

BEST RANDOM PROXIMITY PAIR THEOREMS FOR RELATIVELY U-CONTINUOUS RANDOM OPERATORS WITH APPLICATIONS

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ABSTRACT. It is our purpose in this paper to introduce the concept of *best random proximity pair* for subsets A and B of a separable Banach space E . We prove some best random approximation and best random proximity pair theorems of certain classes of random operators, which is the stochastic verse of the deterministic results of Eldred *et al.* [22], Eldred *et al.* [18] and Eldred and Veeramani [19]. Furthermore, our results generalize and extend recent results of Okeke and Abbas [42] and Okeke and Kim [43]. Moreover, we shall apply our results to study nonlinear stochastic integral equations of the Hammerstein type.

1. Introduction

Let (Ω, Σ, μ) be a complete probability measure space and $(E, B(E))$ measurable space, where E a separable Banach space, $B(E)$ is Borel sigma algebra of E , (Ω, Σ) is a measurable space (Σ – sigma algebra) and μ a probability measure on Σ , that is, a measure with total measure one. A mapping $\xi : \Omega \rightarrow E$ is called (a) E – valued random variable if ξ is $(\Sigma, B(E))$ – measurable (b) strongly μ – measurable if there exists a sequence $\{\xi_n\}$ of μ – simple functions converging to ξ μ – almost everywhere. Due to the separability of a Banach space E , the sum of two E – valued random variables is E – valued random variable. A mapping $T : \Omega \times E \rightarrow E$ is called a random operator if for each fixed e in E , the mapping $T(\cdot, e) : \Omega \rightarrow E$ is measurable. Denote by $F(T) = \{\xi^* : \Omega \rightarrow E \text{ such that } T(\omega, \xi^*(\omega)) = \xi^*(\omega) \text{ for each } \omega \in \Omega\}$ (the random fixed point set of T). Clearly, the necessary condition for the existence of a random fixed point of T is that $T(\Omega \times E) \cap E \neq \emptyset$ (but not sufficient). If the random fixed point $T(\omega, \xi^*(\omega)) = \xi^*(\omega)$ does not possess a solution, then $d(\xi^*(\omega), T\xi^*(\omega)) > 0$ for all $\xi^* \in E$ and $\omega \in \Omega$. In such a situation, it is our purpose to find an element $\xi^* \in E$ such that $d(\xi^*(\omega), T\xi^*(\omega))$ is minimum in

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some sense. Motivated by the above facts, we prove some best random approximation and best random proximity pair theorems of certain classes of random operators, which is the stochastic verse of the deterministic results of Eldred *et al.* [22], Eldred *et al.* [18] and Eldred and Veeramani [19].

The following well-known best approximation theorem is due to Ky Fan [24].

Theorem 1.1. ([24]). *Let A be a nonempty compact convex subset of a normed linear space X and $T : A \rightarrow X$ be a continuous function. Then there exists $x \in A$ such that $\|x - Tx\| = \text{dist}(Tx, A) := \inf\{\|Tx - a\| : a \in A\}$.*

The element $x \in A$ in Theorem 1.1 is called a best approximant of T in A . We note that if $x \in A$ is a best approximant, then $\|x - Tx\|$ need not be the optimum. Well known mathematicians have explored best proximity point theorems to find sufficient conditions so that the minimization problem

$$\min_{x \in A} \|x - Tx\| \tag{1.1}$$

has at least one solution.

In order to have a concrete lower bound, we consider two nonempty subsets A, B of a separable Banach space E and a random mapping $T : \Omega \times A \rightarrow B$. The natural question is whether one can find an element $x_0(\omega) \in \Omega \times A$ such that $d(x_0(\omega), Tx_0(\omega)) = \min\{d(x(\omega), Tx(\omega)) : x(\omega) \in \Omega \times A\}$. Since $d(x(\omega), Tx(\omega)) \geq \text{dist}(A, B)$, the optimal solution to the problem of minimizing the random map $x(\omega) \mapsto d(x(\omega), Tx(\omega))$ over the domain $\Omega \times A$ of the random mapping T will be the one for which the valued $\text{dist}(A, B)$ is attained. A point $x_0(\omega) \in \Omega \times A$ is called a best random proximity point of T if $d(x_0(\omega), Tx_0(\omega)) = \text{dist}(A, B)$. We note that if $\text{dist}(A, B) = 0$, then the best random proximity point reduces to a random fixed point of T .

The class of *relatively nonexpansive mappings* was introduced by Eldred *et al.* [18]. They studied the existence of best proximity points for such class of nonlinear mappings. Their deterministic results generalize the celebrated fixed point theorem for nonexpansive mappings due to Browder [15], Göhde [27], Kirk [34] and Goebel and Kirk [26]. The concept of relatively nonexpansive mappings is defined as follows by Eldred *et al.* [18].

Definition 1. ([18]). Let A, B be nonempty subsets of a metric space (X, d) . A mapping $T : A \cup B \rightarrow A \cup B$ is said to be a relatively nonexpansive mapping if

1. $T(A) \subseteq B, T(B) \subseteq A$;
2. $d(Tx, Ty) \leq d(x, y)$, for all $x \in A, y \in B$.

Note that a relatively nonexpansive mapping need not be a continuous mapping. Moreover, every nonexpansive self-map can be considered as a relatively nonexpansive mapping (see, [18], [22]). Eldred *et al.* [18] introduced and used a geometric notion called *proximal normal structure* to prove the existence of a best proximity point for relatively nonexpansive mappings. They proved the following proximity pair theorem.

Theorem 1.2. ([18]). *Let (A, B) be a nonempty, weakly compact convex pair in a Banach space X . Let $T : A \cup B \rightarrow A \cup B$ be a relatively nonexpansive mapping and suppose (A, B) has a proximal normal structure. Then there exists $(x_0, y_0) \in A \times B$ such that $\|x_0 - Tx_0\| = \|Ty_0 - y_0\| = \text{dist}(A, B)$.*

Eldred and Veeramani [19] proved some existence of best proximity points results for the class of *cyclic contraction mappings*. The concept of *cyclic contraction mappings* is defined as follows.

Definition 2. ([19]). Let A, B be nonempty subsets of a metric space X . A mapping $T : A \cup B \rightarrow A \cup B$ is said to be a cyclic contraction if there exists $k \in [0, 1)$ such that

1. $T(A) \subseteq B, T(B) \subseteq A$;
2. $d(Tx, Ty) \leq kd(x, y) + (1 - k)\text{dist}(A, B)$, for all $x \in A, y \in B$.

Note that the class of *cyclic contraction mappings* defined on $A \cup B$, where A, B are nonempty subsets of a metric space, is strictly contained in the class of relatively nonexpansive mappings on $A \cup B$ (see [22]). The authors in [22] introduced and studied the class of *relatively u -continuous mappings* which properly contains the class of *relatively nonexpansive mappings*. They studied sufficient conditions for the existence of a best proximity point for this class of nonlinear mappings. The following theorem was proved by the authors in [22].

Theorem 1.3. ([22]). *Let A, B be nonempty compact convex subsets of a strictly convex Banach space X and $T : A \cup B \rightarrow A \cup B$ be a relatively u -continuous mapping. Then there exists $(x_0, y_0) \in A \times B$ such that $\|x_0 - Tx_0\| = \|y_0 - Ty_0\| = \text{dist}(A, B)$.*

Real world problems are embedded with uncertainties and ambiguities. To deal with probabilistic models, Probabilistic functional analysis has emerged as one of the momentous mathematical discipline and attracted the attention of several mathematicians over the years in view of its applications in diverse areas from pure mathematics to applied sciences. Random nonlinear analysis, an important branch of probabilistic functional analysis, deals with the solution of various classes of random operator equations and the related problems. Of course, the development of random methods have revolutionized the financial markets. Random fixed point theorems are stochastic generalizations of classical or deterministic fixed point theorems and are required for the theory of random equations, random matrices, random partial differential equations and various classes of random operators arising in physical systems (see, Joshi and Bose [30], Beg and Abbas [7], Beg and Abbas [8], Okeke and Abbas [42], Okeke and Kim [43]), Okeke and Kim [44]. Random fixed point theory was initiated in 1950s by Prague school of probabilists. Spacek [55] and Hans [28] established a stochastic analogue of the Banach fixed point theorem in a separable complete metric space. Itoh [29] in 1979 generalized and extended Spacek and Han's theorem to a multivalued contraction random operator. The survey article by Bharucha-Reid [14] in 1976, where he studied sufficient conditions for a stochastic analogue

of Schauder's fixed point theorem for random operators, gave wings to random fixed point theory. Now this area has become full fledged research area and many interesting techniques to obtain the solution of nonlinear random system have appeared in literature (see, [5],[9]-[11], [16], [29]-[30], [41], [42], [43], [48], [54], [55], [57], [59]).

Papageorgiou [48] established an existence of random fixed point of measurable closed and nonclosed valued multifunctions satisfying general continuity conditions and hence improved the results in [23], [29] and [52]. Xu [57] extended the results of Itoh to a nonself-random operator T , where T satisfies weakly inward or the Leray-Schauder condition. Shahzad and Latif [54] proved a general random fixed point theorem for continuous random operators. As applications, they derived a number of random fixed points theorems for various classes of 1-set and 1-ball contractive random operators. Arunchai and Plubtieng [5] obtained some random fixed point results for the sum of a weakly-strongly continuous random operator and a nonexpansive random operator in Banach spaces.

The concept of *best random proximity points*, which is an extension of the notion of *random fixed points* was introduced by Anh [2]. He considered certain random operator equations and extended certain random fixed point theorems. It is our purpose in this paper to study the *best random proximity points* of some new classes of random operators. The results of this study improves and generalizes several deterministic best proximity point results, including the results of the authors [2], [18], [19], [20], [21], [22], [36] among others.

2. Preliminaries

Definition 3. ([2]). Let X, Y be metric spaces, $f, g : \Omega \times X \rightarrow Y$ be random operators. Consider the random equation of the form

$$f(\omega, x) = g(\omega, x). \quad (2.1)$$

We say that the equation (2.1) has a random solution if there exists an X -valued random variable $\xi : \Omega \rightarrow X$ such that, for every ω ,

$$f(\omega, \xi(\omega)) = g(\omega, \xi(\omega)).$$

We call ξ a random solution of the equation (2.1).

Clearly, if the equation (2.1) has a random solution then it has a deterministic solution for each $\omega \in \Omega$. However, the converse is not true (see, e.g. [2], Example 2).

Definition 4. ([2]). Let A, B be two closed subsets of a Polish space X and $f : \Omega \times A \rightarrow B$ a random operator. A measurable mapping $\xi : \Omega \rightarrow A$ is called a *best random proximity point* of f if

$$d(\xi(\omega), f(\omega, \xi(\omega))) = d(A, B)$$

for any $\omega \in \Omega$.

Clearly, the best random proximity point of a random operator f becomes a random fixed point of f if $A \cap B \neq \emptyset$. This means that the concept of best random proximity point is an extension of the concept of random fixed point. Generally, if f has a best random proximity point for each $\omega \in \Omega$ the mapping $f(\omega, \cdot)$ has a best proximity point. However, Anh [2] proved that the converse is not true in the following example.

Example 2.1 ([2]). Let $\Omega = [0, 1]$ and \mathfrak{F} be the family of subsets $A \in \Omega$ with the property that either A is countable or the complement A^c is countable. Define a probability measure P on \mathfrak{F} by

$$P(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{otherwise.} \end{cases}$$

Let $A = B = [0, 1]$. Define a mapping $f : \Omega \times A \rightarrow B$ by

$$f(\omega, x) = \begin{cases} x & \text{if } \omega = x \\ 0 & \text{if } \omega \neq x. \end{cases}$$

Anh [2] showed that $f(\omega, \cdot)$ has a unique best proximity point $x = \omega$. However, f does not have a best random proximity point.

The following theorem by Anh [2] gives a sufficient condition ensuring that the existence of a deterministic solution for each ω implies the existence of a random solution for a general random equation.

Theorem 2.1. ([2]). *Let $f, g : \Omega \times X \rightarrow Y$ be measurable random operators and $F : \Omega \rightarrow C(X)$ a measurable mapping. If for each ω , the random equation $f(\omega, x) = g(\omega, x)$ has a deterministic solution in $F(\omega)$ then it has a random solution in $F(\omega)$.*

Similarly, the following theorem of Anh [2] gives a sufficient condition on f ensuring that the existence of a best proximity point of $f(\omega, \cdot)$ for each ω implies the existence of a best random proximity point of f .

Theorem 2.2. ([2]). *Let A and B be two closed subsets of a Polish space X , $f : \Omega \times A \rightarrow B$ a measurable random operator. If $f(\omega, \cdot)$ has a best proximity point for each $\omega \in \Omega$ then f has a best random proximity point.*

Definition 5. ([36]). Let X be a metric space and let A and B be nonempty subsets of X . Let

$$A_0 = \{x \in A : d(x, y) = \text{dist}(A, B) \text{ for some } y \in B\};$$

$$B_0 = \{x \in B : d(x, y) = \text{dist}(A, B) \text{ for some } y \in A\}.$$

A pair $(x, y) \in A_0 \times B_0$ for which $d(x, y) = \text{dist}(A, B)$ is called a *best proximity pair* for A and B .

Kirk *et al.* [36] gave sufficient conditions which guarantee the nonemptiness of A_0 and B_0 .

Example 2.2 ([36]). Let

$$A = \{(1, y) : 0 \leq y \leq 1\}, B = \{(2, y) : 0 \leq y \leq 1\}.$$

Define $T : A \rightarrow B$ by setting $T(1, y) = (2, 1 - y)$. Then $A_0 = A$, and $B_0 = B$.

Let (Ω, Σ, μ) be a complete probability measure space and A, B be nonempty weakly compact convex subsets of a separable Banach space E . Consider the mapping $P : \Omega \times A \cup B \rightarrow A \cup B$ defined as

$$P(\omega, x) = \begin{cases} P_B(\omega, x), & \text{if } x \in A, \omega \in \Omega \\ P_A(\omega, x), & \text{if } x \in B, \omega \in \Omega. \end{cases} \tag{2.2}$$

If E is a strictly convex separable Banach space, then P is a single valued mapping and satisfies $P(\Omega \times A) \subseteq B, P(\Omega \times B) \subseteq A$.

Definition 6. ([22]). Let A, B be nonempty subsets of a Banach space X . A mapping $T : A \cup B \rightarrow A \cup B$, is said to be a relatively u -continuous mapping if it satisfies

1. $T(A) \subseteq B, T(B) \subseteq A$;
2. for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|Tx - Ty\| < \varepsilon + dist(A, B)$, whenever $\|x - y\| < \delta + dist(A, B)$, for all $x \in A, y \in B$.

Note that every relatively nonexpansive mapping is a relatively u -continuous mapping. The following example given by Eldred *et al.* [22] shows that the converse is not true.

Example 2.3 ([22]). Let us consider $(X = \mathbb{R}^2, \|\cdot\|_2)$. Let $A = \{(0, t) : 0 \leq t \leq 1\}$ and $B = \{(1, s) : 0 \leq s \leq 1\}$. Define $T : A \cup B \rightarrow A \cup B$ by

$$T(x, y) = \begin{cases} (1, \sqrt{y}) & \text{if } x = 0 \\ (0, \sqrt{y}) & \text{if } x = 1. \end{cases} \tag{2.3}$$

Then T is a relatively u -continuous mapping but not a relatively nonexpansive mapping.

Motivated by the results of Eldred *et al.* [22], we now give the stochastic verse of the definition of *relatively u -continuous mappings* as follows.

Definition 7. Let (Ω, Σ, μ) be a complete probability measure space and A, B be nonempty subsets of a separable Banach space E . A mapping $T : \Omega \times A \cup B \rightarrow A \cup B$, is said to be a *relatively u -continuous random mapping* if it satisfies

1. $T(\Omega \times A) \subseteq B, T(\Omega \times B) \subseteq A$;
2. for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|T(\omega, x) - T(\omega, y)\| < \varepsilon + dist(A, B)$, whenever $\|x(\omega) - y(\omega)\| < \delta + dist(A, B)$, for all $x \in A, y \in B$ and $\omega \in \Omega$.

The following definitions are needed in this study and can be found in Beg *et al.* [12].

Definition 8. A mapping $x : \Omega \rightarrow E$ is said to be a *finitely valued random variable*, if it is constant on each finite number of disjoint sets $A_i \in \Sigma$ and is equal to 0 on $\Omega - (\bigcup_{i=1}^n A_i)$. A mapping x is called a *simple random variable* if it is finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.

Definition 9. A mapping $x : \Omega \rightarrow E$ is said to be *E-valued random variable*, if the inverse image under the mapping x of every Borel subset β of E belongs to Σ ; that is $x^{-1}(\beta) \in \Sigma$ for all $\beta \in B(E)$.

Definition 10. A mapping $x : \Omega \rightarrow E$ is said to be a *strong random variable*, if there exists a sequence $\{x_n(\omega)\}$ of simple random variables which converges to $x(\omega)$ almost surely, i.e., there exists a set $A_0 \in \Sigma$ with $\mu(A_0) = 0$ such that $\lim_{n \rightarrow \infty} x_n(\omega) = x(\omega)$, $\omega \in \Omega - A_0$.

Definition 11. A mapping $x : \Omega \rightarrow E$ is said to be a *weak random variable*, if the function $x^*(x(\omega))$ is real valued random variables for each $x^* \in E^*$, the space E^* denoting the first normed dual space of E .

In a separable Banach space X , the notions of strong and weak random variables $x : \Omega \rightarrow X$ coincide and in respect of such a space X , x is called a random variable (see, Joshi and Bose [30], Corollary 1).

Let Y be another Banach space, Joshi and Bose [30] gave the following definitions which will be needed in this study.

Definition 12. A mapping $F : \Omega \times X \rightarrow Y$ is said to be a *continuous random mapping* if the set of all $\omega \in \Omega$ for which $F(\omega, x)$ is a continuous function of x has measure one.

Definition 13. A mapping $F : \Omega \times X \rightarrow Y$ is said to be a *random mapping* if $F(\omega, x) = y(\omega)$ is a Y -valued random variable for every $x \in X$.

Definition 14. A random mapping $F : \Omega \times X \rightarrow Y$ is said to be *demi-continuous* at $x \in X$ if

$$\|x_n - x\| \rightarrow 0 \text{ implies } F(\omega, x_n) \rightarrow F(\omega, x) \text{ almost surely.}$$

The following definitions is also due to Joshi and Bose [30].

Definition 15. An equation of the type $F(\omega, x(\omega)) = x(\omega)$ where $F : \Omega \times X \rightarrow X$ is a random mapping is called a random fixed point equation.

Definition 16. Any mapping $x : \Omega \rightarrow X$ which satisfies random fixed point equation $F(\omega, x(\omega)) = x(\omega)$ almost surely is said to be a *wide sense solution* of the fixed point equation.

Definition 17. Any X -valued random variable $x(\omega)$ which satisfies

$$\mu\{\omega : F(\omega, x(\omega)) = x(\omega)\} = 1$$

is said to be a *random solution* of the fixed point equation or a random fixed point of F .

Remark 1. It is known that a random solution is a wide sense solution of the fixed point equation. The converse is not true. This was demonstrated in the following example given by Joshi and Bose [30].

Example 2.4 Let X be the set of all real numbers and let E be a non measurable subset of X . Let $F : \Omega \times X \rightarrow Y$ be a random mapping defined as $F(\omega, x) = x^2 + x - 1$ for all $\omega \in \Omega$. In this case, the real valued function $x(\omega)$, defined as $x(\omega) = 1$ for all $\omega \in \Omega$ is a random fixed point of F . However, the real valued function $y(\omega)$ defined as

$$y(\omega) = \begin{cases} -1, & \omega \notin E \\ 1, & \omega \in E \end{cases} \quad (2.4)$$

is a wide sense solution of the fixed point equation $F(\omega, x(\omega)) = x(\omega)$, without being a random fixed point of F .

3. Best proximity points theorems

In this section, we shall prove some common best proximity point results in the setting of hyperconvex metric spaces. We establish that Theorem 4.2 of Eldred *et al.* [22] holds in the setting of hyperconvex metric spaces.

Definition 18. Let A, B be nonempty convex subsets of a normed linear space. A relatively u -continuous mapping $T : A \cup B \rightarrow A \cup B$ is said to be affine if $T(\lambda x + (1 - \lambda)y) = \lambda Tx + (1 - \lambda)Ty$, for all $x, y \in A$ or $x, y \in B$ and $\lambda \in (0, 1)$.

We also define $F_A(T) = \{x \in A : d(x, Tx) = \text{dist}(A, B)\}$, $F_B(T) = \{y \in B : d(y, Ty) = \text{dist}(A, B)\}$.

The following theorems will be needed in the sequel.

Theorem 3.1. (see [39]). *Let X be a paracompact topological space, (M, d) a hyperconvex metric space, and $F : X \rightarrow 2^M$ an almost lower semicontinuous mapping with admissible values. Then F has a continuous selection; that is, there is a continuous mapping $f : X \rightarrow M$ such that $f(x) \in F(x)$ for each $x \in X$.*

Theorem 3.2. (see [31, 35]). *Let (M, d) be a compact hyperconvex metric space and $f : M \rightarrow M$ a continuous mapping. Then f has a fixed point.*

Theorem 3.3. *Let A, B be admissible subsets of a hyperconvex metric space (M, d) . Let A_0 be a compact subset of M and $\mathbb{F} = \{T_1, T_2, \dots, T_n\}$ be a family of commuting, affine, relatively u -continuous mappings on $A \cup B$ such that $T(A) \subset B$, and $T(B) \subset A$. Then there exists $x_0 \in A$ such that $d(x_0, T_i x_0) = \text{dist}(A, B)$, for all $i = 1, 2, \dots, n$.*

Proof. We define

$$F_A(T_i) = \{x \in A : d(x, T_i(x)) = \text{dist}(A, B)\}$$

and

$$F_B(T_i) = \{y \in B : d(y, T_i(y)) = \text{dist}(A, B)\}; \forall i = 1, 2, \dots, n.$$

By a result of Kirk *et al.* [36], the sets A_0 and B_0 are nonempty and hyperconvex. For each $x \in A_0$, choose $y \in B_0$ such that $d(x, y) = \text{dist}(A, B)$. Hence, by u-continuity of $T_i \forall i = 1, 2, \dots, n$, for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $u \in A, v \in B$,

$$d(u, v) < \delta + \text{dist}(A, B)$$

implies that

$$d(T_i(u), T_i(v)) < \varepsilon + \text{dist}(A, B), \forall i = 1, 2, \dots, n. \tag{3.1}$$

Hence, $d(T_i(x), T(y)) = \text{dist}(A, B)$. This implies that $T(x) \in B_0$ for each $x \in A_0$. We define an open neighborhood of x in A_0 by $U(x) = \{u \in A_0 : d(u, x) < \delta\}$. Then $u \in U(x)$ implies that

$$d(u, y) \leq d(u, x) + d(x, y) < \delta + \text{dist}(A, B). \tag{3.2}$$

Since T_i is u-continuous for each $i = 1, 2, \dots, n$, we have

$$d(T_i(u), T_i(y)) < \varepsilon + \text{dist}(A, B). \tag{3.3}$$

We define a multivalued function $F : A_0 \rightarrow 2^{A_0}$ by

$$F(v) = B(T_i(v); \text{dist}(A, B)) \cap A, \forall v \in A_0. \tag{3.4}$$

Since $T(v) \in B_0$ for all $v \in A_0$, $F(v)$ is a nonempty subset of A_0 . Since A is admissible, it follows that $F(v)$ is admissible.

Next, we prove that F is almost lower semicontinuous by showing that $B(T_i(y); \varepsilon) \cap F(u) \neq \emptyset$ for $u \in U(x)$. Using (3.3) and the hyperconvexity of M , for all $u \in U(x)$, we have

$$B(T_i(y); \varepsilon) \cap B(T_i(u); \text{dist}(A, B)) \neq \emptyset, \forall i = 1, 2, \dots, n. \tag{3.5}$$

Since $T_i(u) \in B_0$, for each $i = 1, 2, \dots, n$, we have

$$B(T_i(u); \text{dist}(A, B)) \cap A \neq \emptyset. \tag{3.6}$$

We have that for each $x^* \in B(T_i(u); \text{dist}(A, B)) \cap A$ implies that $x^* \in A_0$, since $d(x^*, T_i(u)) = \text{dist}(A, B)$. Therefore,

$$B(T_i(u); \text{dist}(A, B)) \cap A \subset A_0. \tag{3.7}$$

Using (3.5) and (3.6) and the fact that $T_i(y) \in A_0$ for all $i = 1, 2, \dots, n$, the sets $B(T_i(y); \varepsilon)$, $B(T_i(u); \text{dist}(A, B))$ and A have pairwise nonempty intersection. Since all of these sets are ball intersections, the hyperconvexity of the space M implies that

$$B(T_i(y); \varepsilon) \cap B(T_i(u); \text{dist}(A, B)) \cap A \neq \emptyset. \tag{3.8}$$

Moreover by (3.7) $B(T_i(y); \varepsilon) \cap B(T_i(u); \text{dist}(A, B)) \cap A \subset A_0$. It follows by (3.8) that

$$B(T_i(y); \varepsilon) \cap F(u) \neq \emptyset, \forall u \in U(x).$$

This implies that the mapping F is almost lower semicontinuous.

Using the selection theorem of Markin [39] (see Theorem 3.1) and almost lower semicontinuous mapping on a hyperconvex space with nonempty admissible values has a continuous selection, that is, there is a continuous $f : A_0 \rightarrow A_0$ such that $f(x) \in F(x)$ for each $x \in A_0$. Using Theorem 3.2, a continuous self-mapping on a compact hyperconvex space has a fixed point. Hence, there is a $c \in A_0$ such that $c = f(c) \in F(c)$. Using the definition of F ,

$$d(c, T_i(c)) = \text{dist}(A, B). \quad (3.9)$$

The proof of Theorem 3.3 is completed. \square

4. Best random proximity points theorems

We begin this section by defining the sets A_0 and B_0 in a complete probability measure space. We also introduce the concept of *best random proximity pair* for subsets A and B of a separable Banach space E .

Definition 19. Let (Ω, \sum, μ) be a complete probability measure space, A, B be nonempty subsets of a separable Banach space E . Let

$$A_0 = \{x : \Omega \rightarrow A : \|x(\omega) - y(\omega)\| = \text{dist}(A, B), \text{ for some mapping } y : \Omega \rightarrow B\}$$

and

$$B_0 = \{y : \Omega \rightarrow B : \|x(\omega) - y(\omega)\| = \text{dist}(A, B), \text{ for some mapping } x : \Omega \rightarrow A\}.$$

A pair $(x(\omega), y(\omega)) \in A_0 \times B_0$ for which $\|x(\omega) - y(\omega)\| = \text{dist}(A, B)$ is called a *best random proximity pair* for A and B .

Motivated by the results of Anh [2] (see Theorem 2.1 and Theorem 2.2), we now prove the following theorem which gives a sufficient condition to establish that the existence of a *best proximity pair* of $A_0 \times B_0$ for each $\omega \in \Omega$ implies the existence of a *best random proximity pair* of $A_0 \times B_0$.

Theorem 4.1. Let (Ω, \sum, μ) be a complete probability measure space, A, B be nonempty subsets of a separable Banach space E and $\varphi : \Omega \times (A_0 \times B_0) \rightarrow \mathbb{R}$ be a measurable random operator. If $\varphi(\omega, \cdot)$ has a *best proximity pair* for each $\omega \in \Omega$, then φ has a *best random proximity pair*.

Proof. Define $\varphi : \Omega \times (A_0 \times B_0) \rightarrow \mathbb{R}$ by $\varphi(\omega, (x, y)) = \|x(\omega) - y(\omega)\|$. Then φ is a measurable random operator. Clearly if $\varphi(\omega, (x, y))$ has a best proximity pair for each $\omega \in \Omega$, then the random equation $\varphi(\omega, (x, y)) = \text{dist}(A, B)$ has a deterministic solution in $F(\omega) = A_0 \times B_0$. By the results of Anh ([2], Theorem 2.3), the random equation $\varphi(\omega, (x, y)) = \text{dist}(A, B)$ has a random solution. Hence $\|x(\omega) - y(\omega)\| = \text{dist}(A, B)$ for each $\omega \in \Omega$. i.e. $(x(\omega), y(\omega))$ is a best random proximity pair of φ . The proof of Theorem 4.1 is completed. \square

Next, we prove the following lemma.

Lemma 4.2. *Let (Ω, Σ, μ) be a complete probability measure space, A, B be nonempty subsets of a separable Banach space E . Then A_0 and B_0 (as defined in Definition 4.1) are nonempty and satisfy $P_B(\Omega \times A_0) \subseteq B_0$ and $P_A(\Omega \times B_0) \subseteq A_0$.*

Proof. Now

$$dist(A, B) = \inf\{\|x(\omega) - y(\omega)\| : x \in A, y \in B, \omega \in \Omega\},$$

there exist sequences $\{x_n(\omega)\}$ in $\Omega \times A$ and $\{y_n(\omega)\}$ in $\Omega \times B$ such that

$$\|x_n(\omega) - y_n(\omega)\| \longrightarrow dist(A, B).$$

Since A and B are separable Banach spaces, we suppose that $x_n(\omega)$ converges to $x_0(\omega) \in \Omega \times A$ and $y_n(\omega)$ converges to $y_0(\omega) \in \Omega \times B$. Hence,

$$\|x_0(\omega) - y_0(\omega)\| \leq \lim \|x_n(\omega) - y_n(\omega)\| = dist(A, B),$$

establishing the fact that A_0 and B_0 are nonempty.

Next if $y(\omega) \in P_B(\Omega \times A_0)$, then $y(\omega) = P_B(x(\omega))$ for some $x(\omega) \in A_0$. Clearly,

$$\|x(\omega) - y(\omega)\| = dist(A, B).$$

Therefore, $P_B(\Omega \times A_0) \subseteq B_0$. Similarly, we can show that $P_A(\Omega \times B_0) \subseteq A_0$. The proof of Lemma 4.2 is completed. □

Proposition 4.3. *Let (Ω, Σ, μ) be a complete probability measure space, A, B be nonempty subsets of a separable Banach space E and $T : \Omega \times A \cup B \rightarrow A \cup B$ be a relatively u-continuous random mapping. Then $T(\Omega \times A_0) \subseteq B_0$, $T(\Omega \times B_0) \subseteq A_0$.*

Proof. Now

$$A_0 = \{x : \Omega \rightarrow A : \|x(\omega) - y(\omega)\| = dist(A, B), \text{ for some mapping } y : \Omega \rightarrow B\}$$

and

$$B_0 = \{y : \Omega \rightarrow B : \|x(\omega) - y(\omega)\| = dist(A, B), \text{ for some mapping } x : \Omega \rightarrow A\}.$$

Suppose that $A_0 = \emptyset$, then $B_0 = \emptyset$. Otherwise, we need to show that $\forall x \in A_0$ and $\omega \in \Omega$, $Tx(\omega) \in B_0$. Now for arbitrary $x \in A$ and $\omega \in \Omega$, there exists $y \in B$ such that $\|x(\omega) - y(\omega)\| = dist(A, B)$. Since T is a relatively u-continuous random mapping, we have that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(\omega, a) - T(\omega, b)\| < \varepsilon + dist(A, B)$, whenever $\|a(\omega) - b(\omega)\| < \delta + dist(A, B)$ for all $a \in A, b \in B$ and $\omega \in \Omega$. Since $\|x(\omega) - y(\omega)\| = dist(A, B)$, for each $\delta > 0$. This implies that $\|T(\omega, x) - T(\omega, y)\| = dist(A, B)$, hence $T(x(\omega)) \in B_0$. This means that $T(\Omega \times A_0) \subseteq B_0$. Similarly, we can show that $T(\Omega \times B_0) \subseteq A_0$. This completes the proof of Proposition 4.3. □

Proposition 4.4. *Let (Ω, Σ, μ) be a complete probability measure space, A, B be nonempty subsets of a weakly compact convex subsets of a strictly convex separable Banach space E . Let $T : \Omega \times A \cup B \rightarrow A \cup B$ be a relatively u -continuous random mapping and $P : A \cup B \rightarrow A \cup B$ be a random mapping defined as in (2.2). Then $TP(\omega, x) = P(T(\omega, x))$, for all $x \in A_0 \cup B_0$ and $\omega \in \Omega$, i.e. $P_A(T(\omega, x)) = T(P_B(\omega, x))$, $x \in A_0$ and $\omega \in \Omega$.*

Proof. Take a countable dense subset $\{x_n\}$ of $A \cup B$, define the map $T(\omega)$ by $T(\omega) = \{\omega \in \Omega, x \in A \cup B : T(\omega, x) = x\}$, then by a fixed point theorem of Furi and Vignoli [25], $T(\omega) \in K(A \cup B)$, where $K(A \cup B)$ denotes the non-empty compact subset of $A \cup B$. Moreover by ([29] Theorem 2.1), T is measurable. Let $x \in A_0$, then there exists a unique $y \in B_0$ such that $\|x(\omega) - y(\omega)\| = \text{dist}(A, B)$. Hence, $y = P_B(\omega, x)$ and $x = P_A(\omega, y)$. But T is a relatively u -continuous mapping, then by Proposition 4.1, $T(\omega, x) \in B_0$ and $T(\omega, y) \in A_0$ with $\|T(\omega, x) - T(\omega, y)\| = \text{dist}(A, B)$. Since E is a strictly convex separable Banach space, and the uniqueness of metric projection operator, we have $P_A(T(\omega, x)) = T(\omega, y) = T(P_B(\omega, x))$. Hence, for each $x \in A_0$, $\omega \in \Omega$, we have $P_A(T(\omega, x)) = T(\omega, y) = T(P_B(\omega, x))$. Similarly, we can show that $y \in B_0$, $T(P_A(\omega, y)) = P_B(T(\omega, y))$. Hence, $TP(\omega, x) = P(T(\omega, x))$, for each $x \in A_0 \cup B_0$, $\omega \in \Omega$. This completes the proof of Proposition 4.4. \square

Theorem 4.5. *Let (Ω, Σ, μ) be a complete probability measure space, A, B be nonempty separable closed convex subsets of a Hilbert space X , and let $T : \Omega \times A \cup B \rightarrow X$ be a relatively u -continuous random mapping such that $T(\omega, A \cup B)$ is bounded, for any $\omega \in \Omega$. Then there exists a measurable map $\varphi : \Omega \rightarrow A \cup B$ such that*

$$\|\varphi(\omega) - T(\omega, \varphi(\omega))\| = d(A, B), \text{ for each } \omega \in \Omega.$$

Proof. Consider the metric projection operator $P_A : X \rightarrow A$ on A . Since $T(\Omega \times A_0) \subseteq B_0$ and $P_A(\Omega \times B_0) \subseteq A_0$ (by Proposition 4.1 and Lemma 4.1), the composite mapping $P_A \circ T$ restricted to A_0 is a self-map. i.e. $P_A \circ T : A_0 \rightarrow A_0$. Assume that the measurable map $x_0(\omega) \in \Omega \times A_0$ is a random fixed point of the mapping $P_A \circ T$, i.e. $P_A(T(\omega, x_0(\omega))) = x_0(\omega)$, then $\|x_0(\omega) - T(\omega, x_0(\omega))\| = \text{dist}(Tx_0(\omega), A)$. Since $Tx_0 \in B_0$ implies that there exists $x^* \in A_0$ such that $\|x^*(\omega) - T(\omega, x_0)\| = \text{dist}(A, B)$. Hence, $\text{dist}(T(\omega, x_0), A) = \text{dist}(A, B)$, since

$$\text{dist}(A, B) \leq \text{dist}(T(\omega, x_0), A) \leq \|T(\omega, x_0) - x^*(\omega)\| = \text{dist}(A, B).$$

Now, let $y \in X$ we have

$$\|P_A(y) - y\| = d(y, A \cup B).$$

Since P_A is nonexpansive in Hilbert space X , then $P_A \circ T : \Omega \times A \cup B \rightarrow A \cup B$. Clearly, $P_A \circ T$ is a relatively u -continuous random mapping and $P_A \circ T(\omega, A \cup B)$ is bounded, for each $\omega \in \Omega$. From Itoh ([29], Theorem 2.1) there exists a random fixed point of $P_A \circ T$. That is, there exists a measurable map $\varphi : \Omega \rightarrow A \cup B$ such that $P_A \circ T(\omega, \varphi(\omega)) = \varphi(\omega)$, for all $\omega \in \Omega$. Hence,

$$\begin{aligned} \|\varphi(\omega) - T(\omega, \varphi(\omega))\| &= \|P_A \circ T(\omega, \varphi(\omega)) - T(\omega, \varphi(\omega))\| \\ &= d(T(\omega, \varphi(\omega)), A) \\ &= \text{dist}(A, B), \text{ for each } \omega \in \Omega. \end{aligned}$$

The proof of Theorem 4.5 is completed. □

Remark 2. Observe that $\varphi(\omega)$ in Theorem 4.1 is a *best random proximity point* of the random operator T . Moreover, the best random proximity point and the random fixed point of T coincides if $A = B$.

Corollary 4.6. *Let (Ω, Σ, μ) be a complete probability measure space, A, B be nonempty separable closed convex subsets of a Hilbert space X , and let $T : \Omega \times A \cup B \rightarrow X$ be a relatively nonexpansive random mapping such that $T(\omega, A \cup B)$ is bounded, for any $\omega \in \Omega$. Then there exists a measurable map $\varphi : \Omega \rightarrow A \cup B$ such that*

$$\|\varphi(\omega) - T(\omega, \varphi(\omega))\| = d(A, B), \text{ for each } \omega \in \Omega.$$

Proof. Since T is a relatively nonexpansive random mapping, then for each $x \in A, y \in B$ and $\omega \in \Omega$, we have

1. $T(\Omega \times A) \subseteq B, T(\Omega \times B) \subseteq A$.
2. $\|T(\omega, x) - T(\omega, y)\| \leq \|x(\omega) - y(\omega)\|$.

It follows that for any $\varepsilon > 0$ and $\omega \in \Omega$, there exists a $\delta > 0$ such that $\|T(\omega, x) - T(\omega, y)\| < \varepsilon + \text{dist}(A, B)$, whenever $\|x(\omega) - y(\omega)\| < \delta + \text{dist}(A, B)$, for all $x \in A, y \in B$ and $\omega \in \Omega$. This implies that T is a *relatively u-continuous random mapping*. Hence, this corollary follows immediately from Theorem 4.1. □

Remark 3. Since every nonexpansive self-map can be considered as a relatively nonexpansive mapping (see, e.g. Eldred *et al.* [22]), it follows that Corollary 4.6 is valid if T is a *nonexpansive mapping*.

Corollary 4.7. *Let (Ω, Σ, μ) be a complete probability measure space, A, B be nonempty separable closed convex subsets of a Hilbert space X , and let $g, h : \Omega \times A \cup B \rightarrow X$ be random operators such that g is a cyclic contraction and h is compact and relatively u-continuous random operator. Then there exists a measurable map $\varphi : \Omega \rightarrow A \cup B$ such that*

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(A, B), \text{ for each } \omega \in \Omega,$$

where $f = g + h$. If additionally $f(\Omega \times \partial(A \cup B)) \subseteq A \cup B$, then φ is a random fixed point of f .

Proof. Since the class of cyclic contraction mappings defined on $A \cup B$ is strictly contained in the class of relatively nonexpansive mappings on $A \cup B$ and every relatively nonexpansive mapping is a relatively u-continuous mapping (see, e.g. Eldred *et al.* [22]). It remains to be shown that $f + g$ is a relatively u-continuous random mapping.

Since the random mappings $g, h : \Omega \times A \cup B \rightarrow A \cup B$ are relatively u-continuous random mappings, it follows that the following conditions are satisfied:

1. $g(\Omega \times A) \subseteq B, g(\Omega \times B) \subseteq A$;
2. for each $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that $\|g(\omega, x) - g(\omega, y)\| < \frac{\varepsilon}{2} + dist(A, B)$, whenever $\|x(\omega) - y(\omega)\| < \delta_1 + dist(A, B)$, for all $x \in A, y \in B$ and $\omega \in \Omega$.
and
3. $h(\Omega \times A) \subseteq B, h(\Omega \times B) \subseteq A$;
4. for each $\varepsilon > 0$, there exists a $\delta_2 > 0$ such that $\|h(\omega, x) - h(\omega, y)\| < \frac{\varepsilon}{2} + dist(A, B)$, whenever $\|x(\omega) - y(\omega)\| < \delta_2 + dist(A, B)$, for all $x \in A, y \in B$ and $\omega \in \Omega$.

Take $\delta := \min\{\delta_1, \delta_2\}$. Then for all $x \in A, y \in B$ and $\omega \in \Omega$ such that

$$\begin{aligned} \|x(\omega) - y(\omega)\| < \delta + dist(A, B) \text{ we have,} \\ \|(g + h)(\omega, x) - (g + h)(\omega, y)\| &= \|g(\omega, x) - g(\omega, y) + h(\omega, x) - h(\omega, y)\| \\ &< \|g(\omega, x) - g(\omega, y)\| + \|h(\omega, x) - h(\omega, y)\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $g + h$ is a relatively u-continuous random mapping. Hence, this corollary follows immediately from Theorem 4.5. □

Remark 4. Corollary 4.7 improves and generalizes the results of Sehgal and Waters ([53], Theorem 3) and several related results in literature.

5. Application to a random nonlinear integral equation of the Hammerstein type

In this section, we shall use our results to prove the existence of a solution in a Banach space of a random nonlinear integral equation of the form:

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu_0(s) \tag{5.1}$$

where

- (i) S is a locally compact metric space with a metric d on $S \times S$ equipped with a complete σ -finite measure μ_0 defined on the collection of Borel subsets of S ;
- (ii) $\omega \in \Omega$, where ω is a supporting element of a set of probability measure space (Ω, β, μ) ;
- (iii) $x(t; \omega)$ is the unknown vector-valued random variable for each $t \in S$;
- (iv) $h(t; \omega)$ is the stochastic free term defined on $t \in S$;
- (v) $k(t, s; \omega)$ is the stochastic kernel defined for t and s in S and
- (vi) $f(t, x)$ is a vector-valued function of $t \in S$ and x .

The integral equation (5.1) is interpreted as a Bochner integral, (see Padgett [47]).

Furthermore, we shall assume that S is the union of a decreasing sequence of countable family of compact family of compact sets $\{C_n\}$ such that for any other compact set in S there is a C_i , which contains it, (see, Arens [4]).

Definition 20. We define the space $C(S, L_2(\Omega, \beta, \mu))$ to be the space of all continuous functions from S into $L_2(\Omega, \beta, \mu)$ with the topology of uniform convergence on compact sets of S that is for each fixed $t \in S$, $x(t; \omega)$ is a vector valued random variable such that

$$\|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}^2 = \int_{\Omega} |x(t; \omega)|^2 d\mu(\omega) < \infty.$$

Note that $C(S, L_2(\Omega, \beta, \mu))$ is a locally convex space, whose topology is defined by a countable family of semi-norms (see, Yosida [58]) given by

$$\|x(t; \omega)\|_n = \sup_{t \in C_n} \|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}, \quad n = 1, 2, \dots$$

Moreover, $C(S, L_2(\Omega, \beta, \mu))$ is complete relative to this topology, since $L_2(\Omega, \beta, \mu)$ is complete.

We define $BC = BC(S, L_2(\Omega, \beta, \mu))$ to be the Banach space of all bounded continuous functions from S into $L_2(\Omega, \beta, \mu)$ with norm

$$\|x(t; \omega)\|_{BC} = \sup_{t \in S} \|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}.$$

The space $BC \subset C$ is the space of all second order vector valued stochastic process defined on S , which is bounded and continuous in mean square. We will consider the function $h(t; \omega)$ and $f(t, x(t; \omega))$ to be in the space $C(S, L_2(\Omega, \beta, \mu))$ with respect to the stochastic kernel. We assume that for each pair (t, s) , $k(t, s, \omega) \in L_{\infty}(\Omega, \beta, \mu)$ and denote the norm by

$$\begin{aligned} \| \|k(t, s; \omega)\| \| &= \|k(t, s; \omega)\|_{L_{\infty}(\Omega, \beta, \mu)} \\ &= \mu - \text{ess sup}_{\omega \in \Omega} |k(t, s; \omega)|. \end{aligned}$$

Suppose that $k(t, s; \omega)$ is such that $\| \|k(t, s; \omega)\| \| \cdot \|x(s; \omega)\|_{L_2(\Omega, \beta, \mu)}$ is μ_0 -integrable with respect to s for each $t \in S$ and $x(s; \omega)$ in $C(S, L_2(\Omega, \beta, \mu))$ and there exists a real valued function G defined μ_0 -a.e. on S , so that $G(S)\|x(s; \omega)\|_{L_2(\Omega, \beta, \mu)}$ is μ_0 -integrable and for each pair $(t, s) \in S \times S$,

$$\| \|k(t, u; \omega) - k(s, u; \omega)\| \| \cdot \|x(u, \omega)\|_{L_2(\Omega, \beta, \mu)} \leq G(u)\|x(u, \omega)\|_{L_2(\Omega, \beta, \mu)}$$

μ_0 - a.e. Furthermore, for almost all $s \in S$, $k(t, s; \omega)$ will be continuous in t from S into $L_{\infty}(\Omega, \beta, \mu)$.

Now, we define the random integral operator T on $C(S, L_2(\Omega, \beta, \mu))$ by

$$(Tx)(t; \omega) = \int_S k(t, s; \omega)x(s; \omega)d\mu_0(s) \tag{5.2}$$

where the integral is a Bochner integral. Moreover, we have that for each $t \in S$, $(Tx)(t; \omega) \in L_2(\Omega, \beta, \mu)$ and that $(Tx)(t; \omega)$ is continuous in mean square by Lebesgue dominated convergence theorem. So $(Tx)(t; \omega) \in C(S, L_2(\Omega, \beta, \mu))$.

Definition 21. ([37]). Let B and D be two Banach spaces. The pair (B, D) is said to be admissible with respect to a random operator $T(\omega)$ if $T(\omega)(B) \subset D$.

Lemma 5.1. ([30]). *The linear operator T defined by (5.2) is continuous from $C(S, L_2(\Omega, \beta, \mu))$ into itself.*

Lemma 5.2. ([30] and [37]). *If T is a continuous linear operator from $C(S, L_2(\Omega, \beta, \mu))$ into itself and $B, D \subset C(S, L_2(\Omega, \beta, \mu))$ are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ such that (B, D) is admissible with respect to T , then T is continuous from B into D .*

Remark 5. ([30]). The operator T defined by (5.2) is a bounded linear operator from B into D .

A random solution of equation (5.1) will mean a function $x(t; \omega)$ in $C(S, L_2(\Omega, \beta, \mu))$ which satisfies the equation (5.1) μ - a.e.

We now prove the following theorem.

Theorem 5.3. *We consider the stochastic integral equation (5.1) subject to the following conditions*

(a) *B and D are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ such that (B, D) is admissible with respect to the integral operator defined by (5.2);*

(b) *$x(t; \omega) \rightarrow f(t, x(t; \omega))$ is an operator from the set*

$Q(\rho) = \{x(t; \omega) : x(t; \omega) \in D, \|x(t; \omega)\|_D \leq \rho\}$ *into the space B satisfying*

$$f(\Omega \times B) \subset D, \quad f(\Omega \times D) \subset B; \tag{5.3}$$

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \leq \|x(t; \omega) - y(t; \omega)\|_D, \tag{5.4}$$

for all $x(t; \omega), y(t; \omega) \in Q(\rho)$.

(c) $h(t; \omega) \in D$.

Then there exists a unique random solution of (5.1) in $Q(\rho)$, provided $c(\omega) < 1$ and

$$\|h(t; \omega)\|_D + c(\omega)\|f(t, 0)\|_B \leq \rho(1 - c(\omega)),$$

where $c(\omega)$ is the norm of $T(\omega)$.

Proof. Since the pair (B, D) is admissible with respect to the integral operator defined by (5.2), then condition (5.3) is satisfied.

Define the operator $U(\omega)$ from $Q(\rho)$ into D by

$$(Ux)(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu_0(s). \tag{5.5}$$

Using conditions of the theorem, we have

$$\begin{aligned} \|(Ux)(t; \omega)\|_D &\leq \|h(t; \omega)\|_D + c(\omega)\|f(t, x(t; \omega))\|_B \\ &\leq \|h(t; \omega)\|_D + c(\omega)\|f(t, 0)\|_B \\ &\quad + c(\omega)\|f(t, x(t; \omega)) - f(t, 0)\|_B. \end{aligned} \tag{5.6}$$

Now from condition (5.4), we have

$$\|f(t, x(t; \omega)) - f(t, 0)\|_B \leq \|x(t; \omega)\|_D. \tag{5.7}$$

Using (5.7) in (5.6), we have

$$\|(Ux)(t; \omega)\|_D \leq \|h(t; \omega)\|_D + c(\omega)\|f(t, 0)\|_B + c(\omega)\|x(t; \omega)\|_D \leq \rho. \tag{5.8}$$

Hence, $(Ux)(t; \omega) \in Q(\rho)$.

$$\begin{aligned}
\|(Ux)(t; \omega) - (Uy)(t; \omega)\|_D &= \left\| \int_S k(t, s; \omega) [f(s, x(s; \omega)) - f(s, y(s; \omega))] d\mu_0(s) \right\|_D \\
&\leq c(\omega) \|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \\
&\leq c(\omega) \|x(t; \omega) - y(t; \omega)\|_D.
\end{aligned}$$

Since $c(\omega) < 1$, U is a contraction on $Q(\rho)$. Therefore, by ([18], Theorem 2.2) there exists a unique $x^*(t; \omega) \in Q(\rho)$ which is a fixed point of U . This means that $x^*(t; \omega)$ is the unique random solution of equation (5.1). The proof of Theorem 5.3 is completed. \square

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