AN ERDŐS-KO-RADO THEOREM FOR MINIMAL COVERS

CHENG YEAW KW AND KOK BIN WONG

Abstract. Let \([n] = \{1, 2, \ldots, n\}\). A set \(A = \{A_1, A_2, \ldots, A_l\}\) is a minimal cover of \([n]\) if \(\bigcup_{1 \leq i \leq l} A_i = [n]\) and \(\bigcup_{1 \leq i \leq l, i \neq j_0} A_i \neq [n]\) for all \(j_0 \in [l]\).

Let \(C(n)\) denote the collection of all minimal covers of \([n]\), and write \(C_n = |C(n)|\). Let \(A \in C(n)\). An element \(u \in [n]\) is critical in \(A\) if it appears exactly once in \(A\). Two minimal covers \(A, B \in C(n)\) are said to be restricted \(t\)-intersecting if they share at least \(t\) sets each containing an element which is critical in both \(A\) and \(B\).

A family \(A \subseteq C(n)\) is said to be restricted \(t\)-intersecting if every pair of distinct elements in \(A\) are restricted \(t\)-intersecting. In this paper, we prove that there exists a constant \(n_0 = n_0(t)\) depending on \(t\), such that for all \(n \geq n_0\), if \(A \subseteq C(n)\) is restricted \(t\)-intersecting, then \(|A| \leq C_n - t\). Moreover, the bound is attained if and only if \(A\) is isomorphic to the family \(D_0(t)\) consisting of all minimal covers which contain the singleton parts \(\{1\}, \ldots, \{t\}\). A similar result also holds for restricted \(r\)-cross intersecting families of minimal covers.

1. Introduction

Let \([n] = \{1, \ldots, n\}\), and let \(\binom{[n]}{k}\) denote the family of all \(k\)-subsets of \([n]\). A family \(A\) of subsets of \([n]\) is \(t\)-intersecting if \(|A \cap B| \geq t\) for all \(A, B \in A\). One of the most beautiful results in extremal combinatorics is the Erdős-Ko-Rado theorem.

Theorem 1.1 (Erdős, Ko, and Rado [11], Frankl [13], Wilson [38]). Suppose \(A \subseteq \binom{[n]}{k}\) is \(t\)-intersecting and \(n > 2k - t\). Then for \(n \geq (k - t + 1)(t + 1)\), we have

\[|A| \leq \binom{n - t}{k - t},\]

Moreover, if \(n > (k - t + 1)(t + 1)\), then equality holds if and only if \(A = \{A \in \binom{[n]}{k} : T \subseteq A\}\) for some \(t\)-set \(T\).

Received April 18, 2016; Revised August 8, 2016.
2010 Mathematics Subject Classification. 05D05.
Key words and phrases. \(t\)-intersecting family, Erdős-Ko-Rado, set partitions.

©2017 Korean Mathematical Society
Let \( A_i \subseteq \binom{[n]}{k_i} \) for \( i = 1, 2, \ldots, r \). We say that the families \( A_1, A_2, \ldots, A_r \) are \( r \)-cross \( t \)-intersecting if \( |A_1 \cap A_2 \cap \cdots \cap A_r| \geq t \) holds for all \( A_i \in A_i \). When \( t = 1 \), we will just say \( r \)-cross intersecting instead of \( r \)-cross 1-intersecting. When \( r = 2 \) and \( t = 1 \), we will just say cross-intersecting instead of 2-cross intersecting.

**Theorem 1.2** (Bey [3], Matsumoto and Tokushige [32], Pyber [34]). Let \( A_1 \subseteq \binom{[n]}{k_1} \) and \( A_2 \subseteq \binom{[n]}{k_2} \) be cross-intersecting. If \( k_1, k_2 \leq n/2 \), then

\[
|A_1||A_2| \leq \binom{n - 1}{k_1 - 1}\binom{n - 1}{k_2 - 1}.
\]

Equality holds for \( k_1 + k_2 < n \) if and only if \( A_1 \) and \( A_2 \) consist of all \( k_1 \)-element resp. \( k_2 \)-element sets containing a fixed element.

In the celebrated paper [1], Ahlswede and Khachatrian extended the Erdős-Ko-Rado theorem by determining the structure of all \( t \)-intersecting set systems of maximum size for all possible \( n \) (see also [12, 14, 16, 24, 26, 35, 36] for some related results). There have been many recent results showing that a version of the Erdős-Ko-Rado theorem holds for combinatorial objects other than set systems. For example, an analogue of the Erdős-Ko-Rado theorem for the Hamming scheme is proved in [33]. A complete solution for the \( t \)-intersection problem in the Hamming space is given in [2]. Some recent work done on this problem and its variants can be found in [4, 5, 6, 8, 9, 10, 15, 18, 19, 25, 30, 31, 37]. The Erdős-Ko-Rado type results also appear in vector spaces [7, 17], set partitions [20, 22, 21, 29] and weak compositions [23, 27, 28].

In this paper, we consider Erdős-Ko-Rado type results for minimal covers. Let \( \mathcal{P}(n) \) be the set of all subsets of \([n]\), and let \( \mathcal{P}^2(n) \) be the set of all subsets of \( \mathcal{P}(n) \). Let \( Z \subseteq [n] \). A set \( A = \{A_1, A_2, \ldots, A_l\} \subseteq \mathcal{P}(n) \) is a cover of \( Z \) if \( \bigcup_{1 \leq i \leq l} A_i = Z \). It is a minimal cover of \( Z \) if it is a cover of \( Z \) and \( \bigcup_{1 \leq i \leq l, i \neq j_0} A_i \neq Z \) for all \( j_0 \in [l] \).

Let \( \mathcal{C}(Z) \) denote the collection of all minimal covers of \( Z \). Note that \( \mathcal{C}(Z) \subseteq \mathcal{P}^2(n) \). When \( Z = [n] \), we shall write \( \mathcal{C}(n) \) instead of \( \mathcal{C}([n]) \). Let \( C_n = |\mathcal{C}(n)| \).

For \( 1 \leq n \leq 3 \), we have

- \( \mathcal{C}(1) = \{\{\}\} \),
- \( \mathcal{C}(2) = \{\{\{1\}, \{2\}\}, \{\{1, 2\}\}\} \),
- \( \mathcal{C}(3) = \{\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1\}, \{3\}, \{2\}\}, \{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1, 3\}, \{2, 3\}\}, \{\{1, 2\}, \{3\}\}\} \),

and thus \( C_1 = 1 \), \( C_2 = 2 \) and \( C_3 = 8 \).

Let \( \sigma \) be a permutation on \([n]\). For each \( A \subseteq [n] \), we define \( \sigma(A) = \{\sigma(a) : a \in A\} \). For each \( A \subseteq \mathcal{P}(n) \), we define \( \sigma(A) = \{\sigma(A) : A \in A\} \), and for each \( A \subseteq \mathcal{P}^2(n) \), we define \( \sigma(A) = \{\sigma(A) : A \in A\} \). Two families \( A, B \subseteq \mathcal{P}^2(n) \) are
said to be isomorphic, denoted by $A \cong B$, if they are the same up to relabelling of the underlying elements, i.e., $\sigma(A) = B$.

Let
\[
Q_0(t) = \{ A : A \text{ is a minimal cover of } [n] \setminus [t] \},
\]
\[
Q_1(t) = \{ A \in Q_0(t) : \{ t + 1 \} \notin A \},
\]
\[
Q_2(t) = \{ A : A \text{ is a minimal cover of } [n] \setminus [t + 1] \},
\]
\[
D_0(t) = \{ \{ \{ 1 \}, \{ 2 \}, \ldots, \{ t \} \} \cup A : A \in Q_0(t) \}.
\]

For $1 \leq l \leq t$, let
\[
D_l(t) = \{ \{ \{ 1, t + 1 \}, \{ 2, t + 1 \}, \ldots, \{ l, t + 1 \}, \{ t + 1 \}, \ldots, \{ t \} \} \cup A : A \in Q_1(t) \}
\]
\[
\cup \{ \{ \{ 1, t + 1 \}, \{ 2, t + 1 \}, \ldots, \{ l, t + 1 \}, \{ t + 1 \}, \ldots, \{ t \} \} \cup A : A \in Q_2(t) \}.
\]

Notice that when $l = t$, we have
\[
D_t(t) = \{ \{ \{ 1, t + 1 \}, \{ 2, t + 1 \}, \ldots, \{ l, t + 1 \} \} \cup A : A \in Q_1(t) \}
\]
\[
\cup \{ \{ \{ 1, t + 1 \}, \{ 2, t + 1 \}, \ldots, \{ l, t + 1 \} \} \cup A : A \in Q_2(t) \}.
\]

Clearly $D_0(t) \subseteq C(n)$, and $|D_0(t)| = C_{n-t}$. For each $A \in Q_1(t)$, the mapping defined by
\[
\{ \{ 1, t + 1 \}, \{ 2, t + 1 \}, \ldots, \{ l, t + 1 \}, \{ t + 1 \}, \ldots, \{ t \} \} \cup A
\]
\[
\mapsto \{ \{ 1 \}, \{ 2 \}, \ldots, \{ l \}, \{ t + 1 \}, \ldots, \{ t \} \} \cup A,
\]
is one-to-one. For each $A \in Q_2(t)$, the mapping defined by
\[
\{ \{ 1, t + 1 \}, \{ 2, t + 1 \}, \ldots, \{ l, t + 1 \}, \{ t + 1 \}, \ldots, \{ t \} \} \cup A
\]
\[
\mapsto \{ \{ 1 \}, \{ 2 \}, \ldots, \{ l \}, \{ t + 1 \}, \ldots, \{ t \} \} \cup A,
\]
is also one-to-one. Hence
\[
|D_l(t)| = |D_0(t)| = C_{n-t}
\]
for $1 \leq l \leq t$. However, some of the elements in $D_l(t)$ do not lie in $C(n)$. For example, if $n \geq t + 3$, then the set
\[
A' = \{ \{ 1, t + 1 \}, \{ 2, t + 1 \}, \ldots, \{ l, t + 1 \}, \{ t + 1 \}, \ldots, \{ t \}, \{ t + 1, t + 2 \},
\]
\[
\{ t + 2, t + 3, \ldots, n \}\}
\]
is in $D_l(t)$, but it is not in $C(n)$ since removing $\{ t + 1, t + 2 \}$ from $A'$ results in a collection of sets which is still a cover of $[n]$. Therefore, for $n \geq t + 3$,
\[
|D_l(t) \cap C(n)| < |D_l(t)| = C_{n-t}.
\]

A family $A \subseteq C(n)$ is said to be $t$-intersecting if $|A \cap B| \geq t$ for all $A, B \in A$. We suggest the following conjecture on the characterisation of $t$-intersecting families of maximum size.
Conjecture 1.3. There exists a constant \( n_0 = n_0(t) \) depending on \( t \), such that for all \( n \geq n_0 \), if \( A \subseteq C(n) \) is \( t \)-intersecting, then
\[
|A| \leq C_{n-t}.
\]
Moreover, equality holds if and only if \( A \cong D_0(t) \).

In this paper, we prove a weaker version of Conjecture 1.3 (see Theorem 1.4 below). To this end, we require a stronger notion of intersection. For a fixed \( j \in [n] \), \( A \in P^2(n) \), we define
\[
N_j(A) = |\{ A \in A : j \in A \}|
\]
to be the number of times \( j \) appears in \( A \). If \( N_j(A) = 1 \), then \( j \) is said to be critical in \( A \). For example, if \( A = \{\{1,2,3\},\{1,2,4\},\{1,5,6\}\} \in C(6) \), then \( N_2(A) = 2 \) since 2 appears twice in \( A \). Also, 5 is critical in \( A \) since \( N_5(A) = 1 \).

Given any \( A, B \in C(n) \), we write \( \text{Inter}(A, B) \geq t \) if there exist \( t \) distinct elements \( A_1, \ldots, A_t \in A \cap B \) each containing an element which is critical in both \( A \) and \( B \), i.e., for all \( 1 \leq i \leq t \), there exists \( a_i \in A_i \) such that \( N_{a_i}(A) = 1 = N_{a_i}(B) \). For example, if \( A = \{\{1,2,3\},\{1,2,4\},\{1,5,6\}\} \) and \( B = \{\{1,2,3\},\{2,4,6\},\{2,3,5\}\} \), then \( |A \cap B| = 1 \), but \( \text{Inter}(A, B) = 0 \) because \( A \cap B = \{\{1,2,3\}\} \) and none of the elements in \( \{1,2,3\} \) is critical in both \( A \) and \( B \). On the other hand, if \( C = \{\{1,2,3\},\{1,4,5\},\{1,2,6\}\} \), then \( \text{Inter}(A, C) \geq 1 \) since \( \{1,2,3\} \in A \cap C \) and 3 is critical in both \( A \) and \( C \). In general, if \( \text{Inter}(A, B) \geq t \), then \( \text{Inter}(A, B) \geq t + 1 \), then \( \text{Inter}(A, B) \geq t \). Also, \( \text{Inter}(A, B) \geq t \) implies that \( |A \cap B| \geq t \).

A family \( A \subseteq C(n) \) is said to be restricted \( t \)-intersecting if \( \text{Inter}(A, B) \geq t \) for any \( A, B \in A \).

Theorem 1.4. There exists a constant \( n_0 = n_0(t) \) depending on \( t \), such that for all \( n \geq n_0 \), if \( A \subseteq C(n) \) is restricted \( t \)-intersecting, then
\[
|A| \leq C_{n-t}.
\]
Moreover, equality holds if and only if \( A \cong D_0(t) \).

Families \( A_1, A_2, \ldots, A_r \subseteq C(n) \) are said to be \( r \)-cross \( t \)-intersecting if \( |A_1 \cap A_2 \cap \cdots \cap A_r| \geq t \) for all \( A_i \in A_i \). As in the case for sets, we will just say \( r \)-cross intersecting to mean \( r \)-cross 1-intersecting and cross \( t \)-intersecting to mean 2-cross \( t \)-intersecting.

Conjecture 1.5. There exists a constant \( n_0 = n_0(r) \) depending on \( r \), such that for all \( n \geq n_0 \), if \( A_1, A_2, \ldots, A_r \subseteq C(n) \) are \( r \)-cross intersecting, then
\[
\prod_{i=1}^{r} |A_i| \leq C_{n-1}.
\]
Moreover, equality holds if and only if \( A_1 = A_2 = \cdots = A_r \) and \( A_1 \cong D_0(1) \).
We will prove a weaker version of Conjecture 1.5 (Theorem 1.6). Given any $A_1, A_2, \ldots, A_r \in \mathcal{C}(n)$, we write \( \text{Inter}(A_1, A_2, \ldots, A_r) \geq t \) if there exist \( t \) distinct elements \( A_1, A_2, \ldots, A_t \in A_1 \cap A_2 \cap \cdots \cap A_r \) each containing a critical element in all of the \( A_j \). Families \( A_1, A_2, \ldots, A_r \subseteq \mathcal{C}(n) \) are said to be restricted \( r \)-cross \( t \)-intersecting if \( \text{Inter}(A_1, A_2, \ldots, A_r) \geq t \) for all \( A_i \in A_i \). As before, we will just say restricted \( r \)-cross intersecting to mean restricted \( r \)-cross \( 1 \)-intersecting and restricted cross \( t \)-intersecting to mean restricted \( 2 \)-cross \( t \)-intersecting.

**Theorem 1.6.** There exists a constant \( n_0 = n_0(r) \) depending on \( r \), such that for all \( n \geq n_0 \), if \( A_1, A_2, \ldots, A_r \subseteq \mathcal{C}(n) \) are restricted \( r \)-cross intersecting, then

\[
\prod_{i=1}^{r} |A_i| \leq C_r^\ast n^{r-1}.
\]

Moreover, equality holds if and only if \( A_1 = A_2 = \cdots = A_r \) and \( A_1 \cong D_0(1) \).

Theorem 1.4 and Theorem 1.6 are proved in Sections 3 and 4 respectively.

### 2. Splitting operation

**Lemma 2.1.** Every set in a minimal cover of \([n]\) contains a critical element. In particular, if \( A \in \mathcal{C}(n) \) and \( B = \{j\} \) is a singleton in \( A \), then \( j \) is critical in \( A \).

**Proof.** Let \( A \in \mathcal{C}(n) \), and \( A \in A \). By definition, removing \( A \) from \( A \) results in an element of \( 2^n(n) \) which is no longer a cover of \([n]\). So there must be an element in \( A \) which does not appear elsewhere in \( A \). Thus, this element must be critical in \( A \). \( \square \)

Let \( T \subseteq [n] \) and \( |T| \geq 2 \). For each \( A \in \mathcal{C}(n) \) with \( T \in A \), we define

\[
P(T, A) = \{q \in T : q \text{ is critical in } A\}.
\]

By Lemma 2.1, \( P(T, A) \neq \emptyset \). The \( T \)-split of \( A \), denoted by \( s_T(A) \), is defined as follows: If \( T \) is not a set in \( A \), then the \( T \)-split is just \( A \) itself. Otherwise, we replace \( T \) by all the singleton sets each consisting of a critical element found in \( T \). Formally,

1. \( s_T(A) = A \), if \( T \notin A \);
2. \( s_T(A) = (A \setminus \{T\}) \cup P(T, A) \), if \( T \in A \).

**Lemma 2.2.** \( s_T(A) \in \mathcal{C}(n) \) for all \( A \in \mathcal{C}(n) \).

**Proof.** We can assume that \( T \in A \). By removing \( T \) from \( A \) and adding the singleton set \( \{v\} \) for every critical element \( v \in T \), we clearly still have that \( s_T(A) \) covers \([n]\). Furthermore, as we have only reduced the number of occurrences of non-critical elements, every set in \( s_T(A) \) still has a critical element, and so it must be a minimal cover of \([n]\). \( \square \)
For a family \( \mathcal{A} \subseteq \mathcal{C}(n) \), let \( s_T(\mathcal{A}) = \{ s_T(A) : A \in \mathcal{A} \} \). By Lemma 2.2, \( s_T(\mathcal{A}) \subseteq \mathcal{C}(n) \). Any family \( \mathcal{A} \subseteq \mathcal{C}(n) \) can be decomposed with respect to a given \( T \subseteq [n] \) with \( |T| \geq 2 \) as follows:

\[
\mathcal{A} = (\mathcal{A} \setminus \mathcal{A}_T) \cup \mathcal{A}_T,
\]

where \( \mathcal{A}_T = \{ A \in \mathcal{A} : s_T(A) \not\subseteq \mathcal{A} \} \). Define the \( T \)-splitting of \( \mathcal{A} \) to be the family

\[
S_T(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_T) \cup s_T(\mathcal{A}_T).
\]

**Lemma 2.3.** \( |S_T(\mathcal{A})| = |\mathcal{A}| \) for all \( \mathcal{A} \subseteq \mathcal{C}(n) \).

**Proof.** If \( \mathcal{A}_T = \emptyset \), then \( S_T(\mathcal{A}) = \mathcal{A} \) and the lemma holds. Suppose \( \mathcal{A}_T \neq \emptyset \). Clearly, \( s_T(\mathcal{A}_T) \cap (\mathcal{A} \setminus \mathcal{A}_T) = \emptyset \). So, it is sufficient to show that \( s_T \) is one-to-one on \( \mathcal{A}_T \), i.e., \( s_T(\mathcal{A}) = s_T(\mathcal{B}) \) implies that \( \mathcal{A} = \mathcal{B} \) for \( \mathcal{A}, \mathcal{B} \in \mathcal{A}_T \). Note that \( T \in \mathcal{A} \cap \mathcal{B} \) and both \( s_T(\mathcal{A}) \) and \( s_T(\mathcal{B}) \) are obtained by operation (O2). So,

\[
s_T(\mathcal{A}) = A \setminus \{ T \} \cup P(T, A),
\]

\[
s_T(\mathcal{B}) = B \setminus \{ T \} \cup P(T, B).
\]

If \( P(T, A) \cap B \setminus \{ T \} \neq \emptyset \), then \( \{ q \} \in B \setminus \{ T \} \) for some \( q \in T \). So, \( q \) appears at least 2 times in \( B \) (once in \( \{ q \} \) and once in \( T \)), contradicting Lemma 2.1. Thus, \( P(T, A) \cap B \setminus \{ T \} = \emptyset \). Similarly, \( P(T, B) \cap A \setminus \{ T \} = \emptyset \). Therefore, \( P(T, A) = P(T, B) \) and \( A \setminus \{ T \} = B \setminus \{ T \} \). Hence, \( \mathcal{A} = \mathcal{B} \). \( \square \)

Let \( I(n, t) \) be the set of all restricted cross \( t \)-intersecting families in \( \mathcal{C}(n) \), i.e.,

\[
I(n, t) = \{(A_1, A_2) : A_1, A_2 \subseteq \mathcal{C}(n) \text{ are restricted cross } t \text{-intersecting}\}.
\]

Note that \( (\mathcal{A}, \mathcal{A}) \in I(n, t) \) if and only if \( \mathcal{A} \) is restricted \( t \)-intersecting. Given any \( (A_1, A_2) \in I(n, t) \), the \( T \)-splitting of \( (A_1, A_2) \) is defined to be the set \( (S_T(A_1), S_T(A_2)) \).

For any \( (A_1, A_2) \in I(n, t) \), splitting operations preserve the size (Lemma 2.3) and the intersecting property (Lemma 2.4).

**Lemma 2.4.** Let \( T \subseteq [n] \) with \( |T| \geq 2 \). If \( (A_1, A_2) \in I(n, t) \), then

\[
(S_T(A_1), S_T(A_2)) \in I(n, t).
\]

**Proof.** Note that \( \text{Inter}(\mathcal{A}, \mathcal{B}) \geq t \) for all \( \mathcal{A} \in A_1 \setminus (A_1)_T \) and \( \mathcal{B} \in A_2 \setminus (A_2)_T \), where \( (A_1)_T = \{ A \in A_1 : s_T(\mathcal{A}) \not\subseteq A_1 \} \) and \( (A_2)_T = \{ A \in A_2 : s_T(\mathcal{A}) \not\subseteq A_2 \} \). So, it is sufficient to show that \( \text{Inter}(\mathcal{A}, \mathcal{B}) \geq t \) for any \( \mathcal{A} \in S_T(A_1) \) and \( \mathcal{B} \in s_T((A_2)_T) \) (the case \( A \in S_T(A_2) \) and \( \mathcal{B} \in s_T((A_1)_T) \) can be proved similarly).

**Case 1** Suppose \( A \in A_1 \setminus (A_1)_T \) and \( \mathcal{B} \in s_T((A_2)_T) \).

Let \( \mathcal{B} = s_T(\mathcal{C}) \) for some \( \mathcal{C} \in (A_2)_T \). Then \( T \in \mathcal{C} \) and \( \mathcal{B} = \mathcal{C} \setminus \{ T \} \cup P(T, C) \). Suppose \( T \not\subseteq \mathcal{A} \). Then \( T \not\subseteq \mathcal{A} \cap \mathcal{C} \). Since \( \text{Inter}(\mathcal{A}, \mathcal{C}) \geq t \), there exist \( A_1, \ldots, A_t \in \mathcal{A} \cap \mathcal{C} \) each containing a critical element in both \( \mathcal{A} \) and \( \mathcal{C} \). Since \( T \neq A_i \) for all \( i \), we have \( A_1, \ldots, A_t \in \mathcal{A} \cap \mathcal{C} \setminus \{ T \} \subseteq \mathcal{A} \cap \mathcal{B} \). So, \( \text{Inter}(\mathcal{A}, \mathcal{B}) \geq t \).
Suppose $T \in A$. Then $A \setminus \{T\} \cup P(T, A) = s_T(A) \in A_1$. Since $C \in A_2$, we have Inter($s_T(A), C) \geq t$, and so there exist $B_1, \ldots, B_t \in s_T(A) \cap C$ each containing a critical element in both $s_T(A)$ and $C$. If $B_{i_0} \in P(T, A)$ for some $i_0$, then $B_{i_0} = \{q_0\}$ for some $q_0 \in T$, and $q_0$ appears at least 2 times in $C$ (once in $B_{i_0}$ and once in $T$), contradicting Lemma 2.1. Thus, $B_i \in A \setminus \{T\}$ for all $i$. This implies that $B_1, \ldots, B_t \in A \cap C \setminus \{T\} \subseteq A \cap B$. Hence, Inter$(A, B) \geq t$.

**Case 2** Suppose $A \in s_T((A_1)_T)$ and $B \in s_T((A_2)_T)$.

Let $A = s_T(C)$ and $B = s_T(D)$ for some $C \in (A_1)_T$ and $D \in (A_2)_T$. Then

$$A = C \setminus \{T\} \cup P(T, C),$$
$$B = D \setminus \{T\} \cup P(T, D).$$

Since Inter$(C, D) \geq t$, there exist $C_1, \ldots, C_t \in C \cap D$ each containing a critical element in $C$ and $D$. If $T \neq C_i$ for all $i$, then $C_1, \ldots, C_t \in (C \setminus \{T\}) \cap (D \setminus \{T\}) \subseteq A \cap B$. Hence, Inter$(A, B) \geq t$. Suppose $T = C_{i_0}$ for some $i_0$. For convenience, we may assume that $T = C_1$. Since $C_i \neq T$ for all $i \neq 1$, we have $C_2, \ldots, C_t \in (C \setminus \{T\}) \cap (D \setminus \{T\}) \subseteq A \cap B$. Let $c_1 \in C_1$ be a critical element in $C_1$. Then $\{c_1\} \in P(T, C) \cap P(T, D) \subseteq A \cap B$, and $c_1$ is critical in both $A$ and $B$. Since $\{c_1\}, C_2, \ldots, C_t \in A \cap B$, we deduce that Inter$(A, B) \geq t$.

This completes the proof of the lemma. □

A family $A \subseteq C(n)$ is **compressed** if for any $T \subseteq [n]$ with $|T| \geq 2$, we have $s_T(A) = A$. For any $A \in C(n)$, define $\beta(A) = |\{A \in A : |A| = 1\}|$, i.e., $\beta(A)$ is the number of singletons in $A$.

**Lemma 2.5.** Let $(A_1, A_2) \in I(n, t)$. By repeatedly applying the splitting operations on $(A_1, A_2)$, we eventually obtain compressed families $A_1^* \subseteq A_1$ and $A_2^* \subseteq A_2$ with $|A_1^*| = |A_1|, |A_2^*| = |A_2|$, and $(A_1^*, A_2^*) \in I(n, t)$.

**Proof.** For any $(A_1, A_2) \in I(n, t)$, let $w((A_1, A_2)) = \sum_{A \in A_1^*} \beta(A) + \sum_{A \in A_2^*} \beta(A)$. Note that if $s_T(A_i) \neq A_i$ for some $i \in \{1, 2\}$, then $\sum_{A \in s_T(A_i)} \beta(A) > \sum_{A \in A_i^*} \beta(A)$. This implies that $w((A_1, A_2)) < w((s_T(A_1), s_T(A_2)))$. So, the splitting operations cannot go on forever. Eventually, we will obtain $A_1^*$ and $A_2^*$ with $|A_1^*| = |A_1|, |A_2^*| = |A_2|$, and $(A_1^*, A_2^*) \in I(n, t)$ (Lemmas 2.3 and 2.4). □

Let $A \in C(n)$. For each $A \in A$, let

$$\gamma(A) = \{x : \{x\} \in A\},$$

i.e., $\gamma(A)$ is the union of all the singletons in $A$. Let $\gamma(A) = \{\gamma(A) : A \in A\}$.

**Lemma 2.6.** If $A_1$ is compressed and $(A_1, A_2) \in I(n, t)$, then $\gamma(A_1), \gamma(A_2)$ are cross $t$-intersecting families of subsets.

**Proof.** Suppose $\gamma(A_1), \gamma(A_2)$ are not cross $t$-intersecting. Then there exist $A \in A_1$ and $B \in A_2$ with $|\gamma(A) \cap \gamma(B)| \leq t - 1$. Let Inter$(A, B) = s$. Note that $s \geq t$ and there exist $A_1, \ldots, A_s \in A \cap B$ each containing a critical element in both $A$ and $B$. The sets $A_1, \ldots, A_s$ cannot be all singletons since...
the condition \( \text{Inter}(B) \subseteq D \). We first prove the following claim.

**Claim 1.** If \( B \in A_1 \cup A_2 \), then \( B \in A_1 \cap A_2 \) and \( A_1 = A_2 = D \).

**Proof.** Without loss of generality, we may assume that \( B \in A_1 \). We first prove that \( A_2 = D \). Assume, for a contradiction, that \( A_2 \neq D \). Then there exists a \( C \in A_2 \) such that \( \{t\} \notin C \) for some \( 1 \leq i \leq t \). Now, the condition \( \text{Inter}(B, C) \geq t \) implies that

\[ C = \{\{1\}, \{2\}, \ldots, \{i - 1\}, V, \{i + 1\}, \ldots, \{t\}, \{t + 1, t + 2, \ldots, n\}\}, \]
where \( i \in V \) and \(|V| \geq 2\). Note that \( V \cap \{1, 2, \ldots, i - 1, i + 1, \ldots, t\} = \emptyset \) for otherwise there exists \( j \in \{1, 2, \ldots, i - 1, i + 1, \ldots, t\} \) such that \( j \) appears at least 2 times in \( C \) (once in \( \{j\} \) and once in \( V \)), contradicting Lemma 2.1. Similarly, \( \{t + 1, t + 2, \ldots, n\} \not\subseteq V \). Since \( s_T(C) = H_0(t) \), we have \( \{i\} \in s_T(C) \). Thus, \( T = V \).

Let \( D = \{(1), \{2\}, \ldots, \{t\}, \{t + 1, t + 2\}, \{t + 1, t + 3, \ldots, n\}\} \in H_0(t) \).

Note that \( \text{Inter}(D, C) = t - 1 \) since \( D \cap C = \{(1), \{2\}, \ldots, \{i - 1\}, \{i + 1\}, \ldots, \{t\}\} \). Therefore, \( D \notin A_1 \). Since \( D \in s_T(A_1) = D_0(t) \), there is \( E \in A_1 \) with \( T \subseteq E \) and \( s_T(E) = D \). By Lemma 2.8, \( \{t+1, t+2\}, \{t+1, t+3, \ldots, n\} \in E \). From \( \text{Inter}(E, C) \geq t \), we must have

\[
E = \{(1), \{2\}, \ldots, \{i - 1\}, V, \{i + 1\}, \ldots, \{t\}, \{t + 1, t + 2\}, \{t + 1, t + 3, \ldots, n\}\}.
\]

Let \( F \in A_2 \setminus \{C\} \) (such an \( F \) exists because \( |A_2| \geq 2 \)). The aim is to arrive at a contradiction by showing that such \( F \) could never exist.

Suppose \( \{t + 1, t + 2, \ldots, n\} \in F \). Then \( \{t + 1, t + 2\}, \{t + 1, t + 3, \ldots, n\} \not\subseteq F \) (otherwise it contradicts Lemma 2.1). From \( \text{Inter}(E, F) \geq t \), we have \( \{(1), \{2\}, \ldots, \{i - 1\}, V, \{i + 1\}, \ldots, \{t\}\} \subseteq F \). Thus, \( F = C \), a contradiction. So, we may assume that \( \{t + 1, t + 2, \ldots, n\} \not\subseteq F \). Now, from \( \text{Inter}(F, F) \geq t \), we have \( \{(1), \{2\}, \ldots, \{t\}\} \subseteq F \). This implies that \( V \not\subseteq F \) (otherwise both \( \{t\}, V \in F \), contradicting Lemma 2.1). From \( \text{Inter}(E, F) \geq t \), we have \( \{t + 1, t + 2\} \in F \) or \( \{t + 1, t + 3, \ldots, n\} \in F \). In either case, we always have \( \{t + 1\} \not\subseteq F \).

Next, we claim that \( \{j\} \not\subseteq F \) for \( t + 1 \leq j \leq n \). Since \( \{t + 1\} \not\subseteq F \) from the preceding paragraph, it remains to show that \( \{j\} \not\subseteq F \) for \( t + 2 \leq j \leq n \). For \( t + 2 \leq j \leq n \), let

\[
G_j = \{(1), \{2\}, \ldots, \{t\}, \{t + 1, j\}, \{t + 2, t + 3, \ldots, n\}\} \in D_0(t).
\]

Note that \( G_j \notin A_1 \) since \( \text{Inter}(G_j, C) = t - 1 \). Since \( G_j \in s_T(A_1) \), there is \( H_j \in A_1 \) with \( T \subseteq H_j \) such that \( s_T(H_j) = G_j \). By Lemma 2.8, \( \{t + 1, j\}, \{t + 2, t + 3, \ldots, n\} \in H_j \). From \( \text{Inter}(H_j, C) \geq t \), we must have

\[
H_j = \{(1), \{2\}, \ldots, \{i - 1\}, V, \{i + 1\}, \ldots, \{t\}, \{t + 1, j\}, \{t + 2, t + 3, \ldots, n\}\}.
\]

Now, \( \text{Inter}(H_j, F) \geq t \) implies that either \( \{t + 1, j\} \in F \) or \( \{t + 2, t + 3, \ldots, n\} \in F \). Thus, \( \{j\} \not\subseteq F \) for \( t + 2 \leq j \leq n \); otherwise \( j \) would appear twice in \( F \), once in \( \{j\} \) and once in either \( \{t + 1, j\} \) or \( \{t + 2, t + 3, \ldots, n\} \) contradicting Lemma 2.1. Hence \( \{j\} \not\subseteq F \) for all \( t + 1 \leq j \leq n \).

For \( t + 1 \leq j \leq t + 3 \), let \( Y_j = \{t + 1, t + 2, \ldots, n\} \setminus \{j\} \) and

\[
Y_j = \{(1), \{2\}, \ldots, \{t\}, \{j\}, Y_j\} \in D_0(t).
\]
Now, \(Y_j \notin A_1\) since \(\text{Inter}(Y_j, C) = t - 1\). Therefore, there exists \(Z_j \in A_1\) with \(T \in Z_j\) and \(\text{sv}(Z_j) = Y_j\). By Lemma 2.8, \(Y_j \in Z_j\). Moreover, \(|Y_j| \geq 2\) since \(n - t \geq 3\). From \(\text{Inter}(Z_j, C) \geq t\), we must have
\[
\{\{1\}, \{2\}, \ldots, \{i - 1\}, V, \{i + 1\}, \ldots, \{t\}\} \subseteq Z_j.
\]
Therefore,
\[
Z_j = \begin{cases} \{\{1\}, \{2\}, \ldots, \{i - 1\}, V, \{i + 1\}, \ldots, \{t\}, Y_j\}, & \text{if } j \in V; \\ \{\{1\}, \{2\}, \ldots, \{i - 1\}, V, \{i + 1\}, \ldots, \{t\}, \{j\}, Y_j\}, & \text{if } j \notin V. \end{cases}
\]
If \(j \in V\), then \(\text{Inter}(Z_j, F) \geq t\) implies that \(Y_j \in F\). If \(j \notin V\), then \(\text{Inter}(Z_j, F) \geq t\) implies that either \(\{j\} \in F\) or \(Y_j \in F\). Since \(\{j\} \notin F\) for \(t + 1 \leq j \leq n\), we can only have \(Y_j \in F\). Hence, \(Y_j \in F\) in all \(t + 1 \leq j \leq t + 3\). In particular, we have \(Y_{t+1} = \{t + 2, t + 3, \ldots, n\}\), \(Y_{t+2} = \{t + 1, t + 3, \ldots, n\}\), \(Y_{t+3} = \{t + 1, t + 2, t + 4, \ldots, n\}\) \(\in F\) and this contradicts Lemma 2.1, because every element in \(\{t + 2, t + 3, \ldots, n\}\) appears at least 2 times in \(F\).

We conclude that no such \(F\) exists. This contradiction shows that \(A_2 = D_0(t)\). Consequently, \(B \not\in A_2\) and thus \(B \not\in A_1 \cap A_2\). By repeating the above argument starting with \(B \not\in A_2\), we deduce that \(A_1 = D_0(t)\). Hence, Claim 1 is proved.

We now proceed to prove the lemma. If \(B \in A_1\), then the result of the lemma holds by Claim 1. So we may suppose that \(B \not\in A_1\).

Then there exists \(Q \in A_1\) with \(\text{sv}(Q) = B\). Note that \(T \in Q\) and
\[
B = Q \setminus \{T\} \cup P(T, Q).
\]
By Lemma 2.8, \(\{t + 1, t + 2, \ldots, n\} \in Q\) and \(\{t + 1, t + 2, \ldots, n\} \not\subseteq T\). Note that \(P(T, Q) \subseteq \{\{1\}, \{2\}, \ldots, \{t\}\}\). If \(|P(T, Q)| \geq 2\), then \(|Q \setminus \{T\}| \leq t - 1\) and \(|Q| \leq t\). Since \((A_1, A_2) \in I(n, t)\), we must have \(A_2 = \{Q\}\), contradicting \(|A_2| > 1\). Thus, \(|P(T, Q)| = 1\) and \(P(T, Q) = \{j_0\}\) for some \(1 \leq j_0 \leq t\).

Suppose \(|T| \geq 3\). Then \(T = \{j_0\} \cup X\) for some \(X \subseteq \{t + 1, t + 2, \ldots, n\}\) with \(|X| \geq 2\). Let \(Y = \{t + 1, t + 2, \ldots, n\} \setminus X\) and
\[
R = \{\{1\}, \{2\}, \ldots, \{t\}, X, Y\}.
\]
Note that \(R \not\in A_2\) since \(\text{Inter}(R, Q) = t - 1\). In fact \(Q \cap R = \{\{1\}, \ldots, \{j_0 - 1\}, \{j_0 + 1\}, \ldots, \{t\}\}\). Since \(R \in \text{sv}(A_2) = D_0(t)\), there exists \(S \in A_2\) with \(T \in S\) and \(\text{sv}(S) = R\). By Lemma 2.8, \(X \in S\) and \(X \not\subseteq T\), a contradiction. Hence, \(|T| = 2\) and \(T = \{i_0, j_0\}\) or some \(i_0 \in \{t + 1, t + 2, \ldots, n\}\). Subsequently, from \(\text{sv}(Q) = B\), we deduce that
\[
Q = \{\{1\}, \{2\}, \ldots, \{j_0 - 1\}, T = \{i_0, j_0\}, \{j_0 + 1\}, \ldots, \{t\}, \{t + 1, t + 2, \ldots, n\}\},
\]
and \(|Q| = t + 1\).

Since \(B \not\in A_1\) and \(Q \in A_1\), it follows from Claim 1 that
\[
B = \{\{1\}, \{2\}, \ldots, \{t\}, \{t + 1, t + 2, \ldots, n\}\} \not\in A_1, \quad \text{and} \quad Q = \{\{1\}, \{2\}, \ldots, \{j_0 - 1\}, T = \{i_0, j_0\}, \{j_0 + 1\}, \ldots, \{t\}, \}.
\]
\[ \{t + 1, t + 2, \ldots, n\} \in \mathcal{A}_i \quad \text{for } i = 1, 2. \]

Let \( U \in \mathcal{A}_2 \setminus \{Q\} \). If \( T \not\subset U \), then \( s_T(U) = U \in \mathcal{D}_0(t) \) and so \( \{\{1\}, \{2\}, \ldots, \{t\}\} \subset U \). Next, \( \text{Inter}(U, Q) \geq t \) implies that \( \{t + 1, t + 2, \ldots, n\} \in U \). Thus, \( U = B \in \mathcal{A}_2 \), a contradiction. Hence \( T \in U \).

Suppose \( \{k\} \not\subset U \) for some \( k \in \{1, 2, \ldots, t\} \setminus \{j_0\} \). Then there is a set \( K \in U \) with \( |K| \geq 2 \) and \( k \in K \). Since \( K \not\subset T \), we have \( K \in s_T(U) \). Also, since \( s_T(U) \in \mathcal{D}_0(t) \), we have \( \{k\} \in s_T(U) \). This contradicts Lemma 2.1 because \( k \) appears twice in \( s_T(U) \) (once in \( \{k\} \) and once in \( K \)). Hence, \( \{k\} \not\subset U \) for all \( k \in \{1, 2, \ldots, t\} \setminus \{j_0\} \). This implies that every element \( U \in \mathcal{A}_2 \) is of the form

\[ \{\{1\}, \{2\}, \ldots, \{j_0 - 1\}, T = \{j_0, j_0\}, \{j_0 + 1\}, \ldots, \{t\}\} \cup \mathcal{W}, \]

where \( \mathcal{W} \) is a minimal cover of \( [n] \setminus [t] \) with \( \{i_0\} \not\subset \mathcal{W} \). Therefore, \( \mathcal{A}_2 \subset \mathcal{D} \), where \( \mathcal{D} \equiv \mathcal{D}_1(t) \). In fact, since not all elements in \( \mathcal{D}_1(t) \) are minimal covers, we have \( \mathcal{A}_2 \subset \mathcal{D} \cap \mathcal{C}(n) \) and so it follows from (2) that

\[ |\mathcal{A}_2| \leq |\mathcal{D}_1(t) \cap \mathcal{C}(n)| < C_{n-t}, \]

contradicting the assumption that \( |\mathcal{A}_2| \geq |\mathcal{D}_0(t)| = C_{n-t} \).

This completes the proof of the lemma. \( \square \)

3. Proof of Theorem 1.4

For each \( Z \subseteq [n] \), let \( \tilde{\mathcal{C}}(Z) = \{A \in \mathcal{C}(Z) : A \) does not contain any singleton\}.

When \( Z = [n] \), we shall write \( \tilde{\mathcal{C}}(n) \) instead of \( \tilde{\mathcal{C}}([n]) \). Let \( \tilde{\mathcal{C}}_n = |\tilde{\mathcal{C}}(n)| \).

**Lemma 3.1.** Let \( n \geq 2 \). Then

\[ C_n = \sum_{k=0}^{n} \binom{n}{k} \tilde{C}_{n-k}, \]

\[ \tilde{C}_n \geq \sum_{k=1}^{n-1} \binom{n-1}{k} \tilde{C}_{n-1-k}, \]

with the conventions \( C_0 = \tilde{C}_0 = 1 \).

**Proof.** Let \( T \subseteq [n] \) and \( \mathcal{C}(n)(T) \) be the set of all \( A \in \mathcal{C}(n) \) such that the only singletons in \( A \) are those in \( T \), i.e.,

\[ \mathcal{C}(n)(T) = \{A \in \mathcal{C}(n) : \{x\} \in A \text{ if and only if } x \in T\}. \]

Note that if \( A \in \mathcal{C}(n)(T) \), then every \( x \in T \) is critical in \( A \) (Lemma 2.1). Therefore, \( A \setminus \{\{x\} : x \in T\} \in \tilde{\mathcal{C}}([n] \setminus T) \). Hence, \( |\mathcal{C}(n)(T)| = \tilde{C}_{n-|T|} \).

Note that \( \bigcup_{T \subseteq [n]} \mathcal{C}(n)(T) \subseteq \mathcal{C}(n) \). Now, for each \( A_0 \in \mathcal{C}(n) \), there is a \( T_0 \subseteq [n] \) such that \( \{x\} \in A \) if and only if \( x \in T_0 \). Thus, \( A_0 \in \mathcal{C}(n)(T_0) \) and \( \bigcup_{T \subseteq [n]} \mathcal{C}(n)(T) = \mathcal{C}(n) \).
Note that \( C(n)(T) \cap C(n)(T') = \emptyset \) for \( T \neq T' \). So,
\[
C_n = |C(n)| = \left| \bigcup_{T \subseteq [n]} C(n)(T) \right| = \sum_{k=0}^{n} \binom{n}{k} \tilde{C}_{n-k},
\]
proving (3).

Let \( T \subseteq [n-1], |T| \geq 1 \) and \( V(T) \) be the set of all \( A \in C(n-1) \) such that \( T \in A \) and \( A \setminus \{T\} \) is a minimal cover of \([n-1] \setminus T\) that does not contain any singletons, i.e.,
\[
V(T) = \{ A \in C(n-1) : T \in A \text{ and } A \setminus \{T\} \in \tilde{C}([n-1] \setminus T) \}.
\]
Then \( |V(T)| = \tilde{C}_{n-1-|T|} \). Let
\[
\overline{V}(T) = \{(A \setminus \{T\}) \cup \{T \cup \{n\} \} : A \in V(T)\}.
\]
Note that \( \overline{V}(T) \subseteq \tilde{C}(n) \) and \( |\overline{V}(T)| = |V(T)| = \tilde{C}_{n-1-|T|} \). Furthermore, \( \overline{V}(T) \cap \overline{V}(T') = \emptyset \) for \( T \neq T' \). So, from \( \bigcup_{T \subseteq [n-1], |T| \geq 1} \overline{V}(T) \subseteq \tilde{C}(n) \), we have
\[
\sum_{k=1}^{n-1} \binom{n-1}{k} \tilde{C}_{n-1-k} = \left| \bigcup_{T \subseteq [n-1], |T| \geq 1} \overline{V}(T) \right| \leq |\tilde{C}(n)| = \tilde{C}_n,
\]
proving (4). \( \square \)

Given a real number \( x \), we shall denote the greatest integer less than or equal to \( x \), by \( \lfloor x \rfloor \). Note that \( \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \).

**Lemma 3.2.** Given any positive integers \( m, c \) and \( t \) with \( m \geq 2 \), there is a positive integer \( n_0 = n_0(m, c, t) \) depending on \( m, c \) and \( t \), such that for \( n \geq n_0 \),
\[
\tilde{C}_{n-t} > e^n \sum_{\lfloor \frac{n}{m} \rfloor \leq k \leq n} \binom{n}{k} \tilde{C}_{n-k}.
\]

**Proof.** Since \( \tilde{C}_{n-\lfloor n/m \rfloor} \geq \tilde{C}_{n-k} \) for all \( \lfloor n/m \rfloor \leq k \leq n \), we have
\[
\sum_{\lfloor n/m \rfloor \leq k \leq n} \binom{n}{k} \tilde{C}_{n-k} \leq \tilde{C}_{n-\lfloor n/m \rfloor + 2} \sum_{\lfloor n/m \rfloor \leq k \leq n} \binom{n}{k} \leq 2^n \tilde{C}_{n-\lfloor n/m \rfloor + 2}.
\]
So, it is sufficient to show that \( \frac{\tilde{C}_{n-t}}{\tilde{C}_{n-\lfloor n/m \rfloor + 2}} \geq (2c)^n \).

Now, \( n - \lfloor \frac{n}{m} \rfloor + 4 > (2c)^{4m} + 1 \) provided that \( n \geq \frac{n}{m-1}(2c)^{4m} \). So, by (4),
\[
\frac{\tilde{C}_{n-t}}{\tilde{C}_{n-\lfloor n/m \rfloor + 2}} \geq (2c)^{4m} \text{ for } l \geq n - \lfloor \frac{n}{m} \rfloor + 4.
\]
Therefore,
\[
\frac{\tilde{C}_{n-t}}{\tilde{C}_{n-\lfloor n/m \rfloor + 2}} \geq \left( \frac{\tilde{C}_{n-\lfloor n/m \rfloor + 2} + 4}{\tilde{C}_{n-\lfloor n/m \rfloor + 2}} \right) \left( \frac{\tilde{C}_{n-\lfloor n/m \rfloor + 6}}{\tilde{C}_{n-\lfloor n/m \rfloor + 4}} \right) \left( \frac{\tilde{C}_{n-\lfloor n/m \rfloor + 10}}{\tilde{C}_{n-\lfloor n/m \rfloor + 8}} \right) \frac{\tilde{C}_{n-t}}{\tilde{C}_{n-\lfloor n/m \rfloor + 2}} > (2c)^{4m-1},
\]
where \( u = \left\lfloor \frac{\sqrt{3} - 1}{2} \right\rfloor \). Note that \( u - 1 \geq \frac{1}{2}(n - t - 3) - 2 \geq \frac{n}{4m} \)
provided that \( n \geq 2m(t + 7) \). Hence, for sufficiently large \( n \), \( \tilde{C}_{n-t}/C_{n-\left\lfloor \frac{m}{2} \right\rfloor + 2} > (2c)^n \).

Let \( A \subseteq C(n) \) be compressed. Recall that \( \gamma(A) = \{ \gamma(A) : A \in A \} \), where \( \gamma(A) \) is the union of all the singletons in \( A \). We say \( \gamma(A) \) is trivial if there is a fixed \( t \)-set, say \( T \), such that \( T \subseteq \gamma(A) \) for all \( A \in A \).

**Lemma 3.3.** There is a positive integer \( n_0 = n_0(t) \) depending on \( t \), such that for \( n \geq n_0 \), if \( A \subseteq C(n) \) is compressed and is restricted \( t \)-intersecting, and \( \gamma(A) \) is non-trivial, then

\[ |A| < C_{n-t}. \]

**Proof.** Note that \((A, A) \in I(n, t)\). By Lemma 2.6, \( \gamma(A) \) and \( \gamma(A) \) are cross \( t \)-intersecting, i.e., \( \gamma(A) \) is \( t \)-intersecting. For \( k \geq t \), let \( F_k = \gamma(A) \cap \binom{[n]}{k} \).

Then \( F_k \) is \( t \)-intersecting. If \( F_k \neq \emptyset \), then \( \gamma(A) \) is trivial. So, we may assume that \( F_k = \emptyset \). By using Lemma 2.1, it is not hard to see that for each \( A \in A \),

\[ A \setminus \{ \{ x \} : x \in \gamma(A) \} \in \tilde{C}(n) \setminus \gamma(A). \]

Therefore,

\[
|A| \leq \sum_{t+1 \leq k \leq n} |F_k| \tilde{C}_{n-k}
= \sum_{t+1 \leq k \leq \left\lceil \frac{n-t}{t+2} + t \right\rceil} |F_k| \tilde{C}_{n-k} + \sum_{|\frac{n-t}{t+1} + 1| \leq k \leq n} |F_k| \tilde{C}_{n-k}.
\]

By Theorem 1.1, \( |F_k| \leq \binom{n-t}{k-1} \) for \( t+1 \leq k \leq \left\lceil \frac{n-t}{t+1} + t - 1 \right\rceil \). Therefore,

\[
\sum_{t+1 \leq k \leq \left\lceil \frac{n-t}{t+2} + t \right\rceil} |F_k| \tilde{C}_{n-k} \leq \sum_{t+1 \leq k \leq \left\lceil \frac{n-t}{t+1} + t - 1 \right\rceil} \binom{n-t}{k-t} \tilde{C}_{n-k}
= \sum_{1 \leq k \leq \left\lceil \frac{n-t}{t+1} - t \right\rceil} \binom{n-t}{k-t} \tilde{C}_{n-t-k}
\leq \sum_{1 \leq k \leq n-t} \binom{n-t}{k} \tilde{C}_{n-t-k}
= C_{n-t} - \tilde{C}_{n-t},
\]

where the last equality follows from equation (3).

On the other hand, \( |F_k| \leq \binom{n}{k} \) for \( \left\lceil \frac{n-t}{t+1} + t - 1 \right\rceil + 1 \leq k \leq n \). Therefore,

\[
\sum_{\left\lceil \frac{n-t}{t+1} + t - 1 \right\rceil + 1 \leq k \leq n} |F_k| \tilde{C}_{n-k} \leq \sum_{\left\lceil \frac{n-t}{t+1} + t - 1 \right\rceil + 1 \leq k \leq n} \binom{n}{k} \tilde{C}_{n-k}
\]
Therefore, if $A$ only critical element contained in $A$, conclude that $|A| < C_{n-t}$.

**Proof of Theorem 1.4.** Note that $(D_0(t), D_0(t)) \in I(n, t)$ and $|D_0(t)| = C_{n-t}$ for $0 \leq t \leq n$. Let $A$ be restricted $t$-intersecting of maximum size. Then $(\mathcal{A}, A) \in I(n, t)$ and $|A| \geq C_{n-t}$. Repeatedly apply the splitting operations until we obtain a compressed $A^*$ with $|A^*| = |A|$ and $(A^*, A^*) \in I(n, t)$ (Lemma 2.5). By Lemma 2.6, $\gamma(A^*)$ is $t$-intersecting. If $\gamma(A^*)$ is non-trivial, then by Lemma 3.3, $|A| = |A^*| < C_{n-t}$, a contradiction. Hence, $\gamma(A^*)$ is trivial.

Let $T = \{x_1, \ldots, x_t\}$ be the $t$-set such that $T \subseteq \gamma(A)$ for all $A \in \mathcal{A}$. Let $\sigma$ be a permutation of $[n]$ with $\sigma(x_i) = i$ for all $i$. Then $\sigma(A^*) \subseteq D_0(t)$. Since $|\sigma(A^*)| = |A^*| \geq |D_0(t)|$, we deduce that $\sigma(A^*) = D_0(t)$. By Lemma 2.9, we conclude that $A \cong D_0(t)$.

This completes the proof of Theorem 1.4. □

**4. Proof of Theorem 1.6**

Let

$$C_k(n) = \{ A \in \mathcal{C}(n) : |A| = k \},$$

and $C_{n,k} = |C_k(n)|$. Clearly,

$$C_n = \sum_{k=1}^{n} C_{n,k}. \quad (5)$$

**Lemma 4.1.** For $n \geq 1$, $\tilde{C}_{n+1} \leq 2^{n+1}C_n$.

**Proof.** We first define a function $f : \tilde{C}(n+1) \to \mathcal{C}(n)$. Let $A = \{A_1, A_2, \ldots, A_k\}$

in $\tilde{C}(n+1)$. If $\{A_1 \setminus \{n+1\}, A_2 \setminus \{n+1\}, \ldots, A_k \setminus \{n+1\}\} \in \mathcal{C}(n)$, then we say $A$ is of Type I, otherwise, we say $A$ is of Type II.

If $A$ is of Type I, then we set

$$f(A) = \{A_1 \setminus \{n+1\}, A_2 \setminus \{n+1\}, \ldots, A_k \setminus \{n+1\}\}.$$ 

By Lemma 2.1, every set $A_i$ contains a critical element in $A$. Furthermore, if every $A_i$ contains a critical element different from $n+1$, then $\{A_1 \setminus \{n+1\}, A_2 \setminus \{n+1\}, \ldots, A_k \setminus \{n+1\}\} \subseteq \mathcal{C}(n)$ and only if there exists a unique $i_0 \in \{1, \ldots, k\}$ such that $n+1$ is the only critical element contained in $A_{i_0}$, in which case we have $A_{i_0} \setminus \{n+1\} \subseteq A_1 \cup \cdots \cup A_{i_0-1} \cup A_{i_0+1} \cup \cdots \cup A_k$ and $\{A_1, \ldots, A_{i_0-1}, A_{i_0+1}, \ldots, A_k\} \in \mathcal{C}(n)$. Therefore, if $A$ is of Type II, then we set

$$f(A) = \{A_1, \ldots, A_{i_0-1}, A_{i_0+1}, \ldots, A_k\},$$

where the last inequality follows from Lemma 3.2 for sufficiently large $n$ in terms of $t$. Hence, $|A| < C_{n-t}$. □
which is well-defined by the uniqueness of $i_0$.

Let $B = \{B_1, B_2, \ldots, B_k\} \in \mathcal{C}_k(n)$. Consider $\overline{B} = \{\overline{B}_1, \overline{B}_2, \ldots, \overline{B}_k\}$ where $\overline{B}_i = B_i \cup \{n + 1\}$ if $|B_i| = 1$, and $\overline{B}_i = B_i \cup \{n + 1\}$ or $B_i$ if $|B_i| \neq 1$. Note that $\overline{B} = \{\overline{B}_1, \overline{B}_2, \ldots, \overline{B}_k\} \in \tilde{C}(n + 1)$ and $f(\overline{B}) = B$. Therefore, the number of Type I minimal covers in $f^{-1}(B)$ is at most $2^n \leq 2^n$.

Let $C \in f^{-1}(B)$ be of Type II. Then $|B_i| \geq 2$ for $1 \leq i \leq k$ and $C = \{B_0, B_1, B_2, \ldots, B_k\}$ where $B_0 = A \cup \{n + 1\}$, $A \subseteq [n]$ and $A \neq \emptyset$. So, the number of Type II minimal covers in $f^{-1}(B)$ is at most $2^n$. Hence, $|f^{-1}(B)| \leq 2^n + 2^n = 2^{n+1}$.

Note that

$$\tilde{C}_{n+1} = \sum_{k=1}^{n} \sum_{B \in \mathcal{C}_k(n)} f^{-1}(B)$$

$$\leq 2^{n+1} \sum_{k=1}^{n} \sum_{B \in \mathcal{C}_k(n)} 1$$

$$= 2^{n+1} \sum_{k=1}^{n} C_{n,k} = 2^{n+1} C_n \quad \text{(by equation (5)).} \quad \square$$

Let $A_1, A_2 \subseteq \mathcal{C}(n)$ be compressed. We say that $(\gamma(A_1), \gamma(A_2))$ is trivial if there exists $x \in [n]$, such that $x \in \gamma(A)$ for all $A \in A_1$ and $A \in A_2$.

Lemma 4.2. There is a positive integer $n_0$, such that for $n \geq n_0$, if $A_1, A_2 \subseteq \mathcal{C}(n)$ are compressed, $(A_1, A_2) \in I(n,1)$, and $(\gamma(A_1), \gamma(A_2))$ is non-trivial, then

$$|A_1||A_2| < C_n^2.$$

Proof. For $1 \leq i \leq 2$ and $k \geq 1$, let $F_{ik} = \gamma(A_i) \cap \binom{[n]}{k}$. By Lemma 2.6, $\gamma(A_1), \gamma(A_2)$ are cross intersecting. Therefore, if $F_{ik} \neq \emptyset$ for $i = 1,2$, then $(\gamma(A_1), \gamma(A_2))$ is trivial. So, we may assume that $F_{21} = \emptyset$. By using Lemma 2.1, it is not hard to see that for each $A \in A_i$,

$$A \setminus \{\{x\} : x \in \gamma(A)\} \in \tilde{C}([n] \setminus \gamma(A)).$$

Therefore, $|A_1| \leq \sum_{1 \leq k \leq n} |F_{1k}|\tilde{C}_{n-k}$ and $|A_2| \leq \sum_{2 \leq k \leq n} |F_{2k}|\tilde{C}_{n-k}$. So,

$$|A_1| \leq \sum_{1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor} |F_{1k}|\tilde{C}_{n-k} + \sum_{\left\lceil \frac{n}{2} \right\rceil \leq k \leq n} |F_{1k}|\tilde{C}_{n-k}$$

$$\leq \sum_{1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor} |F_{1k}|\tilde{C}_{n-k} + \sum_{\left\lceil \frac{n}{2} \right\rceil \leq k \leq n} \binom{n}{k} \tilde{C}_{n-k},$$

and

$$|A_2| \leq \sum_{2 \leq k \leq \left\lceil \frac{n}{2} \right\rceil} |F_{2k}|\tilde{C}_{n-k} + \sum_{\left\lceil \frac{n}{2} \right\rceil \leq k \leq n} \binom{n}{k} \tilde{C}_{n-k}.$$
Let

\[ Q = \sum_{\lfloor \frac{n}{2} \rfloor \leq k \leq n} \binom{n}{k} \tilde{C}_{n-k}, \]
\[ M_1 = \sum_{1 \leq k < \lfloor \frac{n}{2} \rfloor} |F_{1k}| \tilde{C}_{n-k}, \]
\[ M_2 = \sum_{2 \leq k < \lfloor \frac{n}{2} \rfloor} |F_{2k}| \tilde{C}_{n-k}. \]

Then

\[ |A_1||A_2| \leq (M_1 + Q)(M_2 + Q) = M_1M_2 + M_1Q + M_2Q + Q^2. \]

By equation (3), \( \tilde{C}_{n-1} \leq C_{n-1} \). By Lemma 4.1, \( \tilde{C}_n \leq 2^n C_{n-1} \). By equation (4), for \( 2 \leq k < \lfloor \frac{n}{2} \rfloor \), \( \tilde{C}_{n-k} \leq \tilde{C}_n \leq 2^n C_{n-1} \). Therefore,

\[
M_1 \leq C_{n-1} |F_{11}| + 2^n C_{n-1} \sum_{2 \leq k < \lfloor \frac{n}{2} \rfloor} |F_{1k}|
\leq 2^{n+1} C_{n-1} \sum_{1 \leq k < \lfloor \frac{n}{2} \rfloor} |F_{1k}|
\leq 2^{n+1} C_{n-1} \sum_{1 \leq k < \lfloor \frac{n}{2} \rfloor} \binom{n}{k}
\leq 2^{2n+1} C_{n-1}.
\]

Similarly, \( M_2 \leq 2^{2n} C_{n-1} \).

By Lemma 3.2, \( Q \leq \frac{1}{8^n} \tilde{C}_{n-1} \leq \frac{1}{8^n} C_{n-1} < C_{n-1} \). Therefore

\[
M_1Q + M_2Q + Q^2 < (2^{2n+1} + 2^{2n} + 1)C_{n-1} \left( \frac{\tilde{C}_{n-1}}{8^n} \right)
\leq 3(2^{2n+1}) C_{n-1} \left( \frac{\tilde{C}_{n-1}}{8^n} \right)
= \frac{6}{2^n} C_{n-1} \tilde{C}_{n-1}
< \frac{1}{2} C_{n-1} \tilde{C}_{n-1}.
\]

By Theorem 1.2,

\[
M_1M_2 \leq \sum_{1 \leq k_1 < \lfloor \frac{n}{2} \rfloor, \ 2 \leq k_2 < \lfloor \frac{n}{2} \rfloor} \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \tilde{C}_{n-k_1} \tilde{C}_{n-k_2}.
\]
Hence, Lemma 4.3.

There exists a constant $A$.

Moreover, equality holds if and only if $I$.

Apply the splitting operations until we obtain compressed families $A$.

Note that $\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)$.

Proof.

$\gamma_{2.6}$, $I$.

This implies that $A$.

Proof of Theorem 1.6.

Let $\gamma$ be a permutation of $\left[\begin{array}{c} n \\ 1 \end{array}\right]$ with $\left[\begin{array}{c} n \\ 1 \end{array}\right]$ be such that $\left[\begin{array}{c} n \\ 1 \end{array}\right]$.

Then by Lemma 3.3, $\left[\begin{array}{c} n \\ 1 \end{array}\right]$.

Therefore, $\left(\prod_{i=1}^{r} |A_i| \right)^{r-1} \leq \prod_{1 \leq i < j \leq r} |A_i||A_j| \leq \prod_{1 \leq i < j \leq r} C_{n-1}^2$.
This proves the first part of Theorem 1.6.

Suppose equality holds. Then in equation (6), we must have

\[ |\mathcal{A}_i| |\mathcal{A}_j| = C^{2}_{n-1}. \]

By Lemma 4.3, \( \mathcal{A}_i = \mathcal{A}_j \) and \( \mathcal{A}_i \cong \mathcal{D}_0(1) \). This completes the proof of Theorem 1.6. \hfill \Box

Acknowledgments. We would like to thank the anonymous referee for the comments and suggestions that helped us make several improvements to this paper. Particularly, the referee’s observation about the family \( \mathcal{D}_t(t) \) of minimal covers has simplified and strengthened our original proofs and results.

References


Cheng Yeaw Ku
Department of Mathematics
National University of Singapore
117543, Singapore
E-mail address: matkcy@nus.edu.sg
Kok Bin Wong  
Institute of Mathematical Sciences  
University of Malaya  
50603 Kuala Lumpur, Malaysia  
E-mail address: kbwong@um.edu.my