East Asian Math. J.
Vol. 33 (2017), No. 3, pp. 323–331
http://dx.doi.org/10.7858/eamj.2017.025

EXISTENCE OF POSITIVE SOLUTIONS FOR EIGENVALUE PROBLEMS OF SINGULAR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

Yong-Hoon Lee and Jinsil Lee

Abstract. In this paper, we consider the existence of positive solutions for eigenvalue problems of nonlinear fractional differential equations with singular weights. We give various conditions on f and apply Krasnoselskii's Cone Fixed Point Theorem. As a result, we obtain several existence and nonexistence results corresponding to λ in certain intervals.

1. Introduction

In this paper, we investigate the existence and nonexistence of positive solutions for fractional differential equations with Dirichlet boundary value problems of the form

$$\begin{cases} D^{\alpha}_{0+}u(t) + \lambda h(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{E_λ}$$

where D^{α}_{0+} is the Riemann-Liouville fractional derivative of order α, which is a real number in $(1, 2]$, λ is a positive real parameter, $f \in C([0,\infty), [0,\infty))$and $h \in L^1_{loc}((0, 1), [0,\infty))$ satisfies the condition

$$(H) \qquad \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}h(s)ds < +\infty.$$

Several authors have widely studied existence of positive solutions for fractional differential equations. In particular, Jiang and Yuan([2]) studied positive solutions of nonlinear fractional boundary value problem

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{1.1}$$

Received February 13, 2017; Accepted March 17, 2017.
2010 *Mathematics Subject Classification.* 34B16, 34B18.
Key words and phrases. fractional differential equation, eigenvalue problem, positive solution, existence of solution, singular weight.
This work was supported by a 2-Year Research Grant of Pusan National University.

with the following hypotheses:
(A) $f(t,u)$ is continuous on $[0,1]\times[0,\infty)$
(B) there exist $g\in C([0,\infty),[0,\infty))$, $q_1,q_2\in C((0,1),(0,\infty))$ such that

$$q_1(t)g(u)\le f(t,t^{\alpha-2}u)\le q_2(t)g(u),$$

and $q_i\in L^1(0,1)\quad i=1,2$.

By means of a fixed point theorem, they proved the existence of positive solutions for (1.1) when $g_0=\lim_{u\to 0}\frac{g(u)}{u}$ and $g_\infty=\lim_{u\to\infty}\frac{g(u)}{u}$ are either 0 or ∞.

On the other hand, Han and Gao([3]) established the existence results for the following type of differential equations

$$\begin{cases} D^{\alpha}_{0+}u(t)+\lambda a(t)f(t,u(t))=0, & t\in(0,1),\\ u(0)=0=u(1), \end{cases} \tag{1.2}$$

under assumptions $a\in C([0,1],[0,\infty))$, (A) and (B). They proved the existence of at least one positive solution for (1.2) if both g_0 and g_∞ are finite.

It is interesting to consider the cases that g_0 and g_∞ are neither 0 nor ∞ and as far as the authors know, there have not been any studies about the cases for eigenvlaue problems specially when the weight a is singular. To focus on the singular effect on t-variable, we simply consider the nonlinear term as a separation of variable type, that is, f is of the form $f(t,u)=h(t)g(u)$. The results to variable dependent case can be extended in obvious way.

For the problem having singular weights, Lee and Lee ([5]) investigate the existence of a positive solution for the following nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}_{0+}u(t)+h(t)f(u(t))=0, & t\in(0,1),\\ u(0)=0=u(1), \end{cases} \tag{1.3}$$

where D^{α}_{0+} is the Riemann-Liouville fractional derivative of order α, which is a real number in $(1,2]$, $h\in L^1_{loc}(0,1)$ satisfies the condition (H) and $f\in C([0,\infty),[0,\infty))$. They show that (1.3) has at least one positive solution if either $f_0=0$, $f_\infty=\infty$ or $f_0=\infty$, $f_\infty=0$.

Reminding that given weight function h in our problem is singular at the boundary which may not be integrable but satisfying (H), we exploit several existence and nonexistence results when the nonlinear term f satisfies several conditions such as f_0 and f_∞ could be 0, ∞ or finite.

Our main idea is to construct a cone in a Banach space and a completely continuous operator defined on this cone based on the corresponding Green's function and then we find fixed points for some λ in a certain interval. In addition, we also prove that (E_λ) has no positive solution when λ is in a particular interval.

2. Preliminaries

In this section, we introduce some definitions of fractional calculus and some important theorems and lemmas which we will use later.

Definition 1. Assume that $f(t) \in C[a,b]$ and let n be a number satisfying $n-1 \leq \alpha < n$. Then ${}_aD_t^\alpha f(t)$ is said to be a Riemann-Liouville fractional derivative which is defined by

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_a^t (t-\tau)^{-\alpha+n-1} f(\tau)d\tau.$$

Remark 1. ([1]) In particular Riemann-Liouville fractional derivative case, let n be a number satisfying $n-1 \leq \alpha < n$. Then we define the derivative $D_{0+}^\alpha f(t)$ as

$$D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_0^t (t-\tau)^{-\alpha+n-1} f(\tau)d\tau.$$

Also, we define the integral $I_{0+}^\alpha f(t)$ as

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad x > 0 \text{ and } \alpha > 0.$$

By definitions, we know $D_{0+}^\alpha f(t) = (\frac{d}{dt})^n I_{0+}^{n-\alpha} f(t)$.

Definition 2. Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that

(1) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$, and
(2) $u, -u \in P$ implies $u = 0$.

Theorem 2.1. (Fixed point theorem of cone expansion/compression type) *Let E be a Banach space and let P be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Assume that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous such that either*

(1) *$\|Tu\| \leq \|u\|$, for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, for $u \in P \cap \partial\Omega_2$, or*
(2) *$\|Tu\| \geq \|u\|$, for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, for $u \in P \cap \partial\Omega_2$.*

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. An Application to Eigenvalue Problems

In this section, we prove our main results. We first consider the solution operator. Since h is singular, we cannot get the operator directly by taking fractional integral. In [5], the authors showed that problem (E_λ) is equivalently written as

$$u(t) = \lambda \int_0^1 G(t,s)h(s)f(u(s))ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{3.1}$$

Remark 2. ([2], [4]) The Green function $G(t,s)$ defined by (3.1) has the following properties

(1) $G(t,s) \in C([0,1] \times [0,1])$, and $G(t,s) > 0$ for $t, s \in (0,1)$,
(2) $\max_{0 \le t \le 1} G(t,s) = G(s,s)$, for s $\in (0,1)$.

Let $E = C[0,1]$ be endowed with the ordering $u \le v$ if $u(t) \le v(t)$ for all $t \in [0,1]$. We define $P \subseteq E$ by

$$P = \{u \in E \mid u(t) \ge 0,\ u(t) \ge (\alpha-1)t(1-t)||u||_\infty\}.$$

Then we can easily see that P is a cone. For $u \in E$, define an operator T given as

$$Tu(t) = \lambda \int_0^1 G(t,s)h(s)f(u(s))ds.$$

Then problem (E_λ) can be equivalently written as

$$u = Tu$$

and it is known ([5]) that $T : E \to P$ is completely continuous. When either $f_0 = 0$, $f_\infty = \infty$ or $f_0 = \infty$, $f_\infty = 0$, it is also known ([5]) that under assumption (H), problem (E_λ) has at least one positive solution for all $\lambda > 0$.

In this paper, we first consider the case that f_0 is finite.

Lemma 3.1. *Assume $0 < f_0 < \infty$ and $f_\infty = \infty$ and assume (H). Then problem (E_λ) has at least one positive solution for $\lambda \in (0, (f_0 \int_0^1 G(s,s)h(s)ds)^{-1})$.*

Proof. Fix λ with

$$\lambda < (f_0 \int_0^1 G(s,s)h(s)ds)^{-1}.$$

Then we may choose $\zeta > 0$ satisfying

$$\lambda = ((f_0 + \zeta) \int_0^1 G(s,s)h(s)ds)^{-1}.$$

From the definition of f_0, we can select $r_1 > 0$ such that $f(u) < u(f_0 + \zeta)$ for $0 < u \leq r_1$. Take $\Omega_{r_1} = \{u \in C[0,1] | ||u||_\infty < r_1\}$. For $u \in P \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)h(s)f(u(s))ds \\ &\leq \lambda \int_0^1 G(s,s)h(s)(f_0+\zeta)u(s)ds \\ &\leq ||u||_\infty \lambda \int_0^1 G(s,s)h(s)(f_0+\zeta)ds \\ &= ||u||_\infty. \end{aligned}$$

Hence, this implies that $||Tu||_\infty \leq ||u||_\infty$ for $u \in P \cap \partial\Omega_{r_1}$.

On the other hand, since $f_\infty = \infty$, we may choose $M, R_1 > 0$ such that $\frac{\alpha-1}{16}\lambda\int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2},s)h(s)Mds \geq 1$ and $f(u) \geq Mu$ for all $u > R_1$. Take $R_* > \frac{\alpha-1}{16}R_1 + r_1$ and define $\Omega_{R_*} = \{u \in C[0,1] | ||u||_\infty < R_*\}$. Then for $u \in P \cap \partial\Omega_{R_*}$, we obtain

$$u(t) \geq \frac{\alpha-1}{16}||u||_\infty > R_1, \quad t \in [\frac{1}{4}, \frac{3}{4}]$$

and thus

$$\begin{aligned} Tu(\frac{1}{2}) &\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2},s)h(s)f(u(s))ds \\ &\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2},s)h(s)Mu(s)ds \\ &\geq \frac{\alpha-1}{16}||u||_\infty \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2},s)h(s)Mds \\ &\geq ||u||_\infty. \end{aligned}$$

This implies that $||Tu||_\infty \geq ||u||_\infty$, for $u \in P \cap \partial\Omega_{R_*}$ and therefore T has a fixed point u in $u \in P \cap (\overline{\Omega}_{R_*} \setminus \Omega_{r_1})$. □

Based on this lemma, we can prove a theorem on the existence and nonexistence of solutions as follows;

Theorem 3.2. *Assume* $0 < f_0 < \infty$ *and* $f_\infty = \infty$. *Also assume* (H). *Then there exist* λ^* *and* λ^{**} *such that problem* (E_λ) *has at least one positive solution for* $0 < \lambda < \lambda^*$ *and no positive solution for* $\lambda > \lambda^{**}$.

Proof. From the above assumptions, we know that there exists $K > 0$ such that $f(u) \geq Ku$, for all $u > 0$. Let u be a solution of (E_λ), then $u \in P$, since

$T : E \to P$ and by the above facts, we obtain

$$\begin{aligned} ||u||_\infty &\geq u(\frac{1}{2}) = \lambda \int_0^1 G(\frac{1}{2}, s)h(s)f(u(s))ds \\ &\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)h(s)Ku(s)ds \\ &\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)h(s)\frac{K(\alpha-1)}{16}||u||_\infty ds \end{aligned}$$

which implies

$$\lambda \leq (\int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)h(s)\frac{K(\alpha-1)}{16}ds)^{-1}.$$

Therefore, it follows that the set $\{\lambda > 0$: there exists nonzero u_λ such that $Tu_\lambda = u_\lambda\}$ is bounded above. Together with Lemma 3.1, we completes the proof. □

Next, we consider the case, $f_0 = \infty$ and $0 < f_\infty < \infty$. By using similar arguments, we obtain the following lemma and theorem.

Lemma 3.3. *Assume $f_0 = \infty$ and $0 < f_\infty < \infty$. Also assume (H). Then problem (E_λ) has at least one positive solution for*

$$\lambda \in (0, (f_\infty \int_0^1 G(s, s)h(s)ds)^{-1}).$$

Theorem 3.4. *Assume $f_0 = \infty$ and $0 < f_\infty < \infty$. Also assume (H). Then there exist λ^* and λ^{**} such that problem (E_λ) has at least one positive solution for $0 < \lambda < \lambda^*$ and no positive solution for $\lambda > \lambda^{**}$.*

Now, we consider the case $0 < f_0 < \infty$ and $f_\infty = 0$. In this case, we obtain the following lemma

Lemma 3.5. *Assume $0 < f_0 < \infty$ and $f_\infty = 0$. Also assume (H). Then problem (E_λ) has at least one positive solution for*

$$\lambda \in ((\frac{f_0(\alpha-1)}{16} \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)h(s)ds)^{-1}, \infty).$$

Proof. Fix λ and then we can take η where

$$\lambda = \Big(\frac{(f_0-\eta)(\alpha-1)}{16} \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)h(s)ds\Big)^{-1}.$$

From the definition of f_0, we may choose $r_2 > 0$ such that $f(u) > u(f_0 - \eta)$ for $0 < u \leq r_2$. Take $\Omega_{r_2} = \{u \in C[0,1] | ||u||_\infty < r_2\}$.

$$\begin{aligned} ||Tu||_\infty &\geq Tu(\frac{1}{2}) = \lambda \int_0^1 G(\frac{1}{2}, s)h(s)f(u(s))ds \\ &\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)h(s)(f_0 - \zeta)u(s)ds \\ &\geq \lambda ||u||_\infty \frac{(f_0 - \eta)(\alpha - 1)}{16} \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)h(s)ds \\ &= ||u||_\infty. \end{aligned}$$

Since $f_\infty = 0$, we pick $N, R_2 > 0$ such that $\lambda \int_0^1 G(s,s)h(s)Nds < 1$ and $f(u) \leq Nu$ for all $u > R_2$. Take $R_{**} > \max\{R_2, \frac{\{\max_{0 \leq u \leq R_2} |f(u)|\} \lambda \int_0^1 G(s,s)h(s)ds}{1 - N\lambda \int_0^1 (s(1-s))^{\alpha-1} h(s)ds}\}$. Then for $u \in P \cap \partial\Omega_{R_{**}}$,

$$\begin{aligned} Tu(t) &\leq \int_0^1 G(s,s)h(s)f(u(s))ds \\ &\leq \Big[\int_{0 \leq u \leq R_2} G(s,s)h(s)f(u(s))ds \\ &\quad + \int_{R_2 < u \leq R_{**}} G(s,s)h(s)f(u(s))ds\Big] \\ &\leq \Big[\max_{0 \leq u \leq R_2} |f(u)| \int_{0 \leq u \leq R_2} G(s,s)h(s)ds \\ &\quad + \int_{R_2 < u \leq R_{**}} G(s,s)h(s)Nu(s)ds\Big] \\ &\leq \Big(\max_{0 \leq u \leq R_2} |f(u)| + N||u||_\infty\Big) \int_0^1 G(s,s)h(s)ds \\ &\leq R_2 = ||u||_\infty. \end{aligned}$$

Therefore, T has a fixed point u in $u \in P \cap (\overline{\Omega}_{R_{**}} \setminus \Omega_{r_2})$. □

By using similar caculation in the proof of Lemma 3.5 and Theorem 3.2, we get the following existence and nonexistence result.

Theorem 3.6. *Assume* $0 < f_0 < \infty$ *and* $f_\infty = 0$. *Also assume* (H). *Then there exist* λ^* *and* λ^{**} *such that problem* (E_λ) *has no positive solution for* $0 < \lambda < \lambda^*$ *and at least one positive solution for* $\lambda > \lambda^{**}$.

Moreover, we add several results of similar pattern for the cases, $f_0 = 0$ and $0 < f_\infty < \infty$ or $0 < f_0 < \infty$ and $0 < f_\infty < \infty$.

Lemma 3.7. *Assume* $f_0 = 0$ *and* $0 < f_\infty < \infty$. *Also assume* (H). *Then problem* (E_λ) *has at least one positive solution for*

$$\lambda \in ((\frac{f_\infty(\alpha-1)}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)h(s)ds)^{-1}, \infty).$$

Theorem 3.8. *Assume* $f_0 = 0$ *and* $0 < f_\infty < \infty$. *Also assume* (H). *Then there exist* λ^* *and* λ^{**} *such that problem* (E_λ) *has no positive solution for* $0 < \lambda < \lambda^*$ *and at least one positive solution for* $\lambda > \lambda^{**}$.

Lemma 3.9. *Assume* $0 < f_0 < \infty$ *and* $0 < f_\infty < \infty$. *Also assume* (H). *Then problem* (E_λ) *has at least one positive solution for each* λ *satisfying either*

(1) $\big(\frac{f_\infty(\alpha-1)}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)h(s)ds\big)^{-1} \leq \lambda \leq \big(f_0\int_0^1 G(s,s)h(s)ds\big)^{-1}$ *or*

(2) $\big(\frac{f_0(\alpha-1)}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)h(s)ds\big)^{-1} \leq \lambda \leq \big(f_\infty\int_0^1 G(s,s)h(s)ds\big)^{-1}$.

Theorem 3.10. *Assume* $0 < f_0 < \infty$ *and* $0 < f_\infty < \infty$. *Also assume* (H). *Then there exist* λ^*, λ^{**},λ_* *and* λ_{**} *such that problem* (E_λ) *has no positive solution for* $\lambda^* < \lambda < \lambda^{**}$ *and at least one positive solution for* $\lambda_* < \lambda < \lambda_{**}$.

Example 3.11. Consider the boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda t^{-\beta}f(u) = 0, \quad 1 < \beta < \alpha < 2 \\ u(0) = 0 = u(1), \end{cases} \tag{3.2}$$

where

$$f(u) = \begin{cases} \tan u, \quad u \in (0, \frac{\pi}{4}] \\ \frac{16}{\pi^2}u^2, \quad u \in (\frac{\pi}{4}, \infty). \end{cases}$$

We can easily check that $h(t) = t^{-\beta} \notin L^1(0,1)$ satisfying (H) and f satisfies $0 < f_0 < \infty$ and $f_\infty = \infty$ and thus we conclude that there exist λ^* and λ^{**} such that problem (3.2) has at least one positive solution for $0 < \lambda < \lambda^*$ and no positive solution for $\lambda > \lambda^{**}$ from Theorem 3.2. We notice that the advantage of our results in this paper is to figure out λ^* and λ^{**} explicitly. For example, let us take α=1.5, β=1.2 in (3.2). Then by the fact that $f(u) \geq u$ for all $u > 0$, we may choose $K = 1$ and we can calculate $\lambda^* \approx 3.21197$ and $\lambda^{**} \approx 76.39489$.

References

[1] Tom Kisela, *Fractional Differential Equations and Their Applications*, Fakulta strojnho inenrstv. (2008), 50.

[2] Daqing Jiang and Chengjun Yuan, *The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application*, Nonlinear Anal. T.M.A. **72** (2010) 710-719.

[3] Xiaoling Han and Hongliang Gao, *Existence of positive solutions for eigenvalue problem of nonlinear fractional differential equations*, Adv. Difference Equ. 2012, 2012:66, 1-8.

[4] Zhanbing Bai and Haishen Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311** (2005), 495-505.

[5] Jinsil Lee and Yong-Hoon Lee, *An existence result for nonlinear singular fractional differential equations*, preprint.

Yong-Hoon Lee
Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea
E-mail address: yhlee@pusan.ac.kr

Jinsil Lee
Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea
E-mail address: js3716@pusan.ac.kr