Korean J. Math. **25** (2017), No. 2, pp. 181–199
https://doi.org/10.11568/kjm.2017.25.2.181

THE RECURRENCE COEFFICIENTS OF THE ORTHOGONAL POLYNOMIALS WITH THE WEIGHTS $w_\alpha(x) = x^\alpha \exp(-x^3 + tx)$ AND $W_\alpha(x) = |x|^{2\alpha+1} \exp(-x^6 + tx^2)$

Haewon Joung

Abstract. In this paper we consider the orthogonal polynomials with weights $w_\alpha(x) = x^\alpha \exp(-x^3+tx)$ and $W_\alpha(x) = |x|^{2\alpha+1} \exp(-x^6+tx^2)$. Using the compatibility conditions for the ladder operators for these orthogonal polynomials, we derive several difference equations satisfied by the recurrence coefficients of these orthogonal polynomials. We also derive differential-difference equations and second order linear ordinary differential equations satisfied by these orthogonal polynomials.

1. Introduction

The compatibility conditions for the ladder operators for orthogonal polynomials have been derived by many authors. We refer to [2], [3], [5], [6] and references therein. In this section we derive the compatibility conditions for the ladder operators for orthogonal polynomials, for the sake of completeness of paper.

Let $P_n(x)$ be the monic orthogonal polynomials of degree n in x with the weight $w(x)$. Assume that the weight function w vanishes at the end points of the orthogonality interval. Then,

$$\int P_n(x)P_m(x)w(x)dx = h_n\delta_{n,m}, \quad h_n > 0, \tag{1.1}$$

Received May 1, 2017. Revised May 15, 2017. Accepted May 24, 2017.
2010 Mathematics Subject Classification: 42C05, 39A10.
Key words and phrases: Orthogonal polynomials, Recurrence coefficients, Ladder operators.
© The Kangwon-Kyungki Mathematical Society, 2017.
This is an Open Access article distributed under the terms of the Creative commons Attribution Non-Commercial License (http://creativecommons.org/licenses/by-nc/3.0/) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

and P_n's satisfy the three term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \tag{1.2}$$

where

$$\alpha_n = \frac{1}{h_n}\int xP_n^2(x)w(x)dx, \tag{1.3}$$

$$\beta_n = \frac{1}{h_{n-1}}\int xP_n(x)P_{n-1}(x)w(x)dx = \frac{h_n}{h_{n-1}}, \tag{1.4}$$

and the initial condition is $\beta_0 P_{-1} = 0$. Since $\frac{dP_n(x)}{dx}$ is a polynomial of degree $n-1$, it can be written as

$$\frac{dP_n(x)}{dx} = \sum_{k=0}^{n-1} c_{n,k}P_k(x) \tag{1.5}$$

where

$$c_{n,k}h_k = \int \frac{dP_n(y)}{dy}P_k(y)w(y)dy.$$

Using integration by parts and orthogonality relation, we have

$$c_{n,k} = \frac{1}{h_k}\int P_n(y)P_k(y)v'(y)w(y)dy,$$

where

$$v(y) := -\ln w(y).$$

Noting that

$$\int P_n(y)\sum_{k=0}^{n-1}\frac{P_k(y)P_k(x)}{h_k}v'(x)w(y)dy = 0,$$

from (1.5), we have

$$\begin{aligned}\frac{dP_n(x)}{dx} &= \sum_{k=0}^{n-1}\frac{1}{h_k}\left\{\int P_n(y)P_k(y)v'(y)w(y)dy\right\}P_k(x)\\ &= \int P_n(y)\sum_{k=0}^{n-1}\frac{P_k(y)P_k(x)}{h_k}v'(y)w(y)dy\\ &= \int P_n(y)\sum_{k=0}^{n-1}\frac{P_k(y)P_k(x)}{h_k}[v'(y)-v'(x)]w(y)dy.\end{aligned}$$

By the Christoffel-Darboux formula

$$\sum_{k=0}^{n-1} \frac{P_k(y)P_k(x)}{h_k} = \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{(x-y)h_{n-1}},$$

it follows that

$$\frac{dP_n(x)}{dx} = -B_n(x)P_n(x) + \beta_n A_n(x)P_{n-1}(x), \tag{1.6}$$

where

$$A_n(x) := \frac{1}{h_n}\int P_n^2(y)\frac{v'(x)-v'(y)}{x-y}w(y)dy, \tag{1.7}$$

$$B_n(x) := \frac{1}{h_{n-1}}\int P_n(y)P_{n-1}(y)\frac{v'(x)-v'(y)}{x-y}w(y)dy. \tag{1.8}$$

Now we derive the compatibility conditions for the ladder operators to the orthogonal polynomials.

Lemma 1.1. *The functions $A_n(x)$ and $B_n(x)$ satisfy*

$$B_{n+1}(x) + B_n(x) = (x-\alpha_n)A_n(x) - v'(x), \tag{1.9}$$

and

$$B_{n+1}(x) - B_n(x) = \frac{\beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x) - 1}{x-\alpha_n}. \tag{1.10}$$

Proof. By (1.8), (1.2) and using that $h_n = h_{n-1}\beta_n$, we obtain

$$\begin{aligned} B_{n+1}(x) + B_n(x) &= \frac{1}{h_n}\int P_{n+1}(y)P_n(y)\frac{v'(x)-v'(y)}{x-y}w(y)dy \\ &\quad + \frac{1}{h_{n-1}}\int P_n(y)P_{n-1}(y)\frac{v'(x)-v'(y)}{x-y}w(y)dy \\ &= \frac{1}{h_n}\int P_n(y)[P_{n+1}(y)+\beta_n P_{n-1}(y)]\frac{v'(x)-v'(y)}{x-y}w(y)dy \\ &= \frac{1}{h_n}\int P_n(y)[yP_n(y)-\alpha_n P_n(y)]\frac{v'(x)-v'(y)}{x-y}w(y)dy \\ &= \frac{1}{h_n}\int P_n^2(y)\, y\,\frac{v'(x)-v'(y)}{x-y}w(y)dy - \alpha_n A_n(x). \end{aligned}$$

Now, using $\frac{y}{x-y} = \frac{x}{x-y} - 1$, we have

$$B_{n+1}(x) + B_n(x) = (x-\alpha_n)A_n(x) - v'(x) + \frac{1}{h_n}\int P_n^2(y)v'(y)w(y)dy.$$

Using integration by parts and orthogonality relation, we have

$$\begin{aligned}\int P_n^2(y)v'(y)w(y)dy &= -\int P_n^2(y)\frac{dw(y)}{dy}dy \\ &= -\left[P_n^2(y)w(y)\right]_0^\infty + 2\int P_n(y)P_n'(y)w(y)dy \\ &= 0,\end{aligned}$$

which proves (1.9). Similarly

$$\begin{aligned}&(x-\alpha_n)(B_{n+1}(x)-B_n(x)) \\ &= \frac{1}{h_n}\int (x-\alpha_n)P_n(y)[P_{n+1}(y)-\beta_n P_{n-1}(y)]\frac{v'(x)-v'(y)}{x-y}w(y)dy \\ &= \frac{1}{h_n}\int P_n(y)[P_{n+1}(y)-\beta_n P_{n-1}(y)](v'(x)-v'(y))w(y)dy \\ &\quad + \frac{1}{h_n}\int (y-\alpha_n)P_n(y)[P_{n+1}(y)-\beta_n P_{n-1}(y)]\frac{v'(x)-v'(y)}{x-y}w(y)dy \\ &= -1+\frac{1}{h_n}\int [P_{n+1}(y)+\beta_n P_{n-1}(y)][P_{n+1}(y)-\beta_n P_{n-1}(y)]\frac{v'(x)-v'(y)}{x-y}w(y)dy \\ &= -1+\beta_{n+1}A_{n+1}(x)-\beta_n A_{n-1}(x).\end{aligned}$$

□

Remark 1.2. *From* (1.6), (1.9), *and* (1.2), *we can derive the following.*

$$\frac{dP_{n-1}(x)}{dx} = [B_n(x)+v'(x)]P_{n-1}(x) - A_{n-1}(x)P_n(x). \tag{1.11}$$

Also, from (1.9) *and* (1.10), *we obtain*

$$B_n^2(x) + v'(x)B_n(x) + \sum_{k=0}^{n-1} A_k(x) = \beta_n A_n(x)A_{n-1}(x). \tag{1.12}$$

Equations (1.9), (1.10), and (1.12) are called the compatibility conditions for the ladder operators (see [5]).

Lemma 1.3. *The monic orthogonal polynomials $P_n(x)$ satisfy the differential equation*

$$\frac{d^2P_n(x)}{dx^2} + C_n(x)\frac{dP_n(x)}{dx} + D_n(x)P_n(x) = 0, \tag{1.13}$$

where

$$C_n(x) = -\frac{dv(x)}{dx} - \frac{1}{A_n(x)}\frac{dA_n(x)}{dx}, \tag{1.14}$$

and

$$D_n(x) = \beta_n A_n(x) A_{n-1}(x) - B_n^2(x) - B_n(x)\frac{dv(x)}{dx} + \frac{dB_n(x)}{dx} - \frac{B_n(x)}{A_n(x)}\frac{dA_n(x)}{dx}. \tag{1.15}$$

Proof. Differentiating both sides of (1.6) with respect to x, we have

$$\begin{aligned} \frac{d^2P_n(x)}{dx^2} &= -B_n(x)\frac{dP_n(x)}{dx} - \frac{dB_n(x)}{dx}P_n(x) \\ &\quad + \beta_n \frac{dA_n(x)}{dx}P_{n-1}(x) + \beta_n A_n(x)\frac{dP_{n-1}(x)}{dx}. \end{aligned} \tag{1.16}$$

Substituting (1.11) into (1.16) yields

$$\begin{aligned} \frac{d^2P_n(x)}{dx^2} &= -B_n(x)\frac{dP_n(x)}{dx} - \left(\beta_n A_n(x)A_{n-1}(x) + \frac{dB_n(x)}{dx}\right)P_n(x) \\ &\quad + \beta_n\left(A_n(x)B_n(x) + A_n(x)\frac{dv(x)}{dx} + \frac{dA_n(x)}{dx}\right)P_{n-1}(x), \end{aligned} \tag{1.17}$$

and the lemma follows by substituting $P_{n-1}(x)$ in (1.17) using (1.6). □

2. Orthogonal polynomials with the weight $w_\alpha(x) = x^\alpha \exp(-x^3 + tx)$

In this section we consider the weight

$$w_\alpha(x) = x^\alpha \exp(-x^3 + tx) \quad (\alpha > 0, \quad t \in \mathbb{R})$$

on the positive real axis $\mathbb{R}^+$. It satisfies the Pearson equation

$$[x w_\alpha(x)]' = (-3x^3 + tx + \alpha + 1) w_\alpha(x).$$

Now for the weight $w_\alpha(x) = x^\alpha \exp(-x^3 + tx)$ we have

$$v(x) = -\ln w_\alpha(x) = -\alpha \ln x + x^3 - tx, \tag{2.1}$$

hence,

$$\frac{v'(x) - v'(y)}{x - y} = 3(x + y) + \frac{\alpha}{xy}.$$

From (1.7) and (1.8), we obtain

$$A_n(x) = 3(x+\alpha_n) + \frac{M_n}{x}, \qquad B_n(x) = 3\beta_n + \frac{m_n}{x}, \tag{2.2}$$

where

$$M_n = \frac{\alpha}{h_n}\int_0^\infty \frac{P_n^2(y)}{y} w_\alpha(y)dy, \tag{2.3}$$

and

$$m_n = \frac{\alpha}{h_{n-1}}\int_0^\infty \frac{P_n(y)P_{n-1}(y)}{y} w_\alpha(y)dy. \tag{2.4}$$

Substituting (2.2) into (1.9) and comparing the coefficients of x^0 and x^{-1}, we obtain

$$M_n = 3(\beta_{n+1} + \alpha_n^2 + \beta_n) - t, \tag{2.5}$$

$$m_{n+1} + m_n = \alpha - \alpha_n M_n. \tag{2.6}$$

Similarly, from (1.10) and (1.12), we have six more conditions.

$$1 + m_{n+1} - m_n = 3\beta_{n+1}(\alpha_{n+1} + \alpha_n) - 3\beta_n(\alpha_n + \alpha_{n-1}), \tag{2.7}$$

$$\alpha_n(m_n - m_{n+1}) = \beta_{n+1}M_{n+1} - \beta_n M_{n-1}, \tag{2.8}$$

$$m_n = 3\beta_n(\alpha_n + \alpha_{n-1}) - n, \tag{2.9}$$

$$3\beta_n^2 - t\beta_n + \sum_{j=0}^{n-1}\alpha_j = \beta_n(M_{n-1} + 3\alpha_n\alpha_{n-1} + M_n), \tag{2.10}$$

$$\sum_{j=0}^{n-1} M_j - tm_n = 3\beta_n(\alpha_n M_{n-1} + \alpha_{n-1}M_n - 2m_n + \alpha), \tag{2.11}$$

$$m_n^2 - \alpha m_n = \beta_n M_n M_{n-1}. \tag{2.12}$$

Substituting (2.5) and (2.9) into (2.6), (2.8), (2.10), (2.11), and (2.12), we have the following nonlinear difference equations for the recurrence coefficients.

THEOREM 2.1. *Recurrence coefficients* α_n *and* β_n *in* (1.2) *with the weight* $w_\alpha(x) = x^\alpha \exp(-x^3 + tx)$ *satisfy*

$$\begin{aligned}(2.13)\quad & 2n + 1 + \alpha + \alpha_n t \\ = \; & 3\beta_{n+1}(\alpha_{n+1} + \alpha_n) + 3\beta_n(\alpha_n + \alpha_{n-1}) + 3\alpha_n(\beta_{n+1} + \alpha_n^2 + \beta_n),\end{aligned}$$

$$\begin{aligned}(2.14)\quad & \alpha_n + t(\beta_{n+1} - \beta_n) \\ = \; & 3\beta_{n+1}(\alpha_{n+1}^2 + \alpha_{n+1}\alpha_n + \alpha_n^2 + \beta_{n+2} + \beta_{n+1}) \\ & -3\beta_n(\alpha_n^2 + \alpha_n\alpha_{n-1} + \alpha_{n-1}^2 + \beta_n + \beta_{n-1}),\end{aligned}$$

$$\begin{aligned}(2.15)\quad & \sum_{j=0}^{n-1} \alpha_j \\ = \; & \beta_n(3\beta_{n+1} + 3\beta_n + 3\beta_{n-1} + 3\alpha_n^2 + 3\alpha_n\alpha_{n-1} + 3\alpha_{n-1}^2 - t),\end{aligned}$$

$$\begin{aligned}(2.16)\quad & \sum_{j=0}^{n-1}(\alpha_j^2 + \beta_{j+1} + \beta_j) \\ = \; & \beta_n[\alpha_n(3\alpha_{n-1}^2 - 3\beta_n + 3\beta_{n-1}) + \alpha_{n-1}(3\alpha_n^2 + 3\beta_{n+1} - 3\beta_n) + 2n + \alpha],\end{aligned}$$

$$\begin{aligned}(2.17)\quad & [3\beta_n(\alpha_n + \alpha_{n-1}) - n][3\beta_n(\alpha_n + \alpha_{n-1}) - n - \alpha] \\ = \; & \beta_n(3\alpha_n^2 + 3\beta_{n+1} + 3\beta_n - t)(3\alpha_{n-1}^2 + 3\beta_n + 3\beta_{n-1} - t).\end{aligned}$$

REMARK 2.2. *The quantities* M_n *in* (2.3), (2.5) *and* m_n *in* (2.4), (2.9) *can be computed directly as follows. Using the orthogonality relation and integration by parts, we have*

$$0 = \int_0^\infty \frac{dP_n(y)}{dy} P_n(y) w_\alpha(y) dy = -\int_0^\infty P_n^2(y) \left(\frac{\alpha}{y} - 3y^2 + t\right) w_\alpha(y) dy,$$

hence,

$$\begin{aligned} M_n &= \frac{\alpha}{h_n} \int_0^\infty \frac{P_n^2(y)}{y} w_\alpha(y) dy \\ &= \frac{1}{h_n} \int_0^\infty (3y^2 - t) P_n^2(y) w_\alpha(y) dy \\ &= 3(\beta_{n+1} + \alpha_n^2 + \beta_n) - t.\end{aligned}$$

Similarly, we have

$$\begin{aligned} nh_{n-1} &= \int_0^\infty \frac{dP_n(y)}{dy} P_{n-1}(y) w_\alpha(y) dy \\ &= -\int_0^\infty P_n(y) P_{n-1}\left(\frac{\alpha}{y} - 3y^2 + t\right) w_\alpha(y) dy, \end{aligned}$$

therefore,

$$\begin{aligned} m_n &= \frac{\alpha}{h_{n-1}} \int_0^\infty \frac{P_n(y)P_{n-1}(y)}{y} w_\alpha(y) dy - n \\ &= \frac{1}{h_{n-1}} \int_0^\infty (3y^2 - t) P_n(y) P_{n-1}(y) w_\alpha(y) dy - n \\ &= 3\beta_n(\alpha_n + \alpha_{n-1}) - n. \end{aligned}$$

Note that the coefficients α_n and β_n in the recurrence relation (1.2) with the weight w_α are now functions of t. It is well known that the coefficients $\alpha_n(t)$ and $\beta_n(t)$ with the weight $w_\alpha(x) = \exp(tx)x^\alpha \exp(-x^3)$ satisfy the Toda system (see [1])

$$\frac{d\alpha_n}{dt} = \beta_{n+1} - \beta_n, \qquad \frac{d\beta_n}{dt} = \beta_n(\alpha_n - \alpha_{n-1}). \tag{2.18}$$

And the kth moment is

$$\mu_k = \int_0^\infty x^k w_\alpha(x) dx = \frac{d^k}{dt^k}\left(\int_0^\infty w_\alpha(x) dx\right) = \frac{d^k \mu_0}{dt^k}. \tag{2.19}$$

THEOREM 2.3. *The quantities $M_n = M_n(t)$ in (2.5) and $m_n = m_n(t)$ in (2.9) satisfy the following.*

$$\frac{dM_n}{dt} = m_{n+1} - m_n, \qquad \frac{dm_n}{dt} = \beta_n(M_n - M_{n-1}), \tag{2.20}$$

and

$$\frac{d^2 M_n}{dt^2} = \frac{1}{2M_n}\left(\frac{dM_n}{dt}\right)^2 - \frac{M_n^2}{3} + \left(\frac{3\alpha_n^2}{2} - \frac{t}{3}\right) M_n - \frac{\alpha^2}{2M_n}. \tag{2.21}$$

Proof. From (2.5), (2.18), and (2.9), we have

$$\begin{aligned}\frac{dM_n}{dt} &= \frac{d}{dt}[3(\beta_{n+1}+\alpha_n^2+\beta_n)-t]\\ &= 3\left(\frac{d\beta_{n+1}}{dt}+2\alpha_n\frac{d\alpha_n}{dt}+\frac{d\beta_n}{dt}\right)-1\\ &= 3[\beta_{n+1}(\alpha_{n+1}-\alpha_n)+2\alpha_n(\beta_{n+1}-\beta_n)+\beta_n(\alpha_n-\alpha_{n-1})]-1\\ &= 3\beta_{n+1}(\alpha_{n+1}+\alpha_n)-3\beta_n(\alpha_n+\alpha_{n-1})-1\\ &= m_{n+1}-m_n,\end{aligned}$$

similarly

$$\begin{aligned}\frac{dm_n}{dt} &= \frac{d}{dt}[3\beta_n(\alpha_n+\alpha_{n-1})-n]\\ &= 3\frac{d\beta_n}{dt}(\alpha_n+\alpha_{n-1})+3\beta_n\left(\frac{d\alpha_n}{dt}+\frac{d\alpha_{n-1}}{dt}\right)\\ &= 3\beta_n(\alpha_n-\alpha_{n-1})(\alpha_n+\alpha_{n-1})+3\beta_n(\beta_{n+1}-\beta_n+\beta_n-\beta_{n-1})\\ &= \beta_n(M_n-M_{n-1}),\end{aligned}$$

which proves (2.20). Now from (2.6) and (2.12), we have

$$\begin{aligned}m_{n+1} &= \alpha-\alpha_nM_n-m_n,\\ M_{n-1} &= \frac{m_n^2-\alpha m_n}{\beta_nM_n}.\end{aligned}$$

Substituting these into (2.20) yields

$$\frac{dM_n}{dt} = \alpha-\alpha_nM_n-2m_n, \tag{2.22}$$

$$\frac{dm_n}{dt} = \beta_nM_n-\frac{m_n^2-\alpha m_n}{M_n}. \tag{2.23}$$

Solving (2.22) for m_n yields

$$m_n=\frac{1}{2}\left(\alpha-\alpha_nM_n-\frac{dM_n}{dt}\right).$$

Substituting this into (2.23) and using (2.18), we have

$$\frac{d^2M_n}{dt^2}=\frac{1}{2M_n}\left(\frac{dM_n}{dt}\right)^2-(\beta_{n+1}+\beta_n)M_n+\frac{\alpha_n^2}{2}M_n-\frac{\alpha^2}{2M_n}. \tag{2.24}$$

From (2.5), we have

$$\beta_{n+1} + \beta_n = \frac{M_n + t}{3} - \alpha_n^2.$$

Substituting this into (2.24) yields (2.21). □

Let κ_n be the coefficient of x^{n-1} in the monic orthogonal polynomials $P_n(x)$, that is, $P_n(x) = x^n + \kappa_n x^{n-1} + \cdots$. Comparing the coefficients of x^n in (1.2), we have

$$\kappa_{n+1} - \kappa_n = -\alpha_n. \tag{2.25}$$

Taking a telescope sum, we have

$$\kappa_n = -\sum_{j=0}^{n-1} \alpha_j. \tag{2.26}$$

THEOREM 2.4. *Let κ_n be the coefficient of x^{n-1} in the monic orthogonal polynomials $P_n(x)$ with the weight w_α. Then*

$$\kappa_n = -\beta_n(3\beta_{n+1} + 3\beta_n + 3\beta_{n-1} + 3\alpha_n^2 + 3\alpha_n\alpha_{n-1} + 3\alpha_{n-1}^2 - t), \tag{2.27}$$

and

$$\frac{d\kappa_n}{dt} = -\beta_n. \tag{2.28}$$

Proof. From (2.26), (2.15), and (2.18), we have

$$\begin{aligned}
\kappa_n &= -\sum_{j=0}^{n-1} \alpha_j \\
&= -\beta_n(3\beta_{n+1} + 3\beta_n + 3\beta_{n-1} + 3\alpha_n^2 + 3\alpha_n\alpha_{n-1} + 3\alpha_{n-1}^2 - t), \\
\frac{d\kappa_n}{dt} &= -\sum_{j=0}^{n-1} \frac{d\alpha_j}{dt} = -\sum_{j=0}^{n-1}(\beta_{j+1} - \beta_j) = -\beta_n.
\end{aligned}$$

□

The sum of all the zeros of the monic orthogonal polynomials $P_n(x)$ is $-\kappa_n$, therefore, by Theorem 2.4, we have the following.

COROLLARY 2.5. *Let $P_n(x)$ be the monic orthogonal polynomials with the weight w_α. Then the sum of all the zeros of $P_n(x)$ is*

$$\sum_{j=0}^{n-1} \alpha_j = \beta_n(3\beta_{n+1} + 3\beta_n + 3\beta_{n-1} + 3\alpha_n^2 + 3\alpha_n\alpha_{n-1} + 3\alpha_{n-1}^2 - t). \tag{2.29}$$

Substituting (2.5) and (2.9) into (2.2), we have

$$A_n(x) = 3(x+\alpha_n) + \frac{3(\beta_{n+1}+\alpha_n^2+\beta_n)-t}{x}, \tag{2.30}$$

$$B_n(x) = 3\beta_n + \frac{3\beta_n(\alpha_n+\alpha_{n-1})-n}{x}, \tag{2.31}$$

hence, from (1.6) we have the following differential-difference equations.

Theorem 2.6. *The monic orthogonal polynomials $P_n(x)$ with the weight w_α satisfy the differential-difference equation*

$$\begin{aligned} x\frac{dP_n(x)}{dx} &= -\left[3\beta_n x + 3\beta_n(\alpha_n+\alpha_{n-1})-n\right]P_n(x) \\ &\quad + \beta_n\left[3x(x+\alpha_n)+3(\beta_{n+1}+\alpha_n^2+\beta_n)-t\right]P_{n-1}(x). \end{aligned} \tag{2.32}$$

Theorem 2.7. *The monic orthogonal polynomials $P_n(x)$ with the weight w_α satisfy the differential equation*

$$\frac{d^2P_n(x)}{dx^2} + C_n(x)\frac{dP_n(x)}{dx} + D_n(x)P_n(x) = 0, \tag{2.33}$$

where

$$\begin{aligned} C_n(x) &= -3x^2+t+\frac{\alpha}{x} - \frac{3x^2-[3(\beta_{n+1}+\alpha_n^2+\beta_n)-t]}{3x^2(x+\alpha_n)+[3(\beta_{n+1}+\alpha_n^2+\beta_n)-t]x}, \\ D_n(x) &= \beta_n\left(3(x+\alpha_n)+\frac{3(\beta_{n+1}+\alpha_n^2+\beta_n)-t}{x}\right) \\ &\quad \times\left(3(x+\alpha_{n-1})+\frac{3(\beta_n+\alpha_{n-1}^2+\beta_{n-1})-t}{x}\right) \\ &\quad -\left(3x^2-t+3\beta_n+\frac{3\beta_n(\alpha_n+\alpha_{n-1})-n-\alpha}{x}\right) \\ &\quad \times\left(3\beta_n+\frac{3\beta_n(\alpha_n+\alpha_{n-1})-n}{x}\right) \\ &\quad -\left(\frac{3x^2-[3(\beta_{n+1}+\alpha_n^2+\beta_n)-t]}{3x^2(x+\alpha_n)+[3(\beta_{n+1}+\alpha_n^2+\beta_n)-t]x}\right) \\ &\quad \times\left(3\beta_n+\frac{3\beta_n(\alpha_n+\alpha_{n-1})-n}{x}\right) \\ &\quad -\frac{3\beta_n(\alpha_n+\alpha_{n-1})-n}{x^2}. \end{aligned}$$

Proof. From (2.1), (2.30) and (2.31), we have

$$\begin{aligned} v(x) &= -\ln w_\alpha(x) = -\alpha \ln x + x^3 - tx, \\ A_n(x) &= 3(x+\alpha_n) + \frac{3(\beta_{n+1} + \alpha_n^2 + \beta_n) - t}{x}, \\ B_n(x) &= 3\beta_n + \frac{3\beta_n(\alpha_n + \alpha_{n-1}) - n}{x}, \end{aligned}$$

hence, Lemma 1.3 with (1.14) and (1.15) yields the result. □

3. Orthogonal polynomials with the weight $W_\alpha(x) = |x|^{2\alpha+1}\exp(-x^6 + tx^2)$

In this section we consider the weight

$$W_\alpha(x) = |x|^{2\alpha+1}\exp(-x^6 + tx^2) \quad (\alpha > -1, \quad t \in \mathbb{R})$$

in the real line $\mathbb{R}$. First we show that symmetrizing the weight $w_\alpha(x) = x^\alpha \exp(-x^3+tx)$ gives rise to the weight $W_\alpha(x)$. Let $P_n(x)$ be the monic orthogonal polynomials with the weight w_α. It is proved in ([4], Theorem 7.1) that the kernel function $Q_n(x)$ are monic orthogonal polynomials of degree n with respect to the weight $w_{\alpha+1}(x) = x^{\alpha+1}\exp(-x^3 + tx)$. Define

$$R_{2n}(x) = P_n(x^2), \qquad R_{2n+1}(x) = xQ_n(x^2). \tag{3.1}$$

Then

$$\begin{aligned} l_n\delta_{n,m} &= \int_0^\infty P_n(x)P_m(x)x^\alpha \exp(-x^3+tx)dx \\ &= 2\int_0^\infty P_n(x^2)P_m(x^2)x^{2\alpha+1}\exp(-x^6+tx^2)dx \\ &= \int_{-\infty}^\infty P_n(x^2)P_m(x^2)|x|^{2\alpha+1}\exp(-x^6+tx^2)dx \\ &= \int_{-\infty}^\infty R_{2n}(x)R_{2m}(x)|x|^{2\alpha+1}\exp(-x^6+tx^2)dx, \end{aligned}$$

which show that $\{R_{2n}(x)\}_{n=0}^{\infty}$ is a orthogonal sequence with respect to the even weight $W_\alpha(x) = |x|^{2\alpha+1}\exp(-x^6 + tx^2)$ on $\mathbb{R}$. Similarly

$$\begin{aligned} k_n\delta_{n,m} &= \int_0^\infty Q_n(x)Q_m(x)x^{\alpha+1}\exp(-x^3+tx)dx \\ &= 2\int_0^\infty Q_n(x^2)Q_m(x^2)x^{2\alpha+3}\exp(-x^6+tx^2)dx \\ &= \int_{-\infty}^\infty [xQ_n(x^2)][xQ_m(x^2)]|x|^{2\alpha+1}\exp(-x^6+tx^2)dx \\ &= \int_{-\infty}^\infty R_{2n+1}(x)R_{2m+1}(x)|x|^{2\alpha+1}\exp(-x^6+tx^2)dx. \end{aligned}$$

And

$$\int_{-\infty}^\infty R_{2n}(x)R_{2m+1}(x)|x|^{2\alpha+1}\exp(-x^6+tx^2)dx = 0,$$

because the integrand is odd. Therefore $\{R_n(x)\}_{n=0}^{\infty}$ is a sequence of monic orthogonal polynomials with respect to the even weight $W_\alpha(x) = |x|^{2\alpha+1}\exp(-x^6 + tx^2)$ on $\mathbb{R}$. That is,

$$\int_{-\infty}^\infty R_n(x)R_m(x)W_\alpha(x)dx = h_n\delta_{n,m}, \quad h_n > 0. \tag{3.2}$$

Since the weight W_α is even, the three term recurrence relation has the form

$$xR_n(x) = R_{n+1}(x) + \beta_n R_{n-1}(x), \tag{3.3}$$

where

$$\beta_n = \frac{1}{h_{n-1}}\int xR_n(x)R_{n-1}(x)W_\alpha(x)dx, \tag{3.4}$$

and the initial condition is $R_{-1}(x) = 0$. By the three term recurrence relation (3.3), we have

(3.5)
$$y^2R_n(y) = R_{n+2}(y) + (\beta_{n+1}+\beta_n)R_n(y) + \beta_n\beta_{n-1}R_{n-2}(y),$$

(3.6)
$$\begin{aligned} y^3R_n(y) =& R_{n+3}(y) + (\beta_{n+2}+\beta_{n+1}+\beta_n)R_{n+1}(y) \\ &+ \beta_n(\beta_{n+1}+\beta_n+\beta_{n-1})R_{n-1}(y) + \beta_n\beta_{n-1}\beta_{n-2}R_{n-3}(y), \end{aligned}$$

and

(3.7)
$$\begin{aligned} y^4R_n(y) =& R_{n+4}(y) + (\beta_{n+3} + \beta_{n+2} + \beta_{n+1} + \beta_n)R_{n+2}(y) \\ &+ [\beta_{n+1}(\beta_{n+2} + \beta_{n+1} + \beta_n) + \beta_n(\beta_{n+1} + \beta_n + \beta_{n-1})]R_n(y) \\ &+ \beta_n\beta_{n-1}(\beta_{n+1} + \beta_n + \beta_{n-1} + \beta_{n-2})R_{n-2}(y) \\ &+ \beta_n\beta_{n-1}\beta_{n-2}\beta_{n-3}R_{n-4}(y). \end{aligned}$$

Now for the weight $W_\alpha(x) = |x|^{2\alpha+1}\exp(-x^6 + tx^2)$, we have

(3.8)
$$v(x) = -\ln W_\alpha(x) = -(2\alpha + 1)\ln|x| + x^6 - tx^2,$$

hence,

$$\frac{v'(x) - v'(y)}{x - y} = 6(x^4 + x^3y + x^2y^2 + xy^3 + y^4) - 2t + \frac{2\alpha + 1}{xy}.$$

From (1.7) we obtain

$$\begin{aligned} A_n(x) &= \frac{1}{h_n}\int_{-\infty}^{\infty} R_n^2(y)\frac{v'(x) - v'(y)}{x - y}W_\alpha(y)dy \\ &= \frac{6x^4 - 2t}{h_n}\int_{-\infty}^{\infty} R_n^2(y)W_\alpha(y)dy + \frac{6x^3}{h_n}\int_{-\infty}^{\infty} yR_n^2(y)W_\alpha(y)dy \\ &\quad + \frac{6x^2}{h_n}\int_{-\infty}^{\infty} y^2R_n^2(y)W_\alpha(y)dy + \frac{6x}{h_n}\int_{-\infty}^{\infty} y^3R_n^2(y)W_\alpha(y)dy \\ &\quad + \frac{6}{h_n}\int_{-\infty}^{\infty} y^4R_n^2(y)W_\alpha(y) + \frac{2\alpha + 1}{xh_n}\int_{-\infty}^{\infty}\frac{R_n^2(y)}{y}W_\alpha(y)dy. \end{aligned}$$

Noting that

$$\int_{-\infty}^{\infty}\frac{R_n^2(y)}{y}W_\alpha(y)dy = 0,$$

because the integrand is odd, and using (3.3), (3.5), (3.6), and (3.7), we have

(3.9)
$$A_n(x) = 6x^4 + 6x^2(\beta_{n+1} + \beta_n) + 6(s_{n+1} + s_n) - 2t,$$

where

(3.10)
$$s_n := \beta_n(\beta_{n+1} + \beta_n + \beta_{n-1}).$$

Similarly, noting that

$$\frac{1}{h_{n-1}}\int_{-\infty}^{\infty}\frac{R_n(y)R_{n-1}(y)}{y}W_\alpha(y)dy = \frac{[1 - (-1)^n]}{2},$$

from (1.8), we obtain

$$(3.11)\quad B_n(x) = \frac{1}{h_{n-1}}\int_{-\infty}^{\infty} R_n(y)R_{n-1}(y)\frac{v'(x)-v'(y)}{x-y}W_\alpha(y)dy = 6\beta_n x^3 + 6s_n x + \frac{m_n}{x},$$

where

$$(3.12)\qquad m_n := (2\alpha+1)\frac{[1-(-1)^n]}{2}.$$

THEOREM 3.1. *Recurrence coefficients β_n in (3.3) with the weight W_α satisfy the following difference equations.*

$$(3.13)\quad 1+(2\alpha+1)(-1)^n = \beta_{n+1}(6s_{n+2}+6s_{n+1}-2t)-\beta_n(6s_n+6s_{n-1}-2t),$$

$$(3.14)\qquad n+\left(\alpha+\frac{1}{2}\right)[1-(-1)^n]+2\beta_n t = 6\beta_n(s_{n+1}+s_n+s_{n-1})+6\beta_{n+1}\beta_n\beta_{n-1},$$

(3.15)

$$\beta_n(2\alpha+1)(-1)^{n+1}+s_n(6s_n-2t)+\sum_{j=0}^{n-1}(\beta_{j+1}+\beta_j) = \beta_n[(\beta_{n+1}+\beta_n)(6s_n+6s_{n-1}-2t)+(\beta_n+\beta_{n-1})(6s_{n+1}+6s_n-2t)],$$

and

(3.16)

$$\beta_n(6s_n+6s_{n-1}-2t)(6s_{n+1}+6s_n-2t) = 6s_n(2\alpha+1)(-1)^{n+1}-t\{2n+(2\alpha+1)[1-(-1)^n]\}+6\sum_{j=0}^{n-1}(s_{j+1}+s_j),$$

where s_n is defined in (3.10).

Proof. Substituting (3.9) and (3.11) into (1.10) with $\alpha_n = 0$, and comparing the constant terms, we obtain (3.13). Similarly, substituting (3.9) and (3.11) into (1.12) with (3.8), and comparing the coefficients of x^4, x^2, and x^0, we have (3.14), (3.15), and (3.16). □

Theorem 3.2. *Let $\beta_n = \beta_n(t)$ be recurrence coefficients in* (3.3) *with the weight W_α. Then*

$$\frac{d\beta_n}{dt} = \beta_n(\beta_{n+1} - \beta_{n-1}), \tag{3.17}$$

$$\frac{d^2\beta_n}{dt^2} = \frac{1}{6}\left(n + \left(\alpha + \frac{1}{2}\right)[1 - (-1)^n]\right) - \beta_n\left(3\beta_{n+1}\beta_n + 3\beta_{n+1}\beta_{n-1} + 3\beta_n\beta_{n-1} + \beta_n^2 - \frac{t}{3}\right). \tag{3.18}$$

Proof. Differentiating (3.2) for $m = n$ with respect to t yields

$$\frac{dh_n}{dt} = 2\int_{-\infty}^{\infty} R_n(x)\frac{dR_n(x)}{dt}W_\alpha(x)dx + \int_{-\infty}^{\infty} x^2 R_n^2(x)W_\alpha(x)dx.$$

Since $\frac{dR_n(x)}{dt}$ is a polynomial in x of degree $n-1$,

$$2\int_{-\infty}^{\infty} R_n(x)\frac{dR_n(x)}{dt}W_\alpha(x)dx = 0,$$

hence, by (3.5),

$$\frac{dh_n}{dt} = \int_{-\infty}^{\infty} x^2 R_n^2(x)W_\alpha(x)dx = (\beta_{n+1} + \beta_n)h_n.$$

Thus, from (1.4), we have

$$\frac{d\beta_n}{dt} = \frac{d}{dt}\left(\frac{h_n}{h_{n-1}}\right) = \beta_n(\beta_{n+1} - \beta_{n-1}).$$

And

$$\begin{aligned} \frac{d^2\beta_n}{dt^2} &= \frac{d\beta_n}{dt}(\beta_{n+1} - \beta_{n-1}) + \beta_n\left(\frac{d\beta_{n+1}}{dt} - \frac{d\beta_{n-1}}{dt}\right) \\ &= \beta_n(\beta_{n+2}\beta_{n+1} + \beta_{n+1}^2 + \beta_{n-1}^2 + \beta_{n-1}\beta_{n-2}) \\ &\quad - \beta_n(\beta_{n+1}\beta_n + 2\beta_{n+1}\beta_{n-1} + \beta_n\beta_{n-1}). \end{aligned} \tag{3.19}$$

From (3.14), we have

$$\begin{aligned} &\beta_n(\beta_{n+2}\beta_{n+1} + \beta_{n+1}^2 + \beta_{n-1}^2 + \beta_{n-1}\beta_{n-2}) \\ = &\ \frac{1}{6}\left(n + \left(\alpha + \frac{1}{2}\right)[1 - (-1)^n]\right) \\ &- \beta_n\left(2\beta_{n+1}\beta_n + \beta_{n+1}\beta_{n-1} + \beta_n^2 + 2\beta_n\beta_{n-1} - \frac{t}{3}\right). \end{aligned}$$

Substituting this into (3.19) yields (3.18). □

Let λ_n be the coefficient of x^{n-2} in the monic orthogonal polynomials $R_n(x)$ with the weight W_α. Comparing the coefficients of x^{n-1} in (3.3), we obtain

$$\lambda_{n+1} - \lambda_n = -\beta_n \qquad (n \geq 1, \quad \lambda_1 = 0). \tag{3.20}$$

Taking a telescope sum, we have

$$\lambda_n = -\sum_{j=1}^{n-1} \beta_j. \tag{3.21}$$

THEOREM 3.3. *Let λ_n be the coefficient of x^{n-2} in the monic orthogonal polynomials $R_n(x)$ with the weight W_α. Then*

$$\begin{aligned} \lambda_n &= \frac{\beta_n}{2}[1 - (2\alpha+1)(-1)^n] + (t - 3s_n)\beta_n^2 \\ &\quad - 3\beta_n[(\beta_{n+1} + \beta_n)s_{n-1} + (\beta_n + \beta_{n-1})s_{n+1}], \end{aligned} \tag{3.22}$$

and

$$\frac{d\lambda_n}{dt} = -\beta_n \beta_{n-1}. \tag{3.23}$$

Proof. From (3.15), we have

$$\begin{aligned} \sum_{j=0}^{n-1}(\beta_{j+1} + \beta_j) &= -\beta_n(2\alpha+1)(-1)^{n+1} - s_n(6s_n - 2t) \\ &\quad +\beta_n(\beta_{n+1} + \beta_n)(6s_n + 6s_{n-1} - 2t) \\ &\quad +\beta_n(\beta_n + \beta_{n-1})(6s_{n+1} + 6s_n - 2t) \\ &= \beta_n(2\alpha+1)(-1)^n + (6s_n - 2t)\beta_n^2 \\ &\quad +6\beta_n[(\beta_{n+1} + \beta_n)s_{n-1} + (\beta_n + \beta_{n-1})s_{n+1}]. \end{aligned}$$

Since

$$\sum_{j=0}^{n-1}(\beta_{j+1} + \beta_j) = -\lambda_{n+1} - \lambda_n,$$

we have

$$\begin{aligned} -\lambda_{n+1} - \lambda_n &= \beta_n(2\alpha+1)(-1)^n + (6s_n - 2t)\beta_n^2 \\ &\quad +6\beta_n[(\beta_{n+1} + \beta_n)s_{n-1} + (\beta_n + \beta_{n-1})s_{n+1}]. \end{aligned} \tag{3.24}$$

Solving for λ_n in (3.20) and (3.24) yields (3.22). Differentiating (3.21) with respect to t and using (3.17), we obtain (3.23). □

Substituting (3.9) and (3.11) into (1.6) we have the following differential-difference equations.

THEOREM 3.4. *The monic orthogonal polynomials $R_n(x)$ with the weight W_α satisfy the differential-difference equation*

$$
\begin{aligned}
(3.25) \qquad & x\frac{dR_n(x)}{dx} \\
&= -\left(6\beta_n x^4 + 6s_n x^2 + (2\alpha+1)\frac{[1-(-1)^n]}{2}\right) R_n(x) \\
&\quad + \beta_n[6x^5 + 6(\beta_{n+1}+\beta_n)x^3 + 6(s_{n+1}+s_n)x - 2tx]R_{n-1}(x).
\end{aligned}
$$

THEOREM 3.5. *The monic orthogonal polynomials $R_n(x)$ with the weight W_α satisfy the differential equation*

$$
\frac{d^2R_n(x)}{dx^2} + C_n(x)\frac{dR_n(x)}{dx} + D_n(x)R_n(x) = 0, \tag{3.26}
$$

where

$$
\begin{aligned}
C_n(x) &= -6x^5 + 2tx + \frac{2\alpha+1}{x} - \frac{12x^3 + 6x(\beta_{n+1}+\beta_n)}{3x^4 + 3x^2(\beta_{n+1}+\beta_n) + 3(s_{n+1}+s_n) - t}, \\
D_n(x) &= \beta_n\left[6x^4 + 6x^2(\beta_{n+1}+\beta_n) + 6(s_{n+1}+s_n) - 2t\right] \\
&\quad \times \left[6x^4 + 6x^2(\beta_n+\beta_{n-1}) + 6(s_n+s_{n-1}) - 2t\right] \\
&\quad - \left(6\beta_n x^3 + 6s_n x + \frac{1}{x}\left(\alpha+\frac{1}{2}\right)[1-(-1)^n]\right) \\
&\quad \times \left(6x^5 + 6\beta_n x^3 + (6s_n - 2t)x - \frac{1}{x}\left(\alpha+\frac{1}{2}\right)[1+(-1)^n]\right) \\
&\quad -18\beta_n x^2 + 6s_n - \frac{1}{x^2}\left(\alpha+\frac{1}{2}\right)[1-(-1)^n] \\
&\quad - \left(6\beta_n x^3 + 6s_n x + \frac{1}{x}\left(\alpha+\frac{1}{2}\right)[1-(-1)^n]\right) \\
&\quad \times \left(\frac{12x^3 + 6x(\beta_{n+1}+\beta_n)}{3x^4 + 3x^2(\beta_{n+1}+\beta_n) + 3(s_{n+1}+s_n) - t}\right).
\end{aligned}
$$

Proof. From (3.8), (3.9) and (3.11), we have

$$\begin{aligned} v(x) &= -\ln W_\alpha(x) = -(2\alpha+1)\ln|x| + x^6 - tx^2, , \\ A_n(x) &= 6x^4 + 6x^2(\beta_{n+1}+\beta_n) + 6(s_{n+1}+s_n) - 2t, \\ B_n(x) &= 6\beta_n x^3 + 6s_n x + \frac{1}{x}\left(\alpha+\frac{1}{2}\right)[1-(-1)^n], \end{aligned}$$

hence, substituting these into Lemma 1.3 yields the result. □

References

[1] Boelen L, Filipuk G, and Van Assche W, *Recurrence coefficients of generalized Meixner polynomials and Painlevé equations*, J. Phys. A: Math. Theor. **44** (2011) 035202.

[2] Bonan S and Clark D S, *Estimates of the orthogonal polynomials with weight* $\exp(-x^m)$*, m an even positive integer*, J. Approx. Theory **46** (1986), 408–410.

[3] Chen Y and Ismail M, *Jacobi polynomials from compatibility conditions*, Proc. Am. Math. Soc. **133** (2005), 465–472.

[4] Chihara T S, *An introduction to orthogonal polynomials*, Gordon and Breach, New york 1978.

[5] Filipuk G, Van Assche W, and Zhang L, *The recurrence coefficients of semi-classical Laguerre polynomials and the fourth Painlevé equation*, J. Phys. A: Math. Theor. **45** (2012) 205201.

[6] Shohat J, *A differential equation for orthogonal polynomials*, Duke Math. J. **5** (1939), 401–417.

Haewon Joung
Department of mathematics
Inha University
Incheon 22212, Korea
E-mail: hwjoung@inha.ac.kr