ON ZERO DISTRIBUTIONS OF SOME SELF-RECIROCAL POLYNOMIALS WITH REAL COEFFICIENTS

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Abstract. If $q(z)$ is a polynomial of degree $n$ with all zeros in the unit circle, then the self-reciprocal polynomial $q(z) + x^nq(1/z)$ has all its zeros on the unit circle. One might naturally ask: where are the zeros of $q(z) + x^nq(1/z)$ located if $q(z)$ has different zero distribution from the unit circle? In this paper, we study this question when $q(z) = (z - 1)^{n-k}(z - 1 - c_1) \cdots (z - 1 - c_k) + (z + 1)^{n-k}(z + 1 + c_1) \cdots (z + 1 + c_k)$, where $c_j > 0$ for each $j$, and $q(z)$ is a ‘zeros dragged’ polynomial from $(z - 1)^n + (z + 1)^n$ whose all zeros lie on the imaginary axis.

1. Introduction

It what follows, $U$ denotes the unit circle and $n$ is a positive integer. There is an extensive literature concerning zeros of sums of polynomials. Many papers and books([5], [6], [7]) have been written about these polynomials. An immediate question of sums of polynomials, $A + B = C$, is “given zeros of $A$ and $B$, what zeros can be given for $C$?”. For example, all (conjugate) zeros of the polynomial

$$\prod_{l=1}^{n} (z - r_l) + \prod_{l=1}^{n} (z + r_l),$$

where $0 < r_1 \leq r_2 \leq \cdots \leq r_n$, lie on the imaginary axis. For the proof and more, see [3]. Perhaps the most basic form of the polynomial (1) is

$$\begin{align*}
(z + 1)^n + (z - 1)^n,
\end{align*}$$

where, by Fell [2], if all zeros of $A$ and $B$ lie in $[-1, 1]$ with $A, B$ monic and $\deg A = \deg B = n$, then no zero of $C$ can have modulus exceeding $\cot (\pi/2n)$, the largest zero of (2).

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All polynomials in this paper will be assumed to have real coefficients. A polynomial $P(z)$ of degree $n$ is said to be self-inversive if it satisfies $P(z) = \pm P^*(z)$, where $P^*(z) = z^n P(1/z)$. In particular, if $P(z) = P^*(z)$, $P(z)$ is called self-reciprocal. Many questions about the zeros of a family of self-reciprocal polynomials arise naturally in several areas of mathematics - number theory, coding theory, algebraic curves over finite fields, knot theory, but are also of independent interest. The zeros of a self-reciprocal polynomial either lie on $U$ or occur in pairs conjugate to $U$. Since the class of self-inversive polynomials of degree $n$ includes polynomials of degree $n$ which have all their zeros on $U$, it is interesting to mention the condition for a self-reciprocal polynomial having all its zeros on $U$. For example, in [1], Chen proved a following sufficient and necessary condition for a self-inversive polynomial to have all its zeros on $U$.

**Theorem 1.** A necessary and sufficient condition for all the zeros of $f_n(z) = \sum_{k=0}^{n} a_k z^k$ with complex coefficients to lie on $U$ is that there is a polynomial $q_{n-l}(z)$ with all its zeros in or on $U$ such that

$$f_n(z) = z^l q_{n-l}(z) + e^{i\theta} q^*_{n-l}(z)$$

for some nonnegative integer $l$ and real $\theta$.

If $q(z)$ is a polynomial of degree $n$ with all zeros in $U$, then it follows from Theorem 1 that the self-reciprocal polynomial

$$q(z) + q^*(z)$$

has all its zeros on $U$. One might naturally ask: where are the zeros of $q(z) + q^*(z)$ located if $q(z)$ has different zero distribution from $U$? For example, if $q(z)$ is the polynomial (2) whose all zeros are on the imaginary axis, then

$$q(z) + q^*(z) = \begin{cases} 2 \left((z + 1)^n + (z - 1)^n\right) & \text{if } n \text{ is even,} \\ 2(z + 1)^n & \text{if } n \text{ is odd.} \end{cases}$$

In this case, for $n$ even, all zeros of $q(z) + q^*(z)$ lie on the imaginary axis, and for $n$ odd, they lie on $U$.

Suppose we drag the zero $-1, 1$ of each summand of $q(z)$ in (2) to the outward in the same distance, respectively. More specifically, we consider the polynomial

$$q(z) = (z - 1)^{n-k} (z - 1 - c_1) \cdots (z - 1 - c_k) + (z + 1)^{n-k} (z + 1 + c_1) \cdots (z + 1 + c_k),$$

where $c_j > 0$ for each $j$. Our interests in this paper are zero distributions of $q(z) + q^*(z)$, and we will have some results about these. First, we start to study the
polynomial $p_1(z) + p_1^*(z)$, where
\[ p_1(z) = (z - 1)^{n-1}(z - 1 - c) + (z + 1)^{n-1}(z + 1 + c) \]
is a ‘one zero dragged’ polynomial from $(z - 1)^n + (z + 1)^n$. In fact, this polynomial was studied in [4], and very similar results to Theorem 2 below were given there. But our proof here is different from that in [4], and moreover, we describe in detail what the circle is. The theorem below is interesting in that it does not seem obvious how to construct self-reciprocal polynomials with integer coefficients whose zeros all lie on one circle that is not the unit circle.

**Theorem 2.** Let for an odd integer $n$,
\[ p_1(z) = (z - 1)^{n-1}(z - 1 - c) + (z + 1)^{n-1}(z + 1 + c), \]
where $c_j > 0$, $c \neq 0, -1, -2$ for each $j$. Then all zeros of the self-reciprocal polynomial
\[ \frac{p_1(z) + p_1^*(z)}{z + 1} \]
lie on a circle that is not the unit circle. This circle has the center
\[ C \left( 1 + \frac{2}{|k|^2 - 1}, 0 \right) \]
and the radius
\[ r = \left| \frac{2|k|}{|k|^2 - 1} \right|^\frac{1}{n-1}, \]
where $k = \left( \frac{c}{n+2} \right)^{\frac{1}{n-1}} > 0$.

In the next theorem, we consider a generalized form of the polynomial $p_1(z)$ in Theorem 2.

**Theorem 3.** Let for even integers $n$ and $k$,
\[ p_k(z) = (z - 1)^{n-k}(z - 1 - c_1) \cdots (z - 1 - c_k) \]
\[ + (z + 1)^{n-k}(z + 1 + c_1) \cdots (z + 1 + c_k), \]
where $c_j > 0$ for each $j$. If
\[
(3) \quad (2 + c_1) \cdots (2 + c_k) > \sum_{r=2}^{k} \left\{ \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq k} c_{i_1} \cdots c_{i_r} \frac{(2 + c_{i_1}) \cdots (2 + c_{i_r})}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} \right\},
\]
then the self-reciprocal polynomial $p_k(z) + p_k^*(z)$ has all its zeros on the imaginary axis.
In the case of $c_1 = c_2 = \cdots = c_k = c$, (3) becomes
\[
(2 + c)^k > \sum_{r=2}^{k} \left\{ \sum_{1 \leq i_1 < \cdots < i_r \leq k} c^r (2 + c)^{k-r} \right\}.
\]
This is equivalent to
\[
2(2 + c)^k > (2 + c)^k + \sum_{r=2}^{k} \left\{ \sum_{1 \leq i_1 < \cdots < i_r \leq k} c^r (2 + c)^{k-r} \right\} = \frac{(2c + 2)^k + 2^k}{2},
\]
that is,
\[
u(c) := 4(c + 2)^k - (2c + 2)^k - 2^k > 0,
\]
which is true since $u(0) > 0$ and for $k = 2$,
\[
u'(c) = 4k(c + 2)^{k-1} - 2k(2c + 2)^{k-1} = 8 > 0.
\]
This implies the following Corollary 4 that is the special case of $k = 2$ of Theorem 3.

**Corollary 4.** Let for an even integer $n$ and $c > 0$,
\[
p_2(z) = (z - 1)^{n-2}(z - 1 - c)^2 + (z + 1)^{n-2}(z + 1 + c)^2.
\]
Then the self-reciprocal polynomial $p_2(z) + p^*_2(z)$ has all its zeros on the imaginary axis.

We recall the polynomial in Theorem 2 was
\[
p_1(z) = (z - 1)^{n-1}(z - 1 - c) + (z + 1)^{n-1}(z + 1 + c),
\]
where $n$ is an odd integer, and the polynomial in Corollary 4 was
\[
p_2(z) = (z - 1)^{n-2}(z - 1 - c)^2 + (z + 1)^{n-2}(z + 1 + c)^2,
\]
where $n$ is an even integer. By Theorem 2, the self-reciprocal polynomial $p_1(z) + p^*_1(z)$ has all its zeros other than $-1$ lies on a circle that is not the unit circle. Corollary 4 is unexpectedly surprising in that the self-reciprocal polynomial $p_2(z) + p^*_2(z)$ with one more dragging has all its zeros on the imaginary axis.

2. **Proofs**

In this section, we provide the proofs of our results.
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Proof of Theorem 2. With notations of the theorem, the roots of $p_1(z)/(z+1)$ satisfy

$$\left(\frac{z+1}{z-1}\right)^{n-1} = \frac{c}{c+2}.$$ 

Let $k = \left(\frac{c}{c+2}\right)^{1/(n-1)}$ be a real number. Then $\left|\frac{z+1}{z-1}\right| = |k|$ whose locus is a circle of Apollonius that is the set of points with ratio of distances $|k|$ to two points $(-1,0)$ and $(1,0)$ in Figure 1. Let $A$ and $B$ denote the points that the Apollonius circle crosses the real axis.

Then

$$A\left(\frac{|k| - 1}{|k| + 1}, 0\right), \quad B\left(\frac{|k| + 1}{|k| - 1}, 0\right),$$

and the center and the radius of the circle

$$C\left(1 + \frac{2}{|k|^2 - 1}, 0\right), \quad r = \left|\frac{2|k|}{|k|^2 - 1}\right|,$$

respectively.

For the proof of Theorem 3, we will need the following two theorems.

**Theorem 5.** (Cohn) Let $P(z) = \sum_{k=0}^{n} a_k z^k \in \mathbb{C}[z]$, $(a_n \neq 0)$. Then all zeros of $P$ lie on $|z| = 1$ if and only if

(i) $P$ is self-inversive,

(ii) all zeros of $P'$ lie in $|z| \leq 1$. 

Figure 1. Apollonius circle with $k = 2$
Moreover, if $P$ is self-inversive and

$$\tau = \text{the number of zeros on } |z| = 1 \text{ (counted with multiplicity)},$$
$$\nu = \text{the number of critical points in } |z| \leq 1 \text{ (counted with multiplicity)}.$$

Then

$$\tau = 2(\nu + 1) - n.$$  

**Theorem 6.** (Cauchy) All zeros of $P'(z) = na_n z^{n-1} + (n - 1)a_{n-1} z^{n-2} + \cdots + 2a_2 z + a_1$ lie in

$$|z| \leq r,$$

where $r$ is the positive root of the equation

$$n|a_n| z^{n-1} - (n - 1)|a_{n-1}| z^{n-2} - \cdots - 2|a_2| z - |a_1| = 0.$$

For the proofs of above two theorems, see [7, p. 230] and [6, p. 244].

**Proof of Theorem 3.** Let $n$ and $k$ be positive even integers with $n > k$, and

$$P_k(z) = p_k(z) + p_k^\ast(z),$$

where

$$p_k(z) = (z - 1)^{n-k}(z - 1 - c_1)\cdots(z - 1 - c_k)$$
$$+ (z + 1)^{n-k}(z + 1 + c_1)\cdots(z + 1 + c_k),$$

where $c_j > 0$ for each $j$. Then

$$p_k^\ast(z) = (z - 1)^{n-k}(1 - (1 + c_1)z)\cdots(1 - (1 + c_k)z)$$
$$+ (z + 1)^{n-k}(1 + (1 + c_1)z)\cdots(1 + (1 + c_k)z)$$

and

$$P_k(z) = \{(z + 1)^{n-k} + (z - 1)^{n-k}\}(A + C) + \{(z + 1)^{n-k} - (z - 1)^{n-k}\}(B + D),$$

where

$$A = \frac{(z + 1 + c_1)\cdots(z + 1 + c_k) + (z - 1 - c_1)\cdots(z - 1 - c_k)}{2},$$
$$B = \frac{(z + 1 + c_1)\cdots(z + 1 + c_k) - (z - 1 - c_1)\cdots(z - 1 - c_k)}{2},$$
$$C = \frac{(1 + (1 + c_1)z)\cdots(1 + (1 + c_k)z) + (1 - (1 + c_1)z)\cdots(1 - (1 + c_k)z)}{2},$$
$$D = \frac{(1 + (1 + c_1)z)\cdots(1 + (1 + c_k)z) - (1 - (1 + c_1)z)\cdots(1 - (1 + c_k)z)}{2}.$$
Then the zeros of $P_k(z)$ satisfy

\[
(z + 1)^{n-k} + (z - 1)^{n-k} = \frac{B + D}{A + C}.
\]

Write

\[
l = \frac{(z + 1)^{n-k} + (z - 1)^{n-k}}{(z + 1)^{n-k} - (z - 1)^{n-k}}.
\]

Then

\[
\left(\frac{z + 1}{z - 1}\right)^{n-k} = \frac{l + 1}{l - 1} \quad \text{and} \quad \frac{z + 1}{z - 1} = \left(\frac{l + 1}{l - 1}\right)^{1/n} =: L.
\]

So

\[
(4) \quad z = \frac{L + 1}{L - 1} \quad \text{and} \quad l = \frac{L^{n-k} + 1}{L^{n-k} - 1}.
\]

Since

\[
l = \frac{L^{n-k} + 1}{L^{n-k} - 1} = -\frac{B + D}{A + C},
\]

we have

\[(A + B + C + D)L^{n-k} + (A + C - B - D) = 0.
\]

Let

\[
f(L) = \left\{(A + B + C + D)L^{n-k} + (A + C - B - D)\right\}(L - 1)^k,
\]

that is,

\[
f(L) = \left\{(z + 1 + c_1) \cdots (z + 1 + c_k) + (1 + (1 + c_1)z) \cdots (1 + (1 + c_k)z)\right\} L^{n-k}
\]

\[+ (z - 1 - c_1) \cdots (z - 1 - c_k) + (1 - (1 + c_1)z) \cdots (1 - (1 + c_k)z)\}(L - 1)^k.
\]

By (4), $z = \frac{L + 1}{L - 1}$ and put this into the right hand side of above equation so that we have

\[
f(L) = \left\{((2 + c_1)L - c_1) \cdots ((2 + c_k)L - c_k)
\right.
\]

\[+ (2 + c_1)L + c_1) \cdots ((2 + c_k)L + c_k)\} L^{n-k}
\]

\[+ (c_1L - (c_1 + 2)) \cdots (c_kL - (c_k + 2)) + (c_1L + (c_1 + 2)) \cdots (c_kL + (c_k + 2)).
\]

We observe that $L^n f(1/L) = f(L)$ since $k$ is even, that is, $f(L)$ is self reciprocal. We will use Theorem ?? to show that all zeros of $f$ lie on $|L| = 1$. First, we may express $f(L)$ by the sum as follows:

\[
(5) \quad f(L) = 2E(L^n + 1) + 2 \sum_{\substack{r = 2 \text{ even} \atop 1 \leq i_1 < \cdots < i_r \leq k}} c_{i_1} \cdots c_{i_r} \frac{E}{(2 + c_{i_1}) \cdots (2 + c_{i_r})}(L^{n-r} + L^r),
\]
where \( E = (2 + c_1) \cdots (2 + c_k) \). Then
\[
f'(L) = 2nEL^{n-1} + 2 \sum_{r=2}^{k} \left\{ \sum_{1 \leq i_1 < \cdots < i_r \leq k} \frac{E}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} ((n-r)L^{n-r-1} + rL^{r-1}) \right\}.
\]
To use Theorem 5, we let
\[
g(L) = 2nEL^{n-1} - 2 \sum_{r=2}^{k} \left\{ \sum_{1 \leq i_1 < \cdots < i_r \leq k} \frac{c_{i_1} \cdots c_{i_r} E}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} ((n-r)L^{n-r-1} + rL^{r-1}) \right\}.
\]
Then
\[
g'(L) = 2n(n-1)EL^{n-2} - 2 \sum_{r=2}^{k} \left\{ \sum_{1 \leq i_1 < \cdots < i_r \leq k} \frac{E}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} (\frac{1}{(n-r)(n-r-1)L^{n-r-2}} + r(r-1)L^{r-2}) \right\}
\]
and we have
\[
(6) \quad g(0) = 0, \quad g'(0) = -4 \sum_{1 \leq i_1 < i_2 \leq k} \frac{c_{i_1}c_{i_2} E}{(2 + c_{i_1})(2 + c_{i_2})} < 0.
\]
But we observe that
\[
g(1) = 2nE - 2 \sum_{r=2}^{k} \left\{ \sum_{1 \leq i_1 < \cdots < i_r \leq k} \frac{c_{i_1} \cdots c_{i_r} nE}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} \right\} > 0
\]
is equivalent to
\[
(7) \quad (2 + c_1) \cdots (2 + c_k) > \sum_{r=2}^{k} \left\{ \sum_{1 \leq i_1 < \cdots < i_r \leq k} \frac{c_{i_1} \cdots c_{i_r} (2 + c_1) \cdots (2 + c_k)}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} \right\}.
\]
Hence if the inequality (7) holds, \( g(1) > 0 \) and so by (6), \( g(L) = 0 \) has at least one zero \( \alpha \) in the open interval \((0,1)\). In fact, this zero \( \alpha \) is unique in the open interval \((0,1)\) by Theorem 6. It follows from Theorem 5 that all the zeros of \( f(L) = 0 \) lie
on $|L| \leq \alpha < 1$, where $\alpha$ is the positive zero of the equation $g(L) = 0$. Hence by Theorem 5, all zeros of $f$ lie on $|L| = 1$. But by (4),

$$|L| = \left| \frac{z + 1}{z - 1} \right| = 1,$$

where $z$ was the zero of $P_k(z)$. One gets that the distances of $z$ from the point $-1$ equals the distances of $z$ from the point 1. Thus, if $z$ is to the left or to the right of the imaginary axis, one of these distances is bigger. This implies that $z$ lies on the imaginary axis, which completes the proof.

**REFERENCES**


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