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A REMARK ON A TRIPLE POINTS IN THE BOUNDARY OF QUATERNIONIC HYPERBOLIC SPACE

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Abstract. In this paper we consider a triple of distinct points in the boundary of quaternionic hyperbolic space and detect where these points are by using the quaternionic triple product.

1. Introduction

When a triple of distinct points are given on the boundary of quaternionic hyperbolic space, by using the Cartan angular invariant, one can determine whether these three points lie in a same $\mathbb{R}$-circle or in the boundary of $\mathbb{H}$-line.(See [1]) More precisely, B.Apanasov and I.Kim proved the following theorem.

Theorem 1.1. (Theorem 3.5 and 3.6 in [1]) *A triple* $x = (x_1, x_2, x_3) \in (\partial \mathbf{H}^{\mathbf{n}}_{\mathbb{H}})^{\mathbf{3}}$ *lies in the boundary of an* $\mathbb{H}$*-line if and only if* $\mathbb{A}_{\mathbb{H}}(p) = \pi/2$, *and lies in the same* $\mathbb{R}$*-circle if and only if* $\mathbb{A}_{\mathbb{H}}(p) = 0$.

Here $\mathbb{A}_{\mathbb{H}}(p) = \pi/2$ if and only if $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ is purely imaginary and $\mathbb{A}_{\mathbb{H}}(p) = 0$ if and only if $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \in \mathbb{R}$ respectively.(We will define the Cartan angular invariant $\mathbb{A}_{\mathbb{H}}(p)$ and the triple $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ in next chapter.)
In this article, we give answer to the question that where these three

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points are when the triple $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\rangle$ is other values such as a complex number or of the form $a + bj$ or $a + bk$ where $a, b \in \mathbb{R}$.

2. Quaternionic Cartan angular invariant

The projective model of the quaternionic hyperbolic space $H^n_{\mathbb{H}}$ is the set of negative lines in the Hermitian vector space $\mathbb{H}^{n,1}$ with Hermitian structure defined by the indefinite $(n, 1)$-form

$$\langle\langle z, w\rangle\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n} - z_{n+1}\overline{w_{n+1}}.$$

One can obtain the ball model $B^n_{\mathbb{H}}(0, 1) \subset \mathbb{H}^n$ by taking inhomogeneous coordinates. Here, throughout this article, we use the left vector space $\mathbb{H}^{n,1}$, in which multiplication by quaternion numbers is on the left. For more details on quaternionic hyperbolic geometry, we refer [1], [3] or [4]. The Cartan angular invariant is well-known invariant in complex hyperbolic geometry, but in quaternionic hyperbolic geometry, B.N.Apanasov and I.Kim first defined it in [1]. Here we give the definition and some properties.

Let $x = (x_1, x_2, x_3) \in (H^n_{\mathbb{H}} \cup \partial H^n_{\mathbb{H}})^3$ be a triple of distinct points with lifts $\tilde{x}_i \in H^{n,1}_{\mathbb{H}}$ for $i = 1, 2, 3$. Then the quaternionic Cartan angular invariant $\mathbb{A}_{\mathbb{H}}(x)$ of a triple $x = (x_1, x_2, x_3)$ is the angle between the first coordinate line $\mathbb{R}e_0 = (\mathbb{R}, 0, 0, 0) \subset \mathbb{R}^4 \cong \mathbb{H}$ and the radius vector of the quaternion equal to the Hermitian triple product $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\rangle = \langle \tilde{x}_1, \tilde{x}_2\rangle\langle \tilde{x}_2, \tilde{x}_3\rangle\langle \tilde{x}_3, \tilde{x}_1\rangle \in \mathbb{H}$. We list some properties of this invariant. One can check them easily or find the proofs in [1].

(1) $\mathbb{A}_{\mathbb{H}}(x)$ is independent of the choice of the lifts $\tilde{x}_i$ of the x_i.
(2) $\mathbb{A}_{\mathbb{H}}(x)$ is invariant under permutations of the points x_i.
(3) For $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, $\mathbb{A}_{\mathbb{H}}(x) = \mathbb{A}_{\mathbb{H}}(y)$ if and only if there exists an isometry $f \in \mathbf{PSp}(n, 1)$ of $H^n_{\mathbb{H}}$ such that $f(x_i) = y_i$ for $i = 1, 2, 3$.

In addition, B.Apanasov and I.Kim showed the following theorems.

Theorem 2.1. (Theorem 3.5 in [1]) *A triple* $x = (x_1, x_2, x_3) \in (\partial H^n_{\mathbb{H}})^3$ *lies in the same* $\mathbb{R}$*-circle if and only if* $\mathbb{A}_{\mathbb{H}}(x) = 0$.

Theorem 2.2. (Theorem 3.6 in [1]) *A triple* $x = (x_1, x_2, x_3) \in (\partial H^n_{\mathbb{H}})^3$ *lies in the boundary of an* $\mathbb{H}$*-line if and only if* $\mathbb{A}_{\mathbb{H}}(x) = \pi/2$.

REMARK 2.3. In the above theorems, $\mathbb{A}_{\mathbb{H}}(x) = 0$ means that $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \in \mathbb{R}$ and $\mathbb{A}_{\mathbb{H}}(x) = \pi/2$ means that $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ is purely imaginary.

3. Main theorem

From now on, we will focus on the triple product $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ instead of $\mathbb{A}_{\mathbb{H}}$.

THEOREM 3.1. *Let* $K = \{(q_1, q_2) \in H^2_{\mathbb{H}} | q_1 \in \mathbb{H}, q_2 \in \mathbb{C}\}$. *Then a triple points* $x_1, x_2, x_3 \in \partial H^2_{\mathbb{H}}$ *lies in a copy of* K *if and only if the triple product* $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \in \mathbb{C}$.

Proof. First, assume that $x_1, x_2, x_3 \in \partial H^2_{\mathbb{H}}$ lies in a copy of K. Without loss of generality, we may assume that $x_1 = (0, -1), x_2 = (0, 1), x_3 = (q_1, q_2)$, where $q_1 \in \mathbb{H}$, $q_2 \in \mathbb{C}$ and $|q_1|^2 + |q_2|^2 = 1$. Then $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = -2(\overline{q}_2 - 1)(-q_2 - 1) = -2\{|q_1|^2 + (q_2 - \overline{q}_2)\} \in \mathbb{C}$.
Conversely, up to isometry, we can assume that $x_1 = (0, -1), x_2 = (0, 1), x_3 = (q_1, q_2)$ for q_1, q_2 are quaternions, $|q_1|^2 + |q_2|^2 = 1$ and $\tilde{x}_1 = (0, -1, 1), \tilde{x}_2 = (0, 1, 1), \tilde{x}_3 = (q_1, q_2, 1)$. Then $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = -2(|q_1|^2 + 2\mathrm{Im}(q_2)) \in \mathbb{C}$, so $q_2 \in \mathbb{C}$.

□

REMARK 3.2. In the above theorem, when we replace the condition $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \in \mathbb{C}$ with $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ is of the form $a + bj$ or $a + bk$ for $a, b \in \mathbb{R}$, one can easily checked that K is replaced with $K' = \{(q_1, q_2) \in H^2_{\mathbb{H}} | q_1 \in \mathbb{H}, q_2$ is of the form $a + bj$ $\}$ or $K'' = \{(q_1, q_2) \in H^2_{\mathbb{H}} | q_1 \in \mathbb{H}, q_2$ is of the form $a + bk$ $\}$

REMARK 3.3. In the theorem, the set K is similar to the bisector in the complex hyperbolic space.(See [2])

The following theorem is a special case of Theorem 2.2 and also a special case of the above theorem. By the way, it is also analogous of the result in complex hyperbolic Cartan angular invariant.

THEOREM 3.4. *A triple* $x = (x_1, x_2, x_3) \in (\partial H^2_{\mathbb{H}})^3$ *lies in a copy of* $H^1_{\mathbb{C}}$ *if and only if the triple product* $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \in \mathbb{R}i$.

Proof. First, assume that a triple $x = (x_1, x_2, x_3) \in (\partial H^2_{\mathbb{H}})^3$ lies in a copy of $H^1_{\mathbb{C}}$. Without loss of generality, we may assume that $x_1, x_2, x_3 \in \partial H^1_{\mathbb{C}}$ and $x_1 = (0, -1), x_2 = (0, 1), x_3 = (0, z)$, where $|z| = 1$, $z \in \mathbb{C}$. Then $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = -2(z - \overline{z}) \in \mathbb{R}i$.

Conversely, up to isometry, we can assume that $x_1 = (0,-1), x_2 = (0,1), x_3 = (q_1, q_2)$ for q_1, q_2 are quaternions, $|q_1|^2 + |q_2|^2 = 1$ and $\tilde{x}_1 = (0,-1,1), \tilde{x}_2 = (0,1,1), \tilde{x}_3 = (q_1, q_2, 1)$. Then $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = -2(|q_1|^2 + 2\mathrm{Im}(q_2))$, so $q_2 \in \mathbb{C}$ since $|q_1|^2 + 2\mathrm{Im}(q_2) \in \mathbb{R}i$. Hence $q_1 = 0$ and $q_2 \in \mathbb{C}$, so x_3 is also in $H^1_{\mathbb{C}}$. □

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