(\mathbf{A})).$$

Therefore, $|\mathcal{A}_1| \leq \sum_{1 \leq k \leq n} |\mathcal{F}_{1k}| \widetilde{C}_{n-k}$ and $|\mathcal{A}_2| \leq \sum_{2 \leq k \leq n} |\mathcal{F}_{2k}| \widetilde{C}_{n-k}$. So,

$$\begin{aligned}
|\mathcal{A}_1| &\leq \sum_{1 \leq k < \lfloor \frac{n}{2} \rfloor} |\mathcal{F}_{1k}| \widetilde{C}_{n-k} + \sum_{\lfloor \frac{n}{2} \rfloor \leq k \leq n} |\mathcal{F}_{1k}| \widetilde{C}_{n-k} \\
&\leq \sum_{1 \leq k < \lfloor \frac{n}{2} \rfloor} |\mathcal{F}_{1k}| \widetilde{C}_{n-k} + \sum_{\lfloor \frac{n}{2} \rfloor \leq k \leq n} \binom{n}{k} \widetilde{C}_{n-k},
\end{aligned}$$

and

$$|\mathcal{A}_2| \leq \sum_{2 \leq k < \lfloor \frac{n}{2} \rfloor} |\mathcal{F}_{2k}| \widetilde{C}_{n-k} + \sum_{\lfloor \frac{n}{2} \rfloor \leq k \leq n} \binom{n}{k} \widetilde{C}_{n-k}.$$

Let

$$Q = \sum_{\lfloor \frac{n}{2} \rfloor \le k \le n} \binom{n}{k} \widetilde{C}_{n-k},$$
$$M_1 = \sum_{1 \le k < \lfloor \frac{n}{2} \rfloor} |\mathcal{F}_{1k}| \widetilde{C}_{n-k},$$
$$M_2 = \sum_{2 \le k < \lfloor \frac{n}{2} \rfloor} |\mathcal{F}_{2k}| \widetilde{C}_{n-k}.$$

Then

$$\begin{aligned} |\mathcal{A}_1||\mathcal{A}_2| &\le (M_1 + Q)(M_2 + Q) \\ &= M_1 M_2 + M_1 Q + M_2 Q + Q^2. \end{aligned}$$

By equation (3), $\widetilde{C}_{n-1} \le C_{n-1}$. By Lemma 4.1, $\widetilde{C}_n \le 2^n C_{n-1}$. By equation (4), for $2 \le k < \lfloor \frac{n}{2} \rfloor$, $\widetilde{C}_{n-k} \le \widetilde{C}_n \le 2^n C_{n-1}$. Therefore,

$$\begin{aligned} M_1 &\le C_{n-1}|\mathcal{F}_{11}| + 2^n C_{n-1} \sum_{2 \le k < \lfloor \frac{n}{2} \rfloor} |\mathcal{F}_{1k}| \\ &\le 2^{n+1} C_{n-1} \sum_{1 \le k < \lfloor \frac{n}{2} \rfloor} |\mathcal{F}_{1k}| \\ &\le 2^{n+1} C_{n-1} \sum_{1 \le k < \lfloor \frac{n}{2} \rfloor} \binom{n}{k} \\ &\le 2^{2n+1} C_{n-1}. \end{aligned}$$

Similarly, $M_2 \le 2^{2n} C_{n-1}$.

By Lemma 3.2, $Q \le \frac{1}{8^n} \widetilde{C}_{n-1} \le \frac{1}{8^n} C_{n-1} < C_{n-1}$. Therefore

$$\begin{aligned} M_1 Q + M_2 Q + Q^2 &< (2^{2n+1} + 2^{2n} + 1) C_{n-1} \left(\frac{\widetilde{C}_{n-1}}{8^n} \right) \\ &< 3(2^{2n+1}) C_{n-1} \left(\frac{\widetilde{C}_{n-1}}{8^n} \right) \\ &= \frac{6}{2^n} C_{n-1} \widetilde{C}_{n-1} \\ &< \frac{1}{2} C_{n-1} \widetilde{C}_{n-1}. \end{aligned}$$

By Theorem 1.2,

$$M_1 M_2 \le \sum_{\substack{1 \le k_1 < \lfloor \frac{n}{2} \rfloor, \\ 2 \le k_2 < \lfloor \frac{n}{2} \rfloor}} \binom{n-1}{k_1 - 1} \binom{n-1}{k_2 - 1} \widetilde{C}_{n-k_1} \widetilde{C}_{n-k_2}$$

$$= \left(\sum_{1 \le k < \lfloor \frac{n}{2} \rfloor} \binom{n-1}{k-1} \widetilde{C}_{n-k} \right) \left(\sum_{2 \le k < \lfloor \frac{n}{2} \rfloor} \binom{n-1}{k-1} \widetilde{C}_{n-k} \right).$$

By equation (3),

$$\sum_{1 \le k < \lfloor \frac{n}{2} \rfloor} \binom{n-1}{k-1} \widetilde{C}_{n-k} \le C_{n-1},$$

$$\sum_{2 \le k < \lfloor \frac{n}{2} \rfloor} \binom{n-1}{k-1} \widetilde{C}_{n-k} \le C_{n-1} - \widetilde{C}_{n-1}.$$

Hence, $M_1 M_2 \le (C_{n-1} - \widetilde{C}_{n-1}) C_{n-1}$, and

$$\begin{aligned} |\mathcal{A}_1||\mathcal{A}_2| &< C_{n-1}^2 - \widetilde{C}_{n-1} C_{n-1} + \frac{1}{2}(C_{n-1})\widetilde{C}_{n-1} \\ &< C_{n-1}^2. \end{aligned}$$

□

Lemma 4.3. *There exists a constant n_0, such that for all $n \ge n_0$, if $(\mathcal{A}_1, \mathcal{A}_2) \in I(n,1)$, then*

$$|\mathcal{A}_1||\mathcal{A}_2| \le C_{n-1}^2.$$

Moreover, equality holds if and only if $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{A}_1 \cong \mathcal{D}_0(1)$.

Proof. Note that $(\mathcal{D}_0(1), \mathcal{D}_0(1)) \in I(n,1)$ and $|\mathcal{D}_0(1)| = C_{n-1}$. Let $(\mathcal{A}_1, \mathcal{A}_2) \in I(n,1)$ such that $|\mathcal{A}_1||\mathcal{A}_2|$ is maximum. Then $|\mathcal{A}_1||\mathcal{A}_2| \ge C_{n-1}^2$. Repeatedly apply the splitting operations until we obtain compressed families $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ with $|\mathcal{A}_1^*| = |\mathcal{A}_1|$, $|\mathcal{A}_2^*| = |\mathcal{A}_2|$, and $(\mathcal{A}_1^*, \mathcal{A}_2^*) \in I(n,t)$ (Lemma 2.5). By Lemma 2.6, $\gamma(\mathcal{A}_1^*)$ and $\gamma(\mathcal{A}_2^*)$ are cross intersecting. If $(\gamma(\mathcal{A}_1), \gamma(\mathcal{A}_2))$ is non-trivial, then by Lemma 3.3, $|\mathcal{A}_1||\mathcal{A}_2| = |\mathcal{A}_1^*||\mathcal{A}_2^*| < C_{n-1}^2$, a contradiction. Hence, $(\gamma(\mathcal{A}_1), \gamma(\mathcal{A}_2))$ is trivial.

Let $x \in [n]$ be such that $x \in \gamma(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{A}_1^*$ and $\mathbf{A} \in \mathcal{A}_2^*$. Let σ be a permutation of $[n]$ with $\sigma(x) = 1$. Then $\sigma(\mathcal{A}_i^*) \subseteq \mathcal{D}_0(1)$ for $i = 1, 2$. This implies that $|\mathcal{A}_i^*| = |\sigma(\mathcal{A}_i^*)| \le |\mathcal{D}_0(1)| = C_{n-1}$. Since $C_{n-1}^2 \le |\mathcal{A}_1||\mathcal{A}_2| = |\mathcal{A}_1^*||\mathcal{A}_2^*| \le C_{n-1}^2$, we must have $\sigma(\mathcal{A}_i^*) = \mathcal{D}_0(1)$. By Lemma 2.9, we conclude that $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{A}_1 \cong \mathcal{D}_0(1)$. □

Proof of Theorem 1.6. Note that if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$ are restricted r-cross intersecting, then for any i, j with $i \ne j$, we have $\mathcal{A}_i, \mathcal{A}_j$ are restricted cross-intersecting. By Lemma 4.3,

$$|\mathcal{A}_i||\mathcal{A}_j| \le C_{n-1}^2. \tag{6}$$

Therefore,

$$\begin{aligned} \left(\prod_{i=1}^{r} |\mathcal{A}_i| \right)^{r-1} &= \prod_{1 \le i < j \le r} |\mathcal{A}_i||\mathcal{A}_j| \\ &\le \prod_{1 \le i < j \le r} C_{n-1}^2 \end{aligned}$$

$$= C_{n-1}^{r(r-1)}.$$

This proves the first part of Theorem 1.6.

Suppose equality holds. Then in equation (6), we must have

$$|\mathcal{A}_i||\mathcal{A}_j| = C_{n-1}^2.$$

By Lemma 4.3, $\mathcal{A}_i = \mathcal{A}_j$ and $\mathcal{A}_i \cong \mathcal{D}_0(1)$. This completes the proof of Theorem 1.6. □

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