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CHARACTERIZATIONS OF SEVERAL SPLIT REGULAR FUNCTIONS ON SPLIT QUATERNION IN CLIFFORD ANALYSIS

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Abstract. In this paper, we investigate the regularities of the hypercomplex valued functions of the split quaternion variables. We define several differential operators for the split qunaternionic function. We research several left split regular functions for each differential operators. We also investigate split harmonic functions. And we find the corresponding Cauchy-Riemann system and the corresponding Cauchy theorem for each regular functions on the split quaternion field.

1. Introduction

The non-commutative four dimensional real field of the hypercomplex numbers with some properties is called a split quaternion (skew) field $\mathcal{S}$.

Naser [12] described the notation and the properties of regular functions by using the differential operator D in the hypercomplex number system. And Naser [12] investigated conjugate harmonic functions of quaternion variables. In 2011, Koriyama et al. [9] researched properties of regular functions in quaternion field. In 2013, Jung et al. [1] have studied the hyperholomorphic functions of dual quaternion variables. And Jung and Shon [2] have shown hyperholomorphy of hypercomplex functions on dual ternary number system. Kim et al. [8] have investigated regularities of ternary number valued functions in Clifford analysis. Kang and Shon [3] have developed several differential operators for quaternionic functions. And Kang et al. [4] researched some properties of quaternionic regular functions. Kim and Shon [5, 6] obtained properties of hyperholomorphic functions and hypermeromorphic functions in each hypercompelx number system. And Kim and Shon [7] expressed regular functions of hypercomplex varialbes in the polar coordinate in the split quaternion number

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system. Recently, Lim and Shon [10, 11] have studied several regular functions, biregular functions and J-regular functions of non-commutative algebra associated Pauli matrices in Clifford analysis.

We represent the corresponding split Cauchy-Riemann system from the regularities of split quaternionic functions. We also research the split-harmonic functions and the corresponding split Cauchy theorem in split quaternion structure.

2. Preliminaries

The split quaternionic field

$$\mathcal{S} = \{z \mid z = \sum_{p=0}^{3} e_p x_p,\ x_p \in \mathbb{R}\} \approx \mathbb{C}^2$$

is a four-dimensional non-commutative $\mathbb{R}$-division ring (skew field) generated by four basis $\{e_0, e_1, e_2, e_3\}$ with the multiplication rules. Each basis satisfiy the followings:

$$e_0 = id.,\ e_1^2 = -1,\ e_2^2 = e_3^2 = 1\ (e_2, e_3 \neq \pm 1),$$

$$e_1e_2 = -e_2e_1 = e_3,\ e_2e_3 = -e_3e_2 = -e_1,\ e_3e_1 = -e_1e_3 = e_2$$

and

$$e_1e_2e_3 = 1.$$

The base e_1 of $\mathcal{S}$ identifies the imanginary unit $i = \sqrt{-1}$ in the $\mathbb{C}$-field of complex numbers. With the split quaternionic multiplication, a split quaternion z is denoted by

$$z = \sum_{p=0}^{3} e_p x_p = x_0 + e_1x_1 + (x_2 + e_1x_3)e_2 = z_1 + z_2e_2,$$

where $z_1 = x_0 + e_1x_1$ and $z_2 = x_2 + e_1x_3$. And $\overline{z_1} = x_0 - e_1x_1$ and $\overline{z_2} = x_2 - e_1x_3$ are usual complex numbers in $\mathbb{C}$. Then, the split quaternionic conjugate z^* is defined by

$$z^* = x_0 - \sum_{p=1}^{3} e_p x_p = \overline{z_1} - z_2e_2$$

in the similar way of representation of z. Due to the non-commutativity of the split quaternion, we should be careful to the multiplication. We know

$$\begin{aligned} zw &= (z_1 + z_2e_2)(w_1 + w_2e_2) \\ &= (z_1w_1 + z_2\overline{w_2}) + (z_1w_2 + z_2\overline{w_1})e_2 \neq wz \end{aligned}$$

for any split quaternion $z, w \in \mathcal{S}$.

The modulus zz^* of z and z^* is defined by

$$zz^* = x_0^2 + x_1^2 - x_2^2 - x_3^2 = |z_1|^2 - |z_2|^2 = z^*z$$

and any non-zero split quaternion z has a unique inverse

$$z^{-1} = \frac{z^*}{zz^*} \quad (zz^* \neq 0).$$

Let Ω be a bounded open set in $\mathcal{S}$ and $z \in \Omega$. A function $f : \Omega \to \mathcal{S}$ is expressed by

$$\begin{aligned} f(z) &= \sum_{p=0}^{3} e_p u_p(x_0, x_1, x_2, x_3) = u_0 + e_1 u_1 + (u_2 + e_1 u_3) e_2 \\ &= f_1(z_1, z_2) + f_2(z_1, z_2) e_2, \end{aligned}$$

where u_p $(p = 0,\ 1,\ 2,\ 3)$ are real valued functions and $f_1(z_1, z_2),\ f_2(z_1, z_2)$ are complex valued functions of two complex variables.

We use the following quaternionic differential operators:

$$\begin{aligned} D &:= \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial z_2} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}\right), \\ D^* &= \frac{\partial}{\partial \overline{z_1}} - e_2 \frac{\partial}{\partial z_2} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3}\right), \end{aligned} \tag{1}$$

where $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z_j}}$ $(j = 1, 2)$ are usual complex differential operators. And we have

$$\frac{\partial}{\partial z_1} e_2 = \frac{1}{2}(e_2 \frac{\partial}{\partial x_0} - e_1 e_2 \frac{\partial}{\partial x_1}) = \frac{1}{2}(e_2 \frac{\partial}{\partial x_0} + e_2 e_1 \frac{\partial}{\partial x_1}) = e_2 \frac{\partial}{\partial \overline{z_1}}.$$

We should be cautious when we compute some formulas because the split quaternion field $\mathcal{S}$ is non-commutative.

Definition 1. Let Ω be a bounded open set in $\mathcal{S}$. A split quaternionic function $f(z) = f_1(z) + f_2(z) e_2$ is said to be a L-split regular function on Ω if
(a) $f_1, f_2 \in C^1(\Omega)$,
(b) $D^* f = 0$ in Ω.

The above equation (b) is equivalent to

$$D^* f = \left(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial \overline{f_2}}{\partial \overline{z}_2}\right) + \left(\frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial \overline{f_1}}{\partial \overline{z}_2}\right) e_2 = 0.$$

So we have

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial \overline{z_2}},\ \frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial \overline{f}_1}{\partial \overline{z_2}}.$$

This system is called a corresponding split Cauchy-Riemann system in $\mathcal{S}$.

3. L_p-Split Regular Functions

We define more split quaternionic differential operators to compare the variety of split quaternionic differentials. We consider the following differential operators:

$$\begin{aligned}
D_1 &:= \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial \overline{z_2}}, \quad D_1^* = \frac{\partial}{\partial \overline{z_1}} - e_2 \frac{\partial}{\partial \overline{z_2}}, \\
D_2 &:= \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial z_2}, \quad D_2^* = \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial z_2}, \\
D_3 &:= \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial z_2}, \quad D_3^* = \frac{\partial}{\partial \overline{z_1}} - e_2 \frac{\partial}{\partial z_2}, \\
D_4 &:= \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial \overline{z_2}}, \quad D_4^* = \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial \overline{z_2}}.
\end{aligned} \tag{2}$$

Definition 2. Let Ω be a bounded open set in $\mathcal{S}$. A function $f(z)$ is said to be a L_p-*split regular function* on Ω for $p = 1, 2, 3, 4$ if the following two conditions are satisfied:
(a) $f_1, f_2 \in C^1(\Omega)$,
(b) $D_p^* f = 0$ on $\Omega \quad (p = 1, 2, 3, 4)$.

Similarly with the previous case, in the above condition (b) of Definition 2, D_p^* $(p = 1, 2, 3, 4)$ operate to f by the multiplications of the split quaternion. So we have the following remark.

Remark 1. For each differential operators, the following systems are called the corresponding split Cauchy-Riemann system in $\mathcal{S}$:

$$\begin{aligned}
&\text{if } \ p = 1, \quad \frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial z_2}, \ \frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial \overline{f}_1}{\partial z_2}, \\
&\text{if } \ p = 2, \quad \frac{\partial f_1}{\partial z_1} = \frac{\partial \overline{f_2}}{\partial \overline{z_2}}, \ \frac{\partial f_2}{\partial z_1} = \frac{\partial \overline{f_1}}{\partial \overline{z_2}}, \\
&\text{if } \ p = 3, \quad \frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial \overline{z_2}}, \ \frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial \overline{f}_1}{\partial \overline{z_2}}, \\
&\text{if } \ p = 4, \quad \frac{\partial f_1}{\partial z_1} = \frac{\partial \overline{f_2}}{\partial z_2}, \ \frac{\partial f_2}{\partial z_1} = \frac{\partial \overline{f_1}}{\partial z_2}.
\end{aligned} \tag{3}$$

By the simple computation of the operator D_p and D_p^*, we have

$$D_p D_p^* = D_p^* D_p = \frac{1}{4}\left(\frac{\partial^2}{\partial x_0 \partial x_0} + \frac{\partial^2}{\partial x_1 \partial x_1} - \frac{\partial^2}{\partial x_2 \partial x_2} - \frac{\partial^2}{\partial x_3 \partial x_3}\right)$$

for $p = 1,\ 2,\ 3,\ 4$. We denote the operator $D_p D_p^*$ by DD^* for convenience.

Definition 3. Let Ω be a bounded open set in S. A function f is a split harmonic on Ω if f_j $(j = 1, 2)$ are split harmonic on Ω.

Definition 4. Let Ω be a bounded open set in S. For any element $z \in S$, a function f is said to be a split harmonic function on Ω if $DD^* f = 0$.

Theorem 3.1. *If f is a L_p-split regular function on Ω for $p = 1, 2, 3, 4$, then f_j $(j = 1, 2)$ are split harmonic on Ω.*

Proof. By the simple computation, if $p = 1$,

$$\begin{aligned} DD^* f_1 &= \left(\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} - \frac{\partial^2}{\partial z_2 \partial \overline{z_2}} \right) f_1 \\ &= \frac{\partial}{\partial z_1} \left(\frac{\partial f_1}{\partial \overline{z_1}} \right) - \frac{\partial}{\partial \overline{z_2}} \left(\frac{\partial f_1}{\partial z_2} \right) \\ &= \frac{\partial}{\partial z_1} \left(\frac{\partial \overline{f_2}}{\partial \overline{z_2}} \right) - \frac{\partial}{\partial \overline{z_2}} \left(\frac{\partial \overline{f_2}}{\partial z_1} \right) \\ &= 0. \end{aligned}$$

And we can prove that f_2 is split harmonic on Ω.

Similarly, we obtain the results in the cases of $p = 2, 3, 4$. □

Example 3.2. *Let Ω be a bounded open set in $\mathcal{S}$ and $f \in L_p(\Omega)$ for $p = 1, 2, 3, 4$. The function $f(z)$ is defined by*

$$\begin{aligned} f(z) &= \exp(z_1 + z_2) + \sin(z_1)\cos(z_2)e_2 \\ &= \exp(x_0 + e_1x_1 + x_2 + e_1x_3) + \sin(x_0 + e_1x_1)\cos(x_2 + e_1x_3)e_2. \end{aligned}$$

Therefore,

$$\begin{aligned} DD^* f(z) &= \left(\frac{\partial^2}{\partial x_0 \partial x_0} + \frac{\partial^2}{\partial x_1 \partial x_1} - \frac{\partial^2}{\partial x_2 \partial x_2} - \frac{\partial^2}{\partial x_3 \partial x_3} \right) f(z) \\ &= \frac{\partial}{\partial x_0} (\exp(x_0 + e_1x_1 + x_2 + e_1x_3) + \cos(x_2 + e_1x_3)\cos(x_0 + e_1x_1)) \\ &\quad + \frac{\partial}{\partial x_1} (e_1 \exp(x_0 + e_1x_1 + x_2 + e_1x_3) + e_1 \cos(x_2 + e_1x_3)\cos(x_0 + e_1x_1)) \\ &\quad - \frac{\partial}{\partial x_2} (\exp(x_0 + e_1x_1 + x_2 + e_1x_3) - \sin(x_0 + e_1x_1)\sin(x_2 + e_1x_3)) \\ &\quad - \frac{\partial}{\partial x_3} (e_1 \exp(x_0 + e_1x_1 + x_2 + e_1x_3) - e_1 \sin(x_0 + e_1x_1)\sin(x_2 + e_1x_3)) \\ &= \exp(x_0 + e_1x_1 + x_2 + e_1x_3) - \cos(x_2 + e_1x_3)\sin(x_0 + e_1x_1) \\ &\quad - \exp(x_0 + e_1x_1 + x_2 + e_1x_3) + \cos(x_2 + e_1x_3)\sin(x_0 + e_1x_1) \\ &\quad - \exp(x_0 + e_1x_1 + x_2 + e_1x_3) + \sin(x_0 + e_1x_1)\cos(x_2 + e_1x_3) \\ &\quad + \exp(x_0 + e_1x_1 + x_2 + e_1x_3) - \sin(x_0 + e_1x_1)\cos(x_2 + e_1x_3) = 0. \end{aligned}$$

Thus, $f(z)$ is a split harmonic function on Ω.

Example 3.3. *Let Ω be a bounded open set in $\mathcal{S}$ and $f \in L_p(\Omega)$ for $p = 1, 2, 3, 4$.*

(*a*) *A function $f(z) = e^{z_1} + \sin z_2 e_2$ is split harmonic function on Ω.*
(*b*) *A function $f(z) = e^{z_1} + \sin \overline{z_2} e_2$ is split harmonic function on Ω.*
(*c*) *A function $f(z) = e^{\overline{z_1}} + \sin z_2 e_2$ is split harmonic function on Ω.*
(*d*) *A function $f(z) = e^{\overline{z_1}} + \sin \overline{z_2} e_2$ is split harmonic function on Ω.*

4. The Corresponding Split Cauchy Theorem on L_p-Regular Functions

In 1971, Naser [12] has studied the corresponding Cauchy theorem in the quaternion field. Similarly, we find split quaternion forms κ_p $(p = 1, 2, 3, 4)$ to research the corresponding split Cauchy theorem for each differential operator (2) as follows:

$$\begin{aligned}\kappa_1 &= dz_1 \wedge dz_2 \wedge d\overline{z_2} + dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} e_2,\\ \kappa_2 &= dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} + dz_1 \wedge dz_2 \wedge d\overline{z_1} e_2\\ \kappa_3 &= dz_1 \wedge dz_2 \wedge d\overline{z_2} + dz_1 \wedge dz_2 \wedge d\overline{z_1} e_2,\\ \kappa_4 &= dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} + dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} e_2.\end{aligned}$$

And the split quaternion forms κ_p $(p = 1, 2, 3, 4)$ are called kernels for the corresponding split Cauchy theorem.

Theorem 4.1. *Let κ_p $(p = 1, 2, 3, 4)$ be kernels for the split corresponding Cauchy theorem for each differential operator and Ω be an open set in $\mathcal{S}$. If a function f is L_p- split regular in Ω, then for a domain $D \subset \Omega$ with smooth boundary ∂D,*

$$\int_{\partial D} \kappa_p f = 0 \ \ (p = 1, 2, 3, 4), \tag{4}$$

where $\kappa_p f$ is the split quaternion product of form (4) upon the function f.

Proof. In the case of $p = 1$, by the rule of the split quaternionic multiplication, we have

$$\begin{aligned}\kappa_1 f =&(dz_1 \wedge dz_2 \wedge d\overline{z_2} + dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} e_2)(f_1 + f_2 e_2)\\ =&\overline{f_1}(dz_1 \wedge dz_2 \wedge d\overline{z_2}) + f_1(dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2})e_2\\ &+ \overline{f_2}(dz_1 \wedge dz_2 \wedge d\overline{z_2})e_2 + f_1(dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2}).\end{aligned}$$

Thus,

$$\begin{aligned}d(\kappa_1 f) =&(\partial + \overline{\partial})(\kappa_1 f)\\ =&\partial(\kappa_1 f) + \overline{\partial}(\kappa_1 f)\\ =&\left(\frac{\partial}{\partial z_1}dz_1 + \frac{\partial}{\partial z_2}dz_2\right)(\kappa_1 f) + \left(\frac{\partial}{\partial \overline{z_1}}d\overline{z_1} + \frac{\partial}{\partial \overline{z_2}}d\overline{z_2}\right)(\kappa_1 f)\\ =&\left(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial \overline{f_2}}{\partial z_2}\right) dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}\\ &+ \left(\frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial \overline{f_1}}{\partial z_2}\right) dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} e_2.\end{aligned}$$

Since f is a L_1-split regular function, from the corresponding split Cauchy-Riemann system (3), we have $d(\kappa_1 f) = 0$. By Stokes' theorem, we obtain

$$\int_{\partial D} \kappa_1 f = \int_D d(\kappa_1 f) = 0.$$

Similarly, in the cases of $p = 2,\ 3,\ 4$, we have the results. □

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