TRANSLATION SURFACES OF TYPE 2 IN THE THREE DIMENSIONAL SIMPLY ISOTROPIC SPACE $I^3_1$

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Abstract. In this paper, we classify translation surfaces of Type 2 in the three dimensional simply isotropic space $I^3_1$ satisfying some algebraic equations in terms of the coordinate functions and the Laplacian operators with respect to the first, the second and the third fundamental form of the surface. We also give explicit forms of these surfaces.

1. Introduction

Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion of a connected $n$-dimensional manifold in the $m$-dimensional Euclidean space $\mathbb{E}^m$. Denote by $H$ and $\Delta$ the mean curvature and the Laplacian of $M$ with respect to the Riemannian metric on $M$ induced from that of $\mathbb{E}^m$, respectively. Takahashi proved that the submanifolds in $\mathbb{E}^m$ satisfying $\Delta x = \lambda x$, that is, all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue $\lambda \in \mathbb{R}$, are either the minimal submanifolds of $\mathbb{E}^m$ or the minimal submanifolds of hypersphere $S^{m-1}$ in $\mathbb{E}^m$ [15].

As an extension of Takahashi theorem, Garay studied hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated to the same eigenvalue in [8]. He considered hypersurfaces in $\mathbb{E}^m$ satisfying the condition

$$\Delta x = Ax,$$

where $A \in \text{Mat}(m, \mathbb{R})$ is an $m \times m$-diagonal matrix, and proved that such hypersurfaces are minimal ($H = 0$) in $\mathbb{E}^m$ and open pieces of either round hyperspheres or generalized right spherical cylinders. Related to this, Dillen, Pas and Verstraelen investigated surfaces in $\mathbb{E}^3$ whose immersions satisfy the

Received May 2, 2016.
2010 Mathematics Subject Classification. 53A35, 53B30, 53A40.
Key words and phrases. Laplace operator, simply isotropic space, translation surfaces.
The authors would like to thank the referees for their valuable comments which helped to improve the manuscript. The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2015R1D1A1A01060046).

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condition
\begin{equation}
\Delta x = A x + B,
\end{equation}
where $A \in \text{Mat}(3, \mathbb{R})$ is a $3 \times 3$-real matrix and $B \in \mathbb{R}^3$ [6]. In other words, each coordinate function is of 1-type in the sense of Chen [5]. The notion of an isometric immersion $x$ is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with the unit normal vector field to it. Dillen, Pas and Verstraelen studied surfaces of revolution in the three dimensional Euclidean space $\mathbb{E}^3$ such that its Gauss map $G$ satisfies the condition
\begin{equation}
\Delta G = AG,
\end{equation}
where $A \in \text{Mat}(3, \mathbb{R})$ [7]. Yoon studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map $G$ satisfies the condition (1.3) and also translation surfaces in a 3-dimensional Galilean space $G_3$ satisfies the condition
\begin{equation}
\Delta x_i = \lambda x_i,
\end{equation}
where $\lambda_i \in \mathbb{R}$ and provided some examples of new classes of translation surface [16, 17]. Baba-Hamed, Bekkar and Zoubir classified all translation surfaces in the 3-dimensional Lorentz-Minkowski space $\mathbb{R}^3_1$ under the condition (1.4) [2]. Bekkar and Senoussi studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition $\Delta_{III} r_i = \mu r_i$, where $\mu_i \in \mathbb{R}$ and $\Delta_{III}$ denotes the Laplacian of the surface with respect to the third fundamental form $III$ [3]. They showed that in both spaces a translation surface satisfying the preceding relation is a surface of Scherk. Aydin and Sipus studied constant curvatures of translation surfaces in the three dimensional simply Isotropic space $I_1^3$. Karacan, Yoon and Bukcu classified translation surfaces of Type 1 satisfying $\Delta^J x_i = \lambda_i x_i$, $J = 1, 2$ and $\Delta^I x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$ [4, 10].

In this paper, we classify the translation surfaces of Type 2 in the three dimensional simply Isotropic space $I_1^3$ is a Cayley–Klein space defined from the three dimensional projective space $\mathcal{P}(\mathbb{R}^3)$ with the absolute figure which is an ordered triple $(w, f_1, f_2)$, where $w$ is a plane in $\mathcal{P}(\mathbb{R}^3)$ and $f_1, f_2$ are two complex-conjugate straight lines in $w$. The homogeneous coordinates in $\mathcal{P}(\mathbb{R}^3)$ are introduced in such a way that the absolute plane $w$ is given by $x_0 = 0$ and the

2. Preliminaries

A simply isotropic space $I_1^3$ is a Cayley–Klein space defined from the three dimensional projective space $\mathcal{P}(\mathbb{R}^3)$ with the absolute figure which is an ordered triple $(w, f_1, f_2)$, where $w$ is a plane in $\mathcal{P}(\mathbb{R}^3)$ and $f_1, f_2$ are two complex-conjugate straight lines in $w$. The homogeneous coordinates in $\mathcal{P}(\mathbb{R}^3)$ are introduced in such a way that the absolute plane $w$ is given by $x_0 = 0$ and the
absolute lines $f_1, f_2$ by $x_0 = x_1 + ix_2 = 0, x_0 = x_1 - ix_2 = 0$. The intersection point $F(0:0:0:1)$ of these two lines is called the absolute point. The group of motions of the simply isotropic space is a six-parameter group given in the affine coordinates $x = a_1x, y = a_2x, z = a_3x$ by

$$
\begin{align*}
\begin{cases}
\xi = a + x \cos \theta - y \sin \theta, \\
\eta = b + x \sin \theta + y \cos \theta, \\
\zeta = c + cx_1 + c_2y + z,
\end{cases}
\end{align*}
$$

where $a, b, c, c_1, c_2, \theta \in \mathbb{R}$. Such affine transformations are called isotropic congruence transformations or $i$-motions.

Isotropic geometry has different types of lines and planes with respect to the absolute figure. A line is called non-isotropic (resp. completely isotropic) if its point at infinity does not coincide (coincides) with the point $F$. A plane is called non-isotropic (resp. isotropic) if its line at infinity does not contain $F$, otherwise. Completely isotropic lines and isotropic planes in this affine model appear as vertical, i.e., parallel to the $z$-axis. Finally, the metric of the simply isotropic space $\mathbb{R}^3$ is given by

$$
\begin{align*}
\begin{cases}
\frac{ds^2}{\sqrt{EG - F^2}} = dx^2 + dy^2.
\end{cases}
\end{align*}
$$

A surface $M$ immersed in $\mathbb{R}^3$ is called admissible if it has no isotropic tangent planes. For such a surface, the coefficients $E, F, G$ of its first fundamental form are calculated with respect to the induced metric and the coefficients $L, M, N$ of the second fundamental form, with respect to the normal vector field of a surface which is always completely isotropic. The (isotropic) Gaussian and (isotropic) mean curvature are defined by

$$
\begin{align*}
\begin{cases}
K = k_1k_2 = \frac{LN - M^2}{\sqrt{EG - F^2}}, & 2H = k_1 + k_2 = \frac{EN - 2FM + GL}{\sqrt{EG - F^2}},
\end{cases}
\end{align*}
$$

where $k_1, k_2$ are principal curvatures, i.e., extrema of the normal curvature determined by the normal section (in completely isotropic direction) of a surface. Since $EG - F^2 > 0$, for the function in the denominator we often put $W^2 = EG - F^2$. The surface $M$ is said to be isotropic flat (resp. isotropic minimal) if $K$ (resp. $H$) vanishes [1, 11, 13, 14].

It is well known in terms of local coordinates $\{u, v\}$ of $M$ the Laplacian operators $\Delta^I, \Delta^II, \Delta^III$ of the first, the second and the third fundamental form on $M$ are defined by

$$
\begin{align*}
\Delta^I_x = -\frac{1}{\sqrt{EG - F^2}} \left[ \frac{\partial}{\partial u} \left( \frac{Gx_u - Fx_v}{\sqrt{EG - F^2}} \right) - \frac{\partial}{\partial v} \left( \frac{Fx_u - Ex_v}{\sqrt{EG - F^2}} \right) \right],
\end{align*}
$$

$$
\begin{align*}
\Delta^II_x = -\frac{1}{\sqrt{\left| LN - M^2 \right|}} \left[ \frac{\partial}{\partial u} \left( \frac{Nz_u - Mz_v}{\sqrt{\left| LN - M^2 \right|}} \right) - \frac{\partial}{\partial v} \left( \frac{Mz_u - Lz_v}{\sqrt{\left| LN - M^2 \right|}} \right) \right].
\end{align*}
$$
\[\Delta^{III} \mathbf{x} = -\frac{\sqrt{E^G - F^2}}{L^2 - M^2} \left[ \frac{\partial}{\partial u} \left( \frac{2x_x - x_y}{(L^2 - M^2)^{1/2}} \sqrt{E^G - F^2} \right) \right] - \frac{\partial}{\partial v} \left( \frac{Y_x - X_y}{(L^2 - M^2)^{1/2}} \sqrt{E^G - F^2} \right), \]

where

\[X = EM^2 - 2FLM + GL^2,\]
\[Y = EMN - FLN + GLM - FM^2,\]
\[Z = GM^2 - 2FNM + EN^2\]

[2, 3, 4, 9, 10, 12].

3. Translation surfaces in $\mathbb{I}_3$

In order to describe the isotropic analogues of translation surfaces of constant curvatures, we consider translation surfaces obtained by translating two planar curves. The local surface parametrization is given by

\[\mathbf{x}(u, v) = \alpha(u) + \beta(v).\]

Since there are, with respect to the absolute figure, different types of planes in $\mathbb{I}_3$, there are in total three different possibilities for planes that contain translated curves: the translated curves can be curves in isotropic planes (which can be chosen, by means of isotropic motions, as $y = 0$, resp. $x = 0$); or one curve is in a non-isotropic plane ($z = 0$) and one curve in an isotropic plane ($y = 0$); or both curves are curves in non-isotropic perpendicular planes ($y - z = \pi$, resp. $y + z = \pi$). Therefore, the obtained translation surfaces allow the following parametrizations:

**Type 1:** The surface $\mathbf{M}$ is parametrized by

\[\mathbf{x}(u, v) = (u, v, f(u) + g(v)),\]

and the translated curves are $\alpha(u) = (u, 0, f(u))$, $\beta(v) = (0, v, g(v))$.

**Type 2:** The surface $\mathbf{M}$ is parametrized by

\[\mathbf{x}(u, v) = (u, f(u) + g(v), v),\]

and the translated curves are $\alpha(u) = (u, f(u), 0)$, $\beta(v) = (0, g(v), v)$. In order to obtain admissible surfaces, $g'(v) \neq 0$ is assumed (i.e., $g(v) \neq \text{const.}$).

**Type 3:** The surface $\mathbf{M}$ is parametrized by

\[\mathbf{x}(u, v) = \frac{1}{2} \left( f(u) + g(v), u - v + \pi, u + v \right),\]

and the translated curves are

\[\alpha(u) = \frac{1}{2} \left( f(u), u + \frac{\pi}{2}, u - \frac{\pi}{2} \right), \beta(v) = \left( g(v), \frac{\pi}{2} - v, \frac{\pi}{2} + v \right).\]

In order to obtain admissible surfaces, $f'(u) + g'(v) \neq 0$ is assumed (i.e., $f'(u) \neq -g'(v) = a = \text{constant.}$) [13].
In this paper, we will investigate the translation surfaces of Type 2.

4. Translation surfaces of Type 2 satisfying $\Delta^1 x_i = \lambda_i x_i$

In this section, we classify translation surface of Type 2 in $I_3$ satisfying the equation

$$\Delta^1 x_i = \lambda_i x_i,$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ and

$$\Delta^1 x = (\Delta^1 x_1, \Delta^1 x_2, \Delta^1 x_3),$$

where

$$x_1 = u, \quad x_2 = f(u) + g(v), \quad x_3 = v.$$

For the translation surface given by (3.3), the coefficients of the first and second fundamental form are

$$E = 1 + f'^2, \quad F = f'g', \quad G = g'^2,$$

$$L = -\frac{f''}{g'}, \quad M = 0, \quad N = -\frac{g''}{g'},$$

respectively. The Gaussian curvature $K$ and the mean curvature $H$ are

$$K = \frac{f''(u)g''(v)}{g'^2(v)}, \quad H = -\frac{g'^2(v)f''(u) + (1 + f'^2(u))g''(v)}{2g'^3(v)},$$

respectively.

Suppose that the surface has non zero Gaussian curvature, so

$$f''(u)g''(v) \neq 0, \forall u, v \in I.$$

By a straightforward computation, the Laplacian operator on $M$ with the help of (4.2) and (2.3) turns out to be

$$\Delta^1 x = \begin{pmatrix} 0, 0, 0 \end{pmatrix}.$$

Suppose that $M$ satisfies (4.1). Then from (4.5), we have

$$\left( \frac{g'^2 f'' + (1 + f'^2) g''}{g'^3} \right) = \lambda,$$

where $\lambda \in \mathbb{R}$. This means that $M$ is at most of 1-type. First of all, we assume that $M$ satisfies the condition $\Delta^1 x = 0$. We call a surface satisfying that condition a harmonic surface or isotropic minimal. In this case, we get from (4.6)

$$g'^2 f'' + (1 + f'^2) g'' = 0.$$
Here $u$ and $v$ are independent variables, so each side of (4.7) is equal to a constant, call it $p$. Hence, the two equations
\[ \frac{f''}{1 + f'\!\!'} = p = -\frac{g''}{g'\!\!'} . \]
Thus we get
\[ f(u) = c_1 - \frac{\ln (\cos (pu - c_2))}{p}, \]
\[ g(v) = c_3 + \frac{\ln (pv - c_4)}{p}, \]
where $p, c_i \in \mathbb{R}$. In this case, $M$ is parametrized by
\[ x(u, v) = \left( u, \left( c_1 - \frac{\ln (\cos (pu - c_2))}{p} \right) + \left( c_3 + \frac{\ln (pv - c_4)}{p} \right), v \right). \]
In particular, if $p = 0$, we have
\[ f(u) = c_1 u + c_2, \]
\[ g(v) = c_3 v + c_4, \]
where $c_i \in \mathbb{R}$. In this case, $M$ is parametrized by
\[ x(u, v) = (u, (c_1 u + c_2) + (c_3 v + c_4), v) . \]
Using the solution (4.10) gives rise a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form.

**Theorem 4.1.** Let $M$ be a translation surface given by (3.3) in $\Omega_1$. If $M$ is harmonic or isotropic minimal, then it is congruent to an open part of the surface (4.9). The obtained surface is the isotropic Scherk’s surfaces of the second type.

If $\lambda \neq 0$, from (4.6), we have
\[ g^2 f'' + \left( 1 + f'\!\!'^2 \right) g'' = \lambda v g'. \]
This differential equation (4.12) admits the solutions
\[ f(u) = c_1, \quad g(v) = c_2, \]
\[ f(u) = c_1, \quad g(v) = \pm \frac{\arctan \left( \sqrt{\frac{w}{\lambda}} \right)}{\sqrt{\lambda}} + c_2, \]
\[ f(u) = c_1 u + c_2, \quad g(v) = c_3 \pm \frac{\sqrt{-1 - c_1^2 \ln \left( v \lambda + \sqrt{\lambda} \sqrt{v^2 + 2c_4 + 2c_4^2 c_1} \right)}}{\sqrt{\lambda}}, \]
\[ f(u) = \frac{1}{2} c_3 u^2 v\lambda + c_1 u + c_2, \quad g(v) = c_3 v + c_4, \]
TRANSLATION SURFACES OF TYPE 2 IN THE $\mathbf{I}^3_1$

where $c_i \in \mathbb{R}$. But, using the solutions (4.13) give rise a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form. Based on the selection of the function $f(u)$, it is possible to obtain other form of the function $g(v)$, vice versa. For example, if we choose $g(v) = v^2$, we have

$$f(u) = c_1 + 2v^2 \log \left[ \cos \left( \frac{(u + 2v^2c_2)\sqrt{1 - 4\lambda v^4}}{2v^2} \right) \right],$$

where $c_1, c_2 \in \mathbb{R}$. In this case, $M$ is parametrized by

$$(4.14) \quad x(u, v) = \left( u, \left( c_1 + 2v^2 \log \left[ \cos \left( \frac{(u + 2v^2c_2)\sqrt{1 - 4\lambda v^4}}{2v^2} \right) \right] \right) + v^2, v \right).$$

**Theorem 4.2.** Let $M$ be a non harmonic translation surface given by (3.3) in the three dimensional simply isotropic space $\mathbf{I}^3_1$. If the surface $M$ satisfies the condition $\Delta^I x_i = \lambda_i x_i$, where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$, then it is congruent to an open part of the surface (4.14). The obtained surface is the isotropic Scherk’s surface of the second type.

5. Translation surfaces of Type 2 satisfying $\Delta^H x_i = \lambda_i x_i$

In this section, we classify translation surface of Type 2 with non-degenerate second fundamental form in $\mathbf{I}^3_1$ satisfying the equation

$$(5.1) \quad \Delta^H x_i = \lambda_i x_i,$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ and

$$\Delta^H x = (\Delta^H x_1, \Delta^H x_2, \Delta^H x_3),$$

where

$$x_1 = u, \quad x_2 = f(u) + g(v), \quad x_3 = v.$$

By a straightforward computation, the Laplacian operator on $M$ with the help of (4.3) and (2.4) turns out to be

$$\Delta^H x = \left( -\frac{g'f''}{2f'}, \frac{g'}{2} \left( 4 - \frac{f'f'''}{f'^2} - \frac{g'g''}{g'^2} \right), -\frac{g'g''}{2g'} \right).$$

The equation (5.1) by means of (5.2) gives rise to the following system of ordinary differential equations

$$(5.3) \quad -\frac{g'f''}{2f'} = \lambda_1 u,$$

$$(5.4) \quad \frac{g'}{2} \left( 4 - \frac{f'f'''}{f'^2} - \frac{g'g''}{g'^2} \right) = \lambda_2 (f(u) + g(v)),$$

$$(5.5) \quad -\frac{g'g''}{2g'} = \lambda_3 v.$$
where $\lambda_i \in \mathbb{R}$. This means that $M$ is at most of 3-types. Combining equations (5.3), (5.4) and (5.5), we have
(5.6) \[ g'(v) (2 + \lambda_1 u + \lambda_3 v) = \lambda_2 (f(u) + g(v)) \]
We discuss six cases according to constants $\lambda_1, \lambda_2, \lambda_3$.

**Case 1:** Let $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$, from (5.6), we obtain
(5.7) \[ g'(2 + \lambda_3 v) = \lambda_2 f + \lambda_2 g. \]
Here $u$ and $v$ are independent variables, so each side of (5.7) is equal to a constant, call it $p$. Hence, the two equations
(5.8) \[ g'(2 + \lambda_3 v) - \lambda_2 g = p = \lambda_2 f. \]
Their general solutions are
(5.9) \[
    f(u) = \frac{p}{\lambda_2}, \quad g(v) = -\frac{p}{\lambda_2} + c_1 (2 + \lambda_3 v)^{\frac{\lambda_2}{\lambda_3}},
\]
where for some constant $c_1 \neq 0$ and $\lambda_2 \neq 0, \lambda_3 \neq 0$. In particular, if $p = 0$, then we have
(5.10) \[
    f(u) = 0, \quad g(v) = c_1 (2 + \lambda_3 v)^{\frac{\lambda_2}{\lambda_3}},
\]
where $c_1 \in \mathbb{R}$.

**Case 2:** Let $\lambda_1 = \lambda_2 = 0, \lambda_3 \neq 0$, from (5.6), we obtain
(5.11) \[ g'(v) (2 + \lambda_3 v) = 0. \]
Thus we get
(5.12) \[ g(v) = c_1, \]
where $c_1 \in \mathbb{R}$.

**Case 3:** Let $\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0$, from (5.6), we obtain
(5.13) \[ g'(2 + \lambda_1 u) = 0. \]
Thus we get (5.11).

**Case 4:** Let $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 = 0$, from (5.6), we obtain
(5.14) \[ 2g' - \lambda_2 g = \lambda_2 f. \]
Here $u$ and $v$ are independent variables, so each side of (5.14) is equal to a constant, call it $p$. Hence, the two equations
\[ 2g' - \lambda_2 g = p = \lambda_2 f. \]
Their general solutions are
(5.15) \[
    f(u) = \frac{p}{\lambda_2}, \quad g(v) = \frac{p}{\lambda_2} + c_1 e^{\frac{\lambda_2}{\lambda_3}},
\]
where for some constant $c_1 \neq 0$ and $\lambda_2 \neq 0$.

**Case 5:** Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$, from (5.6), we obtain
\[
2g'(v) = 0.
\]
Thus we get (5.11). Here, the function $f(u)$ independent of selection of the function $g(v)$.

**Case 6:** Let $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$, from (5.6), we obtain
\[
g'(2 + \lambda_1 u) = \lambda_2 f + \lambda_2 g.
\]
Here $u$ and $v$ are independent variables, so each side of (5.15) is equal to a constant, call it $p$. Hence, the two equations
\[
2g' - \lambda_1 ug' - \lambda_2 g = p = \lambda_2 f.
\]
Their general solutions are
\[
(5.17) \quad f(u) = \frac{p}{\lambda_2}, \quad g(v) = -\frac{p}{\lambda_2} + c_1 e^{-\frac{2u}{\lambda_1}},
\]
where for some constant $c_1 \neq 0$ and $\lambda_2 \neq 0$.

Using the solutions (5.9), (5.10), (5.11), (5.14) and (5.17) give rise a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form.

**Definition.** A surface of in the three dimensional simple isotropic space is said to be II-harmonic if it satisfies the condition $\Delta II x = 0$.

**Theorem 5.1.** Let $M$ be a translation surface given by (3.3) in the three dimensional simply isotropic space $I_{13}$. Then $M$ is II-harmonic if and only if it is an open part of the following non isotropic plane
\[
\mathbf{x}(u, v) = (u, f(u) + c_1, v).
\]
The obtained surfaces is a cylindrical surface.

**Theorem 5.2.** Let $M$ be a non II-harmonic translation surface with non-degenerate second fundamental form given by (3.3) in the three dimensional simply isotropic space $I_{13}$. There are no surfaces satisfy the condition
\[
\Delta II x_i = \lambda_i x_i,
\]
where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$.

### 6. Translation surfaces of Type 2 satisfying $\Delta III x_i = \lambda_i x_i$

In this section, we classify translation surface of Type 2 in $I_{13}$ satisfying the equation
\[
(6.1) \quad \Delta III x_i = \lambda_i x_i,
\]
where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ and
\[
\Delta III x = (\Delta III x_1, \Delta III x_2, \Delta III x_3),
\]
where \( x_1 = u, x_2 = f(u) + g(v), x_3 = v \).

Suppose that the surface has non zero Gaussian curvature, so \( f''(u)g''(v) \neq 0, \forall u \in I \). By a straightforward computation, the Laplacian operator on \( M \) with the help of (4.2), (4.3) and (2.5) turns out to be

\[
\Delta^{III}x = \left( -\frac{g'f'''}{2f''}, -\frac{4g'f''g'' + f'g'g'' + g'g''}{2f''g''}, -\frac{g'g'''}{2g''} \right).
\]

Equation (6.1) by means of (6.2) gives rise to the following system of ordinary differential equations

\[
\begin{align*}
\left( -\frac{g'f'''}{2f''} \right) = \lambda_1 u, \\
-\frac{4g'f''g'' + f'g'g'' + g'g''}{2f''g''} = \lambda_2 \left( f(u) + g(v) \right), \\
\left( -\frac{g'g'''}{2g''} \right) = \lambda_3 v,
\end{align*}
\]

where \( \lambda_1, \lambda_2 \) and \( \lambda_3 \in \mathbb{R} \). This means that \( M \) is at most of 3- types. Combining equations (6.3), (6.4) and (6.5), we have

\[
2g' + \lambda_1 uf' + \lambda_3 vg' = \lambda_2 \left( f(u) + g(v) \right).
\]

Here \( u \) and \( v \) are independent variables, so each side of (6.6) is equal to constant, call it \( p \). Hence, we have

\[
\lambda_1 uf' - \lambda_2 f = p = -\lambda_3 vg' - 2g' + \lambda_2 g.
\]

Their general solutions are given by

\[
\begin{align*}
f(u) &= \frac{p}{\lambda_2} + c_1 u^{\frac{\lambda_2}{\lambda_1}}, \\
g(v) &= \frac{p}{\lambda_2} + c_2 \left( 2 + \lambda_3 v \right)^{\frac{\lambda_2}{\lambda_3}},
\end{align*}
\]

where \( p, c_1, c_2 \in \mathbb{R} \) with \( \lambda_i \neq 0 \). In this case, \( M \) is parametrized by

\[
x(u, v) = \frac{1}{2} \left( u, \left( -\frac{p}{\lambda_2} + c_1 u^{\frac{\lambda_2}{\lambda_1}} \right), \left( \frac{p}{\lambda_2} + c_2 \left( 2 + \lambda_3 v \right)^{\frac{\lambda_2}{\lambda_3}} \right), v \right).
\]

In particular, if \( p = 0 \),

\[
\begin{align*}
f(u) &= c_1 u^{\frac{\lambda_2}{\lambda_1}}, \\
g(v) &= c_2 \left( 2 + \lambda_3 v \right)^{\frac{\lambda_2}{\lambda_3}}.
\end{align*}
\]

In this case, \( M \) is parametrized by

\[
x(u, v) = \left( u, \left( c_1 u^{\frac{\lambda_2}{\lambda_1}} \right), \left( c_2 \left( 2 + \lambda_3 v \right)^{\frac{\lambda_2}{\lambda_3}} \right), v \right).
\]

We discuss four cases according to constants \( \lambda_1, \lambda_2, \lambda_3 \).
Case 1: Let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda \neq 0$, from (6.7), we obtain
$$\lambda uf' - \lambda f = p = -\lambda vg' - 2g' + \lambda g,$$
and their general solutions are
$$f(u) = -\frac{p}{\lambda} + c_1 u,$$
$$g(v) = \frac{p}{\lambda} + c_2 (2 + \lambda v),$$
where $\lambda, p, c_1 \in \mathbb{R}$. Using the solution (6.11) gives a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form.

Case 2: Let $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$, from (6.7), we obtain
$$p = -\lambda_3 vg' - 2g'.$$
Thus we have
$$f(u), g(v) = c_1 - \frac{p \ln (2 + cv)}{\lambda_3},$$
where $c_1, p \in \mathbb{R}$. Here, the function $f(u)$ is independent of selection of the function $g(v)$. In this case, $M$ is parametrized by
$$x(u, v) = \left( u, f(u) + \left( c_1 - \frac{p \ln (2 + cv)}{\lambda_3} \right), v \right).$$

Case 3: Let $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$, from (6.7), we obtain
$$\lambda_1 uf' - \lambda_2 f = p = -2g' + \lambda_2 g,$$
and
$$f(u) = -\frac{p}{\lambda_2} + c_1 u \frac{\lambda_2}{\lambda_1},$$
$$g(v) = \frac{p}{\lambda_2} + ce \frac{\lambda_2}{\lambda_1},$$
where $c, p \in \mathbb{R}$. In this case, $M$ is parametrized by
$$x(u, v) = \left( u, \left( -\frac{p}{\lambda_2} + c_1 u \frac{\lambda_2}{\lambda_1} \right) + \left( \frac{p}{\lambda_2} + ce \frac{\lambda_2}{\lambda_1} \right), v \right).$$

Case 4: Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$, from (4.12), we obtain
$$g(v) = c_1 + \frac{p}{2} v,$$
where $c_1 \in \mathbb{R}$. Here, the function $f(u)$ is independent of selection of the function $g(v)$. In this case, $M$ is parametrized by
$$x(u, v) = \left( u, v, f(u) + \left( c_1 + \frac{p}{2} v \right) \right).$$

Definition. A surface of in the three dimensional simple isotropic space is said to be $\text{III}$-harmonic if it satisfies the condition $\Delta_{\text{III}} x = 0$. 
Theorem 6.1. Let $M$ be a translation surface by (3.3) in the three dimensional simply isotropic space $I^1_3$. If $M$ is $III$-harmonic, then it is congruent to an open part of the surface (6.20). The obtained surfaces is a cylindrical surface.

Theorem 6.2 (Classification). Let $M$ be a translation surface with non-degenerate second fundamental form given by (3.3) in the three dimensional simply isotropic space $I^1_3$. The surfaces $M$ satisfies the condition $\Delta^{III}x_i=\lambda_ix_i$, where $\lambda_i\in\mathbb{R}$, then it is congruent to an open part of the surfaces (6.9), (6.11), (6.15) and (6.18). The obtained surfaces are the isotropic Scherk’s surfaces of the first type.

References

TRANSLATION SURFACES OF TYPE 2 IN THE $\mathbb{S}^3_1$

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