TWO IDENTITIES INVOLVING THETA FUNCTIONS

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Abstract. We aim to present two identities which reveal certain interesting relationships among three fundamental theta functions arising from the Jacobi’s triple product in an elementary way.

1. Introduction and preliminaries

Jacobi [7] initiated the theory of theta functions which has a long history and many applications in a variety of research fields such as number theory (for example, quadratic forms and elliptic functions) and quantum physics. The Jacobi triple product, which an infinite series is expressed as an infinite product, is given as follows (see [7]):

\[\sum_{k=-\infty}^{\infty} y^k x^{k^2} = \prod_{k=1}^{\infty} \left(1 + y x^{2k-1}\right) \left(1 + \frac{1}{y} x^{2k-1}\right) \left(1 - x^{2k}\right) \quad (1.1)\]

\(|x| < 1, \ y \neq 0\).

From (1.1), it is easy to define the following three fundamental theta functions:

\[f(-x) := \sum_{n=-\infty}^{\infty} (-1)^n x^{n(n+1)/2} = \prod_{k=1}^{\infty} \left(1 - x^{k}\right); \quad (1.2)\]

\[\phi(x) := \sum_{n=-\infty}^{\infty} x^{n^2} = \prod_{k=1}^{\infty} (1 + x^{2k})(1 + x^{2k-1})(1 - x^{2k}); \quad (1.3)\]

\[\psi(x) := \sum_{n=0}^{\infty} x^{n(n+1)/2} = \prod_{k=1}^{\infty} \frac{1 - x^{2k}}{(1 - x^{2k-1})}. \quad (1.4)\]

In this note, we present two identities which reveal certain interesting relationships among three fundamental theta functions arising from the Jacobi’s triple product in an elementary way. For more details and results, the interested reader may be referred to the works [1, 2, 3, 4, 5, 6, 8, 9, 10] and the references therein.

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To do this, we express the functions $f(-x)$, $\phi(x)$, and $\psi(x)$ in the rising powers:

$$f(-x) = 1 + \sum_{n=1}^{\infty} (-1)^n \left( x^{n(3n-1)} + x^{n(3n+1)} \right)$$

$$= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots; \tag{1.5}$$

$$\phi(x) = 1 + 2 \sum_{n=1}^{\infty} x^{n^2} = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \cdots; \tag{1.6}$$

$$\psi(x) = 1 + \sum_{n=1}^{\infty} x^{\frac{n(n+1)}{2}} = 1 + x + x^3 + x^6 + x^{10} + x^{15} + \cdots. \tag{1.7}$$

2. Main results

Here, we present two identities that reveal certain interesting relationships among the three functions $f(-x)$, $\phi(x)$, and $\psi(x)$, which are asserted by the following theorem.

**Theorem 2.1.** Let $f(-x)$, $\phi(x)$, and $\psi(x)$ be the functions in (1.2), (1.3), and (1.4), respectively. Then

$$f(-x^3)f(-x^9) = \phi(x^{54})[\psi(x^4) - x^4\psi(x^{36})] + x^{13}\psi(x^{108})[\phi(x^2) - \phi(x^{18})]$$

$$- x^3\psi(x^{27})[\psi(x) - x\psi(x^9)]$$

and

$$8x^3[\psi(x^4) - x^4\psi(x^{36})][\psi(x^{20}) - x^{20}\psi(x^{180})]$$

$$= [\phi(x^{3})\phi(x^{15}) - \phi(-x^{3})\phi(-x^{15})] + 3[\phi(-x^{27})\phi(-x^{135}) - \phi(x^{27})\phi(x^{135})]$$

$$+ [\phi(x^{3})\phi(x^{135}) - \phi(-x^{3})\phi(-x^{135})] + [\phi(x^{27})\phi(x^{15}) - \phi(-x^{27})\phi(-x^{15})]. \tag{2.2}$$

**Proof.** For (2.1), let $\mathcal{L}(x)$ and $\mathcal{R}(x)$ be the left- and right-sides of (2.1), respectively. By using the expansions in (1.5), (1.6), and (1.7), we can see that

$$\mathcal{L}(x) = 1 - x^3 - x^6 - x^9 + x^{12} + 2x^{15} - x^{18} + 2x^{21} - x^{30}$$

$$- x^{33} - x^{36} - x^{39} + x^{45} - x^{48} - x^{51} + 2x^{54} + x^{60} + 2x^{63}$$

$$+ x^{66} - x^{69} - x^{75} + 2x^{78} - x^{81} - x^{87} - x^{90} - x^{96} - x^{99}$$

$$- x^{105} - 2x^{108} + 2x^{111} + 2x^{114} - x^{120} + x^{123} + x^{129}$$

$$- x^{135} + 2x^{138} + 2x^{141} + x^{144} - 2x^{150} + 2x^{153} - x^{156}$$

$$- x^{162} - x^{165} - x^{168} + x^{171} - \cdots$$

$$= \mathcal{R}(x).$$

This completes the proof of (2.1). Similarly, we can prove (2.2). We omit the details.
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