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A NOTE ON ZEROS OF BOUNDED HOLOMORPHIC FUNCTIONS IN WEAKLY PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$

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Abstract. Let Ω be a bounded, uniformly totally pseudoconvex domain in $\mathbb{C}^2$ with the smooth boundary $b\Omega$. Assuming that Ω satisfies the negative $\bar{\partial}$ property. Let M be a positive, finite area divisor of Ω. In this paper, we will prove that: if Ω admits a maximal type F and the Čeck cohomology class of the second order vanishes in Ω, there is a bounded holomorphic function in Ω such that its zero set is M. The proof is based on the method given by Shaw [27].

1. Introduction. Statement of results

Let Ω be a smooth, bounded domain in $\mathbb{C}^2$ and let M be a positive divisor of Ω. In this paper, we concern with the problem is to find some conditions on Ω, such that there exists a bounded holomorphic function $\mathfrak{g}$ defined on Ω whose zero set $Z(\Omega, \mathfrak{g})$ is M.

In the earlier work [10], the existence of a Nevanlinna holomorphic function defining M is established when Ω is uniformly totally pseudoconvex and admits a maximal type F (see Definition 2.2) at all boundary points. Here, the maximal type F coincides with the notion of finite type in the sense of Range [21] for $F(t) = t^m$ and the notion of infinite type for $F(t) = \exp(\frac{-1}{t^s}), 0 < s < 1/2$.

More precisely, we have:

Theorem 1.1. *Let Ω be a smoothly bounded, uniformly totally pseudoconvex domain which admits the maximal type F at all boundary points, for some function F. Assuming that $\bar{\Omega}$ admits a Stein neighborhood basis and the negative $\bar{\partial}$ property (see Definition 2.4), and the Čech cohomology class of the second degree $H^2(\Omega, \mathbb{Z}) = 0$. Let M be a positive, finite area divisor in Ω. Then, for some bounded holomorphic function $\mathfrak{g}$ on Ω, we have*

$$M = Z(\Omega, \mathfrak{g}).$$

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Theorem 1.1 is a result of the following boundary regularity for solutions to the Poincaré-Lelong equation.

Theorem 1.2. *Let Ω be a smoothly bounded, uniformly totally pseudoconvex domain which admits the maximal type F at all boundary points, for some function F. Assuming that $\bar{\Omega}$ admits a Stein neighborhood basis and the negative $\bar{\partial}$ property holds on Ω, and the DeRham cohomology of the second degree $H^2(\Omega, \mathbb{R}) = 0$. Let α be a positive d-closed, smooth $(1,1)$-form on $\bar{\Omega}$, then the Poincaré - Lelong equation*

$$i\partial\bar{\partial}u = \alpha$$

admits a negative solution u such that:

$$||u||_{L^1(b\Omega)} + ||u||_{L^1(\Omega)} \leq C||\alpha||_{L^1(\Omega)},$$

where C is independent in α.

The paper is organized as follows: In Section 2, we recall and introduce the materials which are used in the paper. In Section 3, we prove Theorem 1.2. Theorem 1.1 is proven in Section 4.

2. Preliminaries

2.1. The tangential Cauchy-Riemann equation

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary $b\Omega$. Let ρ be a defining function for Ω such that $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ and $\nabla\rho \neq 0$ on $b\Omega = \{z \in \mathbb{C}^2 : \rho(z) = 0\}$, and $\nabla\rho \perp b\Omega$. The pseudoconvexity means on $b\Omega$ we have

$$\langle \partial\bar{\partial}\rho, L \wedge \bar{L}\rangle \geq 0,$$

where L is an any nonzero tangential holomorphic vector field. If the strict inequality holds on the boundary, Ω is said to be strongly pseudoconvex.

Definition 2.1 ([21])**.** Ω is said to be uniformly totally pseudoconvex at the point $P \in b\Omega$ if there are positive constants δ, c and a C^1 map $\Psi : U^\delta \times \Omega^\delta \to \mathbb{C}$ such that for all boundary points $\zeta \in b\Omega \cap B(P, \delta)$, the following properties are satisfied:

(1) $\Psi(\zeta, .)$ is holomorphic on Ω;
(2) $\Psi(\zeta, \zeta) = 0$, and $d_z\Psi|_{z=\zeta} \neq 0$;
(3) $\rho(z) > 0$ for all z with $\Psi(\zeta, z) = 0$ and $0 < |z - \zeta| < c$.
By multiplying ρ and Ψ by suitable non-zero functions of ζ, one may assume more
(4) $|\partial\rho(\zeta)| = 1$, and $\partial\rho(\zeta) = d_z\Psi|_{z=\zeta}$,

where $\Omega^\delta = \{z \in \mathbb{C}^2 : \rho(z) < \delta\}$, and $U^\delta = \Omega^\delta \setminus \Omega$.

Definition 2.2 ([9, 10])**.** Let $F : [0, \infty) \to [0, \infty)$ be a smooth, increasing function such that

(1) $F(0) = 0$;

(2) $\int_0^R |\ln F(r^2)|dr < \infty$ for some $R > 0$;
(3) $\frac{F(r)}{r}$ is increasing.

Let $\Omega \subset \mathbb{C}^2$ be uniformly totally pseudoconvex at $P \in b\Omega$. Ω is called a domain admitting maximal type F at the boundary point $P \in b\Omega$ if there are positive constants c, c', such that, for all $\zeta \in b\Omega \cap B(P, c')$ we have

$$\rho(z) \gtrsim F(|z_1 - \zeta_1|^2) \quad \text{for all } z \in B(\zeta, c) \text{ with } \Psi(\zeta, z) = 0.$$

Here and in what follows, the notations $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant, and $\approx$ means the combination of $\lesssim$ and $\gtrsim$.

Example 2.1.

(1) Let $\Omega \subset \mathbb{C}^2$ be pseudoconvex of strict finite type $m(p)$ at every point $p \in b\Omega$ as defined in [17], and generalized by Range in [21, 22], Shaw in [26]. And let $m_0 := \sup_{p \in b\Omega} m(p) < \infty$ and $F(t) = t^{m_0/2}$. We define

$$\Psi(\zeta, z) = \sum_{s+t \leq m_0} \frac{1}{s!t!} \frac{\partial^{s+t}\rho}{\partial \zeta_1^s \partial \zeta_2^k} (z_1 - \zeta_1)^s (z_2 - \zeta_2)^k.$$

Then Ω, in this case, admits the maximal type F. In particular, Ω is finite type in the sense of Range.

(2) Let

$$\Omega^\infty = \{(z_1, z_2) \in \mathbb{C}^2 : \exp(1 + 2/s) \cdot \exp\left(\frac{-1}{|z_1|^s}\right) + |z_2|^2 - 1 < 0\}.$$

Then, for $0 < s < 1/2$, Ω^∞ is a convex domain admitting the maximal type $F(t) = \exp(\frac{-1}{32.t^s})$, see [29].

It is well-known that on infinite type domains in the sense of Range, e.g. the domain Ω^∞, the $\bar\partial$ and $\bar\partial_b$ have no solution in any Hölder class of any positive order. Therefore, the following definitions are necessary to understand pointwise boundary regularities of $\bar\partial_b$-solutions on such domains.

Let f be an increasing function such that $\lim_{t \to +\infty} f(t) = +\infty$. We define the f-Hölder space on $b\Omega$ by

$$\Lambda^f(b\Omega) = \left\{ u \in L^\infty(b\Omega) : ||u||_{L^\infty} + \sup_{\substack{x(.) \in \mathcal{C} \\ 0 \leq t \leq 1}} f(t^{-1})|u(x(t)) - u(x(0))| < +\infty \right\},$$

where the class of curves $\mathcal{C}$ in $b\Omega$ is

$$\mathcal{C} = \left\{ x(t) : t \in [0, 1] \to x(t) \in b\Omega,\ x(t) \text{ is } C^1 \text{ and } |x'(t)| \leq 1 \right\}.$$

That means $\Lambda^f(b\Omega)$ consists all complex-valued functions u such that for each curve $x(\cdot) \in \mathcal{C}$, the function $t \mapsto u(x(t)) \in \Lambda^f([0, 1])$.

For $1 \le p < \infty$, the f-Besov space is denoted by

$$\Lambda_p^f(b\Omega) = \{u \in L^p(b\Omega) : \\ ||u||_{L^p} + \sup_{0\le t\le 1} f(t^{-1}) \left[\left(\int_{b\Omega} |u(x(t)) - u(x(0))|^p dx \right)^{1/p} \right] < +\infty \Bigg\},$$

where the integral is taken in $x = x(t) \in \mathcal{C}$ over the boundary $b\Omega$. It is obvious that $\Lambda_\infty^f(b\Omega) = \Lambda^f(b\Omega)$. In the cases $f(t) = t^m$ for $m = 1, 2, \ldots$, Λ^f are really usual Hölder spaces or Besov spaces on $b\Omega$.

The following global solvability of the tangential Cauchy-Riemann equations on the boundary $b\Omega$ in L^p-spaces was proved in [9]:

Theorem 2.3. *Let Ω be a smoothly bounded, uniformly totally pseudoconvex domain admitting maximal type F, for some function F. Assuming that $\bar\Omega$ admits a Stein neighborhood basis. Let φ belong to $L^p_{0,1}(b\Omega)$, for $1 \le p \le \infty$, and satisfy the compatibility condition*

$$\int_{b\Omega} \varphi \wedge \alpha = 0$$

for every $\bar\partial$-closed $(2,0)$-form α defined on Ω and being continuous up to $b\Omega$.

Let F^* be the inversion of F, and let

$$f(d^{-1}) := \left(\int_0^d \frac{\sqrt{F^*(t)}}{t} dt \right)^{-1}.$$

Then, there exists a function u defined on $b\Omega$ such that $\bar\partial_b u = \varphi$ on $b\Omega$ and

(1) $||u||_{\Lambda^f(b\Omega)} \le C||\varphi||_{L^\infty(b\Omega)}$, *if* $p = \infty$.
(2) $||u||_{L^p(b\Omega)} \le C_p||\varphi||_{L^p_{(0,1)}(b\Omega)}$, *if* $1 \le p < \infty$, *where* $C_p > 0$ *independent on* φ.
(3) $||u||_{\Lambda_p^f(b\Omega)} \le C_p||\varphi||_{L^p(b\Omega)}$ *for every* $1 \le p \le \infty$.

For examples,

- For $m = 1, 2, \ldots$, let
$$\Omega^m = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 < 1\}.$$
Then Ω is smoothly bounded, convex domain admitting the maximal type $F(t) = t^m$, and so $f(t) = t^{1/m}$.
- Let recall
$$\Omega^\infty = \{(z_1, z_2) \in \mathbb{C}^2 : \exp(1 + 2/s) \cdot \exp\left(\frac{-1}{|z_1|^s}\right) + |z_2|^2 - 1 < 0\}$$
for $0 < s < 1/2$, Ω^∞ is a convex domain admitting the maximal type $F(t) = \exp\left(\frac{-1}{32\cdot t^s}\right)$. Then, $f(t) = \frac{1024^s(1-2s)}{2s} (|\ln t|)^{\frac{1}{2s}-1}$.

Definition 2.4. Ω is called satisfying the negative $\bar\partial$ property if and only if for every solution of $\bar\partial u = \phi$, there is a pluriharmonic function λ defined on $\bar\Omega$ such that:

- $(2 \cdot \mathrm{Im}(u) + \lambda)$ is negative.
- $||\lambda||_{L^1(b\Omega)} + ||\lambda||_{L^1(\Omega)} \lesssim ||\phi||_{L^1(\Omega)} + ||\phi||_{L^1(b\Omega)}$.

2.2. Some basic facts of Lelong theory

For the following notions and results, we refer to [28] or [20].

Definition 2.5. Let $M := \{M_j\}$ be a locally finite family of hypersurfaces of Ω. The formal sum

$$\sum_j a_j M_j$$

with $a_j \in \mathbb{Z}$ is called a divisor of Ω.

For a given divisor M of Ω there are uniquely distinct irreducible hypersurfaces $\{M_j\}$ of Ω and $a_j \in \mathbb{Z} \setminus \{0\}$ such that we have the following irreducible decomposition

$$M = \sum_{a_j \neq 0} a_j M_j.$$

If $M = \sum_{a_j \neq 0} a_j M_j$ with $a_j > 0$ for all j, we call M to be a positive divisor of Ω, and write $M > 0$.

For example, let h be a holomorphic function on Ω. Then the hypersurface $M_h := \{z \in \Omega : h = 0\}$ is a positive divisor, and

$$M_h = \sum_{a_j \neq 0} a_j M_j,$$

where $a_j > 0$ is the zero order of h on M_j. In this case, M_h is also called the zero divisor of Ω.

Theorem 2.6 (Cartan)**.** *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. If the cohomology group $H^2(\Omega, \mathbb{Z}) = 0$, and M is a positive divisor of Ω, then*

$$M = Z(\Omega, \mathfrak{g})$$

for some holomorphic function $\mathfrak{g}$ defined on Ω.

Theorem 2.7 (Poincaré-Lelong Formula [20])**.** *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Let $h \neq 0$ be a meromorphic function on Ω and let η be a 2-form of C^2 class on Ω with compact support. Then,*

$$\frac{1}{2\pi} \partial\bar{\partial}[\log |h|^2] = M_h,$$

that is

$$\int_{M_h} \eta = \frac{1}{2\pi} \int_\Omega \log |h|^2 \partial\bar{\partial}\eta = \frac{1}{2\pi} \int_\Omega \partial\bar{\partial}[\log |h|^2] \wedge \eta$$

in this sense of currents.

Definition 2.8. Let $M = \sum_{a_j \neq 0} a_j M_j$ be a divisor of Ω and $d\delta$ be the surface measure on M. Then, M is called to have finite area if

$$\sum_{a_j \neq 0} a_j \int_{z \in M_j} d\delta(z)$$

is finite.

In [1], the negative $\bar{\partial}$ property holds on all balls in $\mathbb{C}^2$. Hence, we have:

Theorem 2.9 ([1])**.** *Assuming that Ω is a ball in $\mathbb{C}^2$. Let M be finite area, positive divisor in Ω. Then M is defined by a bounded holomorphic function.*

Theorem 2.10 ([10])**.** *Let Ω be a smoothly bounded, uniformly totally pseudoconvex domain admitting maximal type F at all boundary points, and the Čech cohomology group of second degree $H^2(\Omega, \mathbb{Z}) = 0$. Assuming that $\bar{\Omega}$ admits a Stein neighborhood basis. If M is a finite area, positive divisor of Ω, then for some Nevanlinna holomorphic function $\mathfrak{g}$, we have*

$$M = Z(\Omega, \mathfrak{g}).$$

3. Proof of Theorem 1.2

For convenience, we recall the following fact which is proved in [10] by using the Bochner-Martinelli-Koppelman kernel. Note that this result was proved without the negative $\bar{\partial}$ property.

Theorem 3.1. *Let Ω be a smoothly bounded, uniformly totally pseudoconvex domain admitting maximal type F at all boundary points, for some function F. Assuming that $\bar{\Omega}$ admits a Stein neighborhood basis. Let φ be a continuous $(0,1)$-form on $\overline{\Omega}$ and satisfy $\bar{\partial}\varphi = 0$, then there exists a function $u \in \Lambda^f(\overline{\Omega})$ such that*

$$\bar{\partial} u = \varphi,$$

where

$$f(d^{-1}) := \left(\int_0^d \frac{\sqrt{F^*(t)}}{t} dt \right)^{-1},$$

with F^* be the inversion of F.

Moreover, we also have:

(i) $||u||_{L^1(\Omega)} \leq C(||\varphi||_{L^1(\Omega)} + ||\varphi||_{L^1(b\Omega)})$.
(ii) $||u||_{L^p(b\Omega)} \leq C_p ||\varphi||_{L^p(b\Omega)}$ *for all* $1 \leq p \leq +\infty$.
(iii) $||u||_{\Lambda_p^f(b\Omega)} \leq C_p ||\varphi||_{L^p(b\Omega)}$ *for all* $1 \leq p \leq +\infty$.

Since $H^2(\Omega, \mathbb{R}) = 0$, we can apply the Poincaré-Cartan lemma, in local sense, from the well-known global construction of Weil [30] for $H^2(\Omega, \mathbb{R})$.

Let $\mathcal{K}$ be the Poincaré-Cartan homotopy operator defined in [5], page 36. Let $\alpha = \sum_{ij} \alpha_{ij} dz_i \wedge d\bar{z}_j$ be a positive, smooth $(1,1)$-form on Ω such that $d\alpha = 0$, then

(3.1)
$$\mathcal{K}\alpha(z) = \sum_j \left(\sum_i \int_0^1 t\alpha_{ij}(tz) dt z_i \right) d\bar{z}_j - \sum_i \left(\sum_j \int_0^1 t\alpha_{ij}(tz) dt \bar{z}_j \right) dz_i,$$

and
$$d\mathcal{K}\alpha(z) = \alpha(z).$$

Because of the positivity of α, we obtain

(3.2)
$$\mathcal{K}\alpha(z) = \sum_j \left(\sum_i \int_0^1 t\alpha_{ij}(tz) dt z_i \right) d\bar{z}_j - \overline{\sum_j \left(\sum_i \int_0^1 t\alpha_{ij}(tz) dt z_i \right) d\bar{z}_j}.$$

In short, $\mathcal{K}\alpha(z) = \mathcal{F}(z) + \overline{\mathcal{F}(z)}$, where
$$\mathcal{F}(z) = \sum_j \left(\sum_i \int_0^1 t\alpha_{ij}(tz) dt z_i \right) d\bar{z}_j.$$

Moreover, as a consequence of the d-closed property of α,
$$\bar{\partial}\mathcal{F} = \partial\mathcal{F} = 0. \tag{3.3}$$

By a changing of coordinates $b\Omega \times [0,1] \to \Omega$, we also obtain
$$||\mathcal{F}||_{L^1(b\Omega)} \lesssim ||\alpha||_{L^1(\Omega)} \quad \text{and} \quad ||\mathcal{F}||_{L^1(\Omega)} \leq ||\alpha||_{L^1(\Omega)}. \tag{3.4}$$

From the estimates (3.3), (3.4) and the existence in Theorem 3.1, there is a function $v \in L^1(\bar{\Omega})$ solving the equation $\bar{\partial} v = \mathcal{F}$ on $\bar{\Omega}$ and satisfying
$$\begin{aligned} ||v||_{L^1(\Omega)} + ||v||_{L^1(b\Omega)} &\lesssim ||\mathcal{F}||_{L^1(\Omega)} + ||\mathcal{F}||_{L^1(b\Omega)} \\ &\lesssim ||\alpha||_{L^1(\Omega)}. \end{aligned} \tag{3.5}$$

Now, we define $U = 2\mathrm{Im}(v)$, then
$$||U||_{L^1(b\Omega)} + ||U||_{L^1(\Omega)} \lesssim ||\alpha||_{L^1(\Omega)}.$$

Then,
$$\begin{aligned} \alpha = d(\mathcal{K}\alpha) &= \partial\mathcal{F} + \bar{\partial}\bar{\mathcal{F}} \\ &= \partial(\bar{\partial} v) + \bar{\partial}(\partial \bar{v}) \\ &= i\partial\bar{\partial}\left(\frac{v - \bar{v}}{i}\right) \\ &= i\partial\bar{\partial} U. \end{aligned} \tag{3.6}$$

Moreover, by the negative $\bar{\partial}$ property, there exists a pluriharmonic function λ such that $u := (2\mathrm{Im}(U) + \lambda)$ is negative, $i\partial\bar{\partial} u = \alpha$ and
$$||u||_{L^1(b\Omega)} + ||u||_{L^1(\Omega)} \lesssim ||\phi||_{L^1(b\Omega)} + ||\phi||_{L^1(b\Omega)}.$$

This completes the proof.

4. Proof of Theorem 1.1

By the Poincaré-Lelong Formula, let α_M be the closed $(1,1)$ positive current associated with M. That means, for some holomorphic function h on Ω which has zero set M, we have

$$\alpha_M = \frac{i}{\pi}\partial\bar{\partial}[\log|h|]$$

in the sense of currents.

Let

$$V_\epsilon(z) = \log|h| * \chi_\epsilon(z)$$

be the smooth regularity of $\log|h(z)|$, where for each $\epsilon > 0$, $\chi_\epsilon \in C_c^\infty(\mathbb{R})$ is a non-negative function such that χ_ϵ is supported on $[-\epsilon/2, \epsilon/2]$, and $\int_{\mathbb{R}} \chi_\epsilon(x)dx = 1$. Then, V_ϵ is smooth on $\Omega_\epsilon = \{\rho(z) < -\epsilon\} \Subset \Omega$ and $V_\epsilon(z) \to \log|h(z)|$ as $\epsilon \to 0^+$.

For convenience, we also denote V_ϵ by the smooth extension of V_ϵ to a neighborhood of Ω, so $V_\epsilon(z) \to \log|h(z)|$ almost everywhere as $\epsilon \to 0^+$. Then the smooth regularity of α_M implies $\alpha_\epsilon = \frac{1}{\pi}\partial\bar{\partial}V_\epsilon \in C^\infty_{(1,1)}(\bar{\Omega})$, and α_ϵ is also d-closed and positive, and $\alpha_\epsilon \to \alpha_M$ in the sense of currents.

Thus, applying Theorem 1.2 to each $\pi\alpha_\epsilon$, we can find a negative function u_ϵ such that

$$\begin{cases} \dfrac{i}{\pi}\partial\bar{\partial}u_\epsilon = \alpha_\epsilon, \\ ||u_\epsilon||_{L^1(b\Omega)} + ||u_\epsilon||_{L^1(\Omega)} \lesssim ||\alpha_\epsilon||_{L^1(\Omega)}, \end{cases}$$

and for some constant $C > 0$, we get

$$\int_\Omega |u_\epsilon(z)|dV(z) < C, \quad \text{uniformly in } \epsilon > 0. \tag{4.1}$$

The plurisubharmonicity of $\log|h(z)|$ implies that it is locally integrable. Hence, for any compact subset $K \subset \Omega$, we have

$$\int_K |V_\epsilon(z)|dV(z) < C_K, \quad C_K > 0 \text{ depending only on } K. \tag{4.2}$$

We define

$$g_\epsilon = u_\epsilon - V_\epsilon,$$

it is easy to see that g_ϵ is a pluriharmonic function on Ω. Since Ω is a domain, $g_\epsilon = \mathrm{Re}[G_\epsilon]$, where G_ϵ is holomorphic on Ω.

Using (4.1), (4.2) and the Montel's Theorem for g_ϵ, there exists a subsequence $\{g_{\epsilon_n}\}$ of $\{g_\epsilon\}$ that converges to a pluriharmonic function g uniformly on compact sets of Ω, where $\lim_{n\to\infty} \epsilon_n = 0$. Moreover, we also have

$$g = \lim_{n\to\infty} g_{\epsilon_n} = \lim_{n\to\infty} \mathrm{Re}[G_{\epsilon_n}] = \mathrm{Re}[G]$$

for some holomorphic function G on Ω.

Now, let $u := \log[|h|] + g = \log[|h|] + \mathrm{Re}[G] = \log[|he^G|]$, then we have

$$\begin{cases} \lim_{n\to\infty} u_{\epsilon_n} = u, \quad \text{in } L^1(\overline{\Omega}), \\ \frac{i}{\pi}\partial\bar{\partial}u = \alpha_M \quad \text{in the sense of currents}, \\ u \in L^1(\overline{\Omega}), \quad \text{by Theorem 1.2.} \end{cases}$$

On the other hand, let $\mathfrak{g}(z) = he^G(z)$, since $\frac{i}{\pi}\partial\bar{\partial}\log[|h|] = \frac{i}{\pi}\partial\bar{\partial}\log[|\mathfrak{g}|] = \alpha_M$, the zero set of $\mathfrak{g}$ is the same as the zero set of h. Finally, $|\mathfrak{g}| = e^u$, that means $\mathfrak{g}$ is bounded holomorphic since u is negative. Thus we complete the proof.

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