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ANALYTIC SOLUTIONS FOR AMERICAN PARTIAL BARRIER OPTIONS BY EXPONENTIAL BARRIERS

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Abstract. This paper concerns barrier option of American type where the underlying price is monitored during only part of the option's life. Analytic valuation formulas of the American partial barrier options are obtained by approximation method. This approximation method is based on barrier options along with exponential early exercise policies. This result is an extension of Jun and Ku [10] where the exercise policies are constant.

1. Introduction

American options are widely traded in the over counter market because American type options give their holders an additional privilege of early exercise. For these reasons, the valuation of the American option price has been very important issue in financial economics.

Brennan and Schwartz [2] and Parkinson [15] have proposed a numerical approach to value American options. They have approached to a solution of the Black-Scholes partial differential equation using finite differences. Cox et al. [4] used the binomial model to reduce the size of errors. Liang et al. [13] found convergence rate of the binomial model.

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They have modeled an American put option as a variational inequality. These numerical methods are very flexible and easy to implement. However, even after employing control variate or convergence extrapolation, they require a very long time.

Many people have tried to reduce the time consuming task. Sullivan [16] approximated the option value function through Chebyshev polynomials and applied a Gaussian quadrature integration scheme at each discrete exercise date. In order to find the value of American put option, Longstaff and Schwartz [14] used the Monte Carlo simulation method. In the Monte Carlo simulation method, optimal stopping is significant problem. They resolved the problem by comparing the conditional expected value of continuing with the value of immediate exercise if the option is currently in the money. Kim et al. [12] introduced a simple iterative method to determine the optimal exercise boundary for American option. They allowed us to compute the values of American options and their Greeks quickly and accurately.

Another flow of the American option pricing is to determine lower and upper bounds for American option value. Kim [11], Jacka [8], and Carr et al. [3] obtained an analytic integral-form solution of American options where the formulas represent the early premium of American option as integral. Broadie and Detemple [1] provided an upper bound on the value of American options using a lower bound for the early exercise boundary. Ju [9] approximated American option price using early exercise boundary as a multi-piece exponential function.

Approximation method of American options is proposed by Ingersoll [7]. He approximated exercise policy employing a simple class of function, and chose a finest policy in that class adopting standard optimization technique. This method is simple and speedy. Concretely, he discussed American put using two types approximate, constant barrier and exponential barrier.

Barrier options have been extensively traded over the counter market since 1967. These options are activated or expired when the price of the underlying asset crosses a barrier level during the option's life. Heynen and Kat [6] studied partial barrier options where the underlying price is monitored during only part of the option's lifetime. If barrier is observed from a fixed date after initial starting date until expiration, it is called forward starting barrier option. If barrier is disappeared at a designated date before the expiry date, it is named early ending barrier option. Heynen and Kat [6] determined pricing formulas for partial barrier options in terms of bivariate normal distribution functions.

For branch of the American barrier option problem, Gao et al. [5] altered standard American option to an American barrier option employing approximation technique. Ingersoll [7] presented American up and in put price by an approximation method based on barrier options using constant and exponential exercise policies. Jun and Ku [10] approximated analytic valuation formulas of the American partial barrier option based on barrier options along with constant early exercise.

This paper concern valuing the American partial barrier option where barrier is observed just from fixed date to expiry date. This paper extend approximation method, derived by Jun and Ku [10], to American partial barrier option as exponential exercise policy. This method includes the case of constant early exercise policy.

Section 2 proposes the approximation of American barrier option based on exponential exercise policies. Section 3 provides the valuation of American partial barrier option using exponential exercise policies.

2. Approximation of American barrier option using exponential barriers

Let r be the risk-free interest rate, q be a dividend rate, and $\sigma > 0$ be a constant. We assume the price of the underlying asset S follows a geometric Brownian motion

$$S_t = S_0 \exp(\mu t + \sigma W_t)$$

where $\mu = r - q - \frac{\sigma^2}{2}$ and W_t is a standard Brownian motion under the risk-neutral probability P.

In this section, we consider the partial barrier option of American type as exponential early exercise policies. American option holders gain more benefit than European option on early exercise. An American up-and-in put option can be exercised before the expiration time when it is in the money, but only after the stock price rises above the knock-in barrier. We deal with the up-and-in put where the barrier appear at a specified time T_1 strictly after the option's initiation. If the underlying asset price never hits the up-barrier over the time period between T_1 and expiration T, payment of option is zero. Otherwise, if the asset price reaches the up-barrier between T_1 and expiration T, this option can early exercise.

In order to obtain the approximation to value American partial barrier option using barrier derivatives under exercise policies, Ingergsoll [7] used following digitals: let $\mathcal{D}(S,t;A)$ be the value at time t of receiving one dollar at time T if and only in the event A occurs, and $\mathcal{DS}(S,t;A)$ be the value at time t of receiving one share of stock at time T if and only if the event A occurs. The $\mathcal{D}$ is said to be a digital or binary option and the $\mathcal{DS}$ is said to be a digital share. The quantity $\mathcal{E}(S,t,K_\tau;A)$ denotes the value at time t of payment $X - K_\tau$ at the first time τ that the stock price S hits the barrier K_τ provided the event A occurs before time T, where X is a strike price. The $\mathcal{E}$ is said to be a first-touch digital.

We consider the class of exercise policies, $\mathcal{K}_e$, is a set of exponential functions whose elements are in the form of $K_t = K_0 e^{\delta t}$ with constant K_0 and $\delta \geq 0$. Since options with exponential barrier have analytical solutions under Black-Scholes conditions, an exponential barrier is a natural choice.

Consider an American up-and-in put expiring T with strike price X. Let us denote by B up barrier and by K_t^* the optimal exercise policy. Let τ_{Y_1} denote the first time the stock price is equal to Y_1 and $\tau_{Y_1Y_2}$ denote the first time after τ_{Y1} that the stock price is equal to Y_2.

Let $E_1 = \{t < \tau_B < T, \tau_{BK_t^*} > T, S_T < X\}$ be the event of exercise at maturity under the optimal policy, and $E_2 = \{t < \tau_B, \tau_{BK_t^*} < T\}$ be the event of early exercise under the optimal policy. Then the value of the up-and-in put can be written as

$$UIP = X \cdot \mathcal{D}(S,t;E_1) - \mathcal{DS}(S,t;E_1) + \mathcal{E}(S,t,K_t^*;E_2)$$

The barrier approximation for this put takes the maximum value within a class of restricted policies. For example, for exponential exercise policies K_t,

$$UIP \geq UIP_{\exp} = \max_{k_t \in \mathcal{K}_e} [X \cdot \mathcal{D}(S,t;E_3) - \mathcal{DS}(S,t;E_3) + \mathcal{E}(S,t,K_t^*;E_4)]$$

where $E_3 = \{t < \tau_B < T, \tau_{BK_t} > T, S_T < X\}$, $E_4 = \{t < \tau_B, \tau_{BK_t} < T\}$, and τ_{BK_t} is the first time the stock price hits the exponential policy barrier K_t after hitting the barrier B. The values for these digitals are given by

$$\mathcal{D}(S,t;E_3) = e^{-r(T-t)}\Bigg[\left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\left\{N\left(h_2\left(\frac{B^2}{S_tK_t}\right)\right) - N\left(h_1\left(\frac{B^2}{S_tX}\right)\right)\right\} + \left(\frac{K_t}{B}\right)^{\frac{2\mu}{\sigma^2}}\left(\frac{B^2}{S_tK_t}\right)^{\frac{2\delta}{\sigma^2}}\left\{N\left(h_1\left(\frac{S_tK_t^2}{B^2X}\right)\right) - N\left(h_2\left(\frac{S_tK_t}{B^2}\right)\right)\right\}\Bigg],$$

$$\mathcal{DS}(S,t;E_3) = S_te^{-q(T-t)}\Bigg[\left(\frac{B}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left\{N\left(h_2\left(\frac{B^2}{S_tK_t}\right)\right) - N\left(h_1\left(\frac{B^2}{S_tX}\right)\right)\right\} + \left(\frac{K_t}{B}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left\{N\left(h_1\left(\frac{S_tK_t^2}{B^2X}\right)\right) - N\left(h_2\left(\frac{S_tK_t}{B^2}\right)\right)\right\}\Bigg],$$

$$\begin{aligned}\mathcal{E}(S,t,K_t;E_4) = X&\Bigg[\left(\frac{S_t}{B}\right)^{q-p}\left(\frac{K_t}{B}\right)^{q+p}\left(\frac{B^2}{S_tK_t}\right)^{\frac{2\delta}{(q-p)\sigma^2}}N\left(g_1\left(\frac{S_tK_t}{B^2}\right)\right) + \left(\frac{B}{K_t}\right)^{q-p}\left(\frac{B}{S_t}\right)^{q+p}N\left(-g_1\left(\frac{B^2}{S_tK_t}\right)\right)\Bigg] \\ - K_t&\Bigg[\left(\frac{S_t}{B}\right)^{q_1-p_1}\left(\frac{K_t}{B}\right)^{q_1+p_1}\left(\frac{B^2}{S_tK_t}\right)^{\frac{2\delta}{(q_1-p_1)\sigma^2}}N\left(\overline{g}_1\left(\frac{S_tK_t}{B^2}\right)\right) + \left(\frac{B}{K_t}\right)^{q_1-p_1}\left(\frac{B}{S_t}\right)^{q_1+p_1}N\left(-\overline{g}_1\left(\frac{B^2}{S_tK_t}\right)\right)\Bigg],\end{aligned}$$

where N is the standard normal distribution function,

$$h_1(z) = \frac{\ln z + \mu(T-t)}{\sigma\sqrt{T-t}},\ h_2(z) = \frac{\ln z + (\mu-\delta)(T-t)}{\sigma\sqrt{T-t}},$$

$$g_1(z) = \frac{\ln z + (q\sigma^2 - \frac{\delta}{q-p})(T-t)}{\sigma\sqrt{T-t}},$$

$$\mu = r - q - \frac{1}{2}\sigma^2,\ \overline{\mu} = r - q + \frac{1}{2}\sigma^2,\ p = \frac{\mu-\delta}{\sigma^2},\ q = \sqrt{p^2 + \frac{2(r-\delta)}{\sigma^2}}.$$

$\overline{h}_i$ and $\overline{g}_1$ are the same as h_i, g_1 except $\overline{\mu} = r - q + \frac{\sigma^2}{2}$ and p_1, q_1 in replacement of μ, p, q for $i = 1, 2$, separately.

3. Valuation of American partial barrier option using exponential exercise policies

Now, we present the valuation of American partial barrier option using exponential exercise polices. Let $X_t = \frac{1}{\sigma}\ln\left(\frac{S_t}{S_0}\right)$ and E^m be the expectation operator under m-measure. Then X_t is a Brownian motion with drift $\frac{\mu}{\sigma}$. Let $k_t = k_0 + \frac{\delta}{\sigma}t$ for k_0, $\delta(\geq 0)$ are constant. Define $\tau_{b(T_1)}$ and $\tau_{bk_t(T_1)}$ by stopping times for this process defined as the first time that $X_t = b > X_0$ after time T_1 and the first time after $\tau_{b(T_1)}$ that $X_t = k_t < b$, respectively.

Lemma 3.1. *For $x \geq k_T$, the probability that the process X_t reaches b after time T_1, and then hits k_t before expiration T, and X_T is greater than x is*

$$\begin{aligned}
&P\left(\tau_{bk_t(T_1)} \leq T, X_T > x | X_0 = 0\right)\\
&= \exp\left(\frac{2\mu}{\sigma}(-b+k_0)\right)\exp\left(-\frac{2\delta}{\sigma}(k_0-2b)\right)G_1(x) + \exp\left(\frac{2(\mu-\delta)}{\sigma}k_0\right)G_2(x)
\end{aligned}$$

where

$$\begin{aligned}
G_1(x) &= N_2\left(\frac{b - \frac{\mu-2\delta}{\sigma}T_1}{\sqrt{T_1}}, \frac{2k_0 - 2b - x + \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right)\\
G_2(x) &= N_2\left(\frac{-b - \frac{\mu-2\delta}{\sigma}T_1}{\sqrt{T_1}}, \frac{2k_0 - x + \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right).
\end{aligned}$$

Proof. If $b > k_t$, then $\left\{X_{T_1} \geq b, \tau_{bk_t(T_1)} \leq T\right\} = \left\{X_{T_1} \geq b, \tau_{k_t(T_1)} \leq T\right\}$.

$$\begin{aligned}
&P\left(\tau_{bk_t(T_1)} \leq T, X_T > x | X_0 = 0\right)\\
&= P\left(X_{T_1} < b, \tau_{bk_t(T_1)} \leq T, X_T > x | X_0 = 0\right)\\
&\quad + P\left(X_{T_1} \geq b, \tau_{k_t(T_1)} \leq T, X_T > x | X_0 = 0\right)\\
&= \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P\left(\tau_{bk_t(T_1)} \leq T, X_T > x | X_{T_1} = x_1\right) dx_1\\
&\quad + \int_{b}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P\left(\tau_{k_t(T_1)} \leq T, X_T > x | X_{T_1} = x_2\right) dx_2.
\end{aligned}$$

First, we calculate $P\left(\tau_{bk_t(T_1)} \le T, X_T > x | X_{T_1} = x_1\right)$. In order to reflect a path at $\tau_{b(T_1)}$, we define $\widetilde{X}_t = \begin{cases} 2b - X_t & \text{if } t \le \tau_{u(T_1)} \\ X_t & \text{if } t > \tau_{u(T_1)} \end{cases}$

$$\begin{aligned}&P\left(\tau_{bk_t(T_1)} \le T, X_T > x | X_{T_1} = x_1\right)\\&= \exp\left(\frac{2\mu}{\sigma}(b - x_1)\right) P\left(\tau_{k_t(T_1)} \le T, \widetilde{X}_T > x | \widetilde{X}_{T_1} = 2b - x_1\right).\end{aligned}$$

Then we reflect this path before its first touch at k_0 again.

$$\begin{aligned}&P\left(\tau_{bk_t(T_1)} \le T, X_T > x | X_{T_1} = x_1\right)\\&\quad= \exp\left(\frac{2\mu}{\sigma}(b - x_1)\right)\exp\left(\frac{2(\mu-\delta)}{\sigma}(k_0 - 2b + x_1 + \frac{\delta}{\sigma}T_1)\right)\\&\qquad\times N\left(\frac{2k_0 - 2b + x_1 - x + \frac{2\delta}{\sigma}T_1 + \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{-\infty}^{b} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P\left(\tau_{bk_t(T_1)} \le T, X_T > x | X_{T_1} = x_1\right) dx_1\\&= \exp\left(\frac{2\mu}{\sigma}(-b + k_0) - \frac{2\delta}{\sigma}(k_0 - 2b)\right) N_2\left(\frac{u - \frac{\mu - 2\delta}{\sigma}T_1}{\sqrt{T_1}}, \frac{2k_0 - 2b - x + \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right)\end{aligned}$$

and

$$\begin{aligned}&\int_{b}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P\left(\tau_{k_t(T_1)} \le T, X_T > x | X_{T_1} = x_2\right) dx_2\\&= \exp\left(\frac{2(\mu-\delta)}{\sigma}k_0\right) N_2\left(\frac{-b - \frac{\mu - 2\delta}{\sigma}T_1}{\sqrt{T_1}}, \frac{2k_0 - x + \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right).\end{aligned}$$

□

LEMMA 3.2. *The probability that the process X_t reaches b after time T_1, and then falls below k_t before time T is*

$$\begin{aligned}&P(\tau_{bk_t(T_1)} \le T | X_0 = 0)\\&= \exp\left(\frac{2\mu}{\sigma}(-b + k_0)\right)\exp\left(-\frac{2\delta}{\sigma}(k_0 - 2b)\right) G_1(k_T)\\&\quad + \exp\left(\frac{2(\mu-\delta)}{\sigma}\ k_0\right) G_2(k_T) + \exp\left(\frac{2b\mu}{\sigma}\right) G_3(k_T) + G_4(k_T)\end{aligned}$$

where

$$G_3(x) = N_2\left(\frac{b+\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{x-2b-\frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right),$$

$$G_4(x) = N_2\left(\frac{-b+\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{x-\frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right).$$

Proof. Since $\{\tau_{bk_t(T_1)} \le T, X_T \le k_T\} = \{\tau_{b(T_1)} \le T, X_T \le k_T\}$

$$P(\tau_{bk_t(T_1)} \le T | X_0 = 0)$$
$$= P(\tau_{bk_t(T_1)} \le T, X_T > k_T | X_0 = 0) + P(\tau_{b(T_1)} \le T, X_T \le k_T | X_0 = 0)$$

When $X_{T_1} > b$, the event $\{\tau_{b(T_1)} \le T, X_T \le k_T\} = \{X_T \le k_T\}$ and

$$P(\tau_{b(T_1)} \le T, X_T \le k_T | X_0 = 0)$$
$$= \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1-\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P\left(\tau_{b(T_1)} \le T, X_T \le k_T | X_{T_1} = x_1\right) dx_1$$
$$+ \int_{b}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2-\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P\left(X_T \le k_T | X_{T_1} = x_2\right) dx_2$$
$$= \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1-\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} \exp\left(\frac{2\mu}{\sigma}(b-x_1)\right)$$
$$\times N\left(\frac{k_0-2b+x_1-\frac{\mu-\delta}{\sigma}T+\frac{\mu}{\sigma}T_1}{\sqrt{T-T_1}}\right) dx_1$$
$$+ \int_{b}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2-\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} N\left(\frac{k_0-x_2-\frac{\mu-\delta}{\sigma}T+\frac{\mu}{\sigma}T_1}{\sqrt{T-T_1}}\right) dx_2$$

$$= \exp\left(\frac{2u\mu}{\sigma}\right) N_2\left(\frac{u+\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{l_0-2u-\frac{\mu-\delta}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right)$$
$$+ N_2\left(\frac{-u+\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{l_0-\frac{\mu-\delta}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right).$$

□

Lemma 3.3. *For $k_T \le x \le b$, the probability that the process X_t reaches b after time T_1, and then does not fall below k_t before expiration T, and its value at time T is less than x is*

$$\begin{aligned}
&P(\tau_{b(T_1)} < T, \tau_{bk_t(T_1)} > T, X_T \le x | X_0 = 0)\\
&\quad = \exp\left(\frac{2\mu}{\sigma}(-b+k_0)\right)\exp\left(-\frac{2\delta}{\sigma}(k_0-2u)\right)[G_1(x)-G_1(k_T)]\\
&\qquad + \exp\left(\frac{2(\mu-\delta)}{\sigma}k_0\right)[G_2(x)-G_2(l_T)]\\
&\qquad + \exp\left(\frac{2b\mu}{\sigma}\right)[G_3(x)-G_3(k_T)] + G_4(x) - G_4(k_T)
\end{aligned}$$

Proof.

$$\begin{aligned}
&P(\tau_{b(T_1)} < T, \tau_{bk_t(T_1)} > T, X_T \le x | X_0 = 0)\\
&\quad = P(\tau_{b(T_1)} < T, X_T \le x | X_0 = 0)\\
&\qquad - P(\tau_{b(T_1)} < T, \tau_{bk_t(T_1)} \le T, X_T \le x | X_0 = 0)\\
&\quad = P(\tau_{b(T_1)} < T, X_T \le x | X_0 = 0) - P(\tau_{bk_t(T_1)} \le T, X_T \le x | X_0 = 0)\\
&\quad = P(\tau_{b(T_1)} < T, X_T \le x | X_0 = 0) - P(\tau_{bk_t(T_1)} \le T | X_0 = 0)\\
&\qquad + P(\tau_{bk_t(T_1)} \le T, X_T > x | X_0 = 0)
\end{aligned}$$

$P(\tau_{b(T_1)} < T, X_T \le x | X_0 = 0)$ can be calculated with $k_T (= l_0 + \frac{\delta}{\sigma}T) = x$. The second and third probabilities are calculated by Lemma 3.2 and Lemma 3.1. □

THEOREM 3.4. *The value of a digital option and a digital share at time t for the event* $E_8 = \{\tau_{BK_t(T_1)} < T\}$ *are*

$$\begin{aligned}
&\mathcal{D}(S,t;E_8)\\
&= e^{-r(T-t)}\left(\frac{K_t}{B}\right)^{\frac{2\mu}{\sigma^2}}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{\sigma^2}} N_2\left(h_4\left(\frac{B}{S_t}\right), h_2\left(\frac{K_tS_t}{B^2}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)\\
&\quad + e^{-r(T-t)}\left(\frac{K_t}{S_t}\right)^{\frac{2(\mu-\delta)}{\sigma^2}} N_2\left(h_4\left(\frac{S_t}{B}\right), h_2\left(\frac{K_t}{S_t}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)\\
&\quad + e^{-r(T-t)}\left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}} N_2\left(h_3\left(\frac{B}{S_t}\right), -h_2\left(\frac{B^2}{K_tS_t}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)\\
&\quad + e^{-r(T-t)} N_2\left(h_3\left(\frac{S_t}{B}\right), -h_2\left(\frac{S_t}{K_t}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{DS}(S,t;E_8)\\
=&S_t e^{-q(T-t)}\left(\frac{K_t}{B}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{\sigma^2}} N_2\left(\overline{h}_4\left(\frac{B}{S_t}\right),\overline{h}_2\left(\frac{K_tS_t}{B^2}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\\
&+S_t e^{-q(T-t)}\left(\frac{K_t}{S_t}\right)^{\frac{2(\overline{\mu}-\delta)}{\sigma^2}} N_2\left(\overline{h}_4\left(\frac{S_t}{B}\right),\overline{h}_2\left(\frac{K_t}{S_t}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\\
&+S_t e^{-q(T-t)}\left(\frac{B}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N_2\left(\overline{h}_3\left(\frac{B}{S_t}\right),-\overline{h}_2\left(\frac{B^2}{K_tS_t}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\\
&+S_t e^{-q(T-t)} N_2\left(\overline{h}_3\left(\frac{S_t}{B}\right),-\overline{h}_2\left(\frac{S_t}{K_t}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)
\end{aligned}
$$

Proof. Apply Lemma 3.3 with $b=\frac{1}{\sigma}\ln\frac{B}{S_t}, k_t=\frac{1}{\sigma}\ln\frac{K_t}{S_t}(=\frac{1}{\sigma}\ln\frac{K_0e^{\delta t}}{S_t})$ and $x=\frac{1}{\sigma}\ln\frac{X}{S_t}$ to derive the risk-neutral probability of early exercise.

We calculate each term of $P(\tau_{BK_t(T_1)}\le T|S_t)$.

$$
\begin{aligned}
&P(\tau_{BK_t(T_1)}\le T|S_t)\\
=&\left(\frac{K_t}{B}\right)^{\frac{2\mu}{\sigma^2}}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{\sigma^2}} N_2\left(h_4\left(\frac{B}{S_t}\right),h_2\left(\frac{K_tS_t}{B^2}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\\
&+\left(\frac{K_t}{S_t}\right)^{\frac{2(\mu-\delta)}{\sigma^2}} N_2\left(h_4\left(\frac{S_t}{B}\right),h_2\left(\frac{K_t}{S_t}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\\
&+\left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}} N_2\left(h_3\left(\frac{B}{S_t}\right),-h_2\left(\frac{B^2}{K_tS_t}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\\
&+N_2\left(h_3\left(\frac{S_t}{B}\right),-h_2\left(\frac{S_t}{K_t}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)
\end{aligned}
$$

Thus, the value of digital option at time t

$$\mathcal{D}(S,t;E_8)=e^{-r(T-t)}P(\tau_{BK_t(T_1)}\le T|S_t)$$

is obtained. □

Lemma 3.5. *If the stock does not pay dividends, the value of a first touch digital for the event* $E_8=\{\tau_{BK_t(T_1)}<T\}$ *is*

$$
\begin{aligned}
&\mathcal{E}(S,t,K_t;E_8)\\
&=\frac{X-K_\tau}{K_\tau}S_t\Bigg[\left(\frac{K_t}{B}\right)^{\frac{2r}{\sigma^2}+1}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{\sigma^2}}N_2\left(\widetilde{h}_4\left(\frac{B}{S_t}\right),\widetilde{h}_2\left(\frac{S_tK_t}{B^2}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\\
&\qquad+\left(\frac{K_t}{S_t}\right)^{\frac{2(r-\delta)}{\sigma^2}+1}N_2\left(\widetilde{h}_4\left(\frac{S_t}{B}\right),\widetilde{h}_2\left(\frac{K_t}{S_t}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\\
&\qquad+\left(\frac{B}{S_t}\right)^{\frac{2r}{\sigma^2}+1}N_2\left(\widetilde{h}_3\left(\frac{B}{S_t}\right),-\widetilde{h}_2\left(\frac{B^2}{S_tK_t}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\\
&\qquad+N_2\left(\widetilde{h}_3\left(\frac{S_t}{B}\right),-\widetilde{h}_2\left(\frac{S_t}{K_t}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\Bigg]
\end{aligned}
$$

where $\widetilde{h}_i$ is the same as h_i except $\widetilde{\mu}=r+\frac{1}{2}\sigma^2$ in replacement of μ for $i=2,3,4$.

Proof. The first-touch digital pays $X-K_\tau$ at time $\tau_{BK_\tau(T_1)}$. This money can be used to purchase $\frac{X-K_\tau}{K_\tau}$ shares of the stock at time. Since the shares do not pay dividends, it is worth $\frac{X-K_T}{K_T}S_T$ at maturity, i.e.,

$$\mathcal{E}(S,t,K_t;E_8)=\frac{X-K_\tau}{K_\tau}\mathcal{DS}(S,t,K_t;E_8)$$

where $\mathcal{DS}(S,t,K_t;E_8)$ is the value when $q=0$ in Theorem 3.4. □

Let K^* denote the optimal exercise policy. We denote the $E_5=\{\tau_{B(T_1)}<T,\tau_{BK_t^*(T_1)}>T,S_T<X\}$ be the event of exercise at maturity under the optimal policy, and $E_6=\{\tau_{BK_t^*(T_1)}<T\}$ be the event of early exercise under the optimal policy. Then the value of this partial up-and-in put is written as

$$PUIP=X\cdot\mathcal{D}(S,t;E_5)-\mathcal{DS}(S,t;E_5)+\mathcal{E}(S,t,K_t^*;E_6)$$

For the barrier approximation of this option, we consider a class of all exponential exercise policies. Let $E_7=\{\tau_{B(T_1)}<T,\tau_{BK_t(T_1)}>T,S_T<X\}$ be the event of exercise at maturity under an exponential policy K_t, and $E_8=\{\tau_{BK_t(T_1)}<T\}$ be the event of early exercise under policy K_t. Then we can approximate the option price as

$$
\begin{aligned}
PUIP&\geq PUIP_{\exp}\\
&=\max_{k_t\in\mathcal{K}_e}[X\cdot\mathcal{D}(S,t;E_7)-\mathcal{DS}(S,t;E_7)+\mathcal{E}(S,t,K_t;E_8)].
\end{aligned}
$$

If the set of policies considered contains all continuous functions, then the resulting put value will be exact. Since the set $\mathcal{K}_e$ is the set of all exponential functions, then the resulting value will be an approximation providing a lower bound to the put price.

We first present the digital options in case of barrier greater than strike price for an American partial barrier option.

THEOREM 3.6. *For $X \le B$, the values of a digital option and a digital share at time $t < T_1$ for the event $E_7 = \{\tau_{B(T_1)} < T, \tau_{BK_t(T_1)} > T, S_T \le X\}$ are*

$$\begin{aligned}
&\mathcal{D}(S,t;E_7)\\
&= e^{-r(T-t)}\Bigg[\left(\frac{K_t}{B}\right)^{\frac{2\mu}{\sigma^2}}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{\sigma^2}}\left(F_1\left(\frac{B}{S_t},\frac{K_t^2S_t}{B^2X}\right)-F_2\left(\frac{B}{S_t},\frac{K_tS_t}{B^2}\right)\right)\\
&\qquad+\left(\frac{K_t}{S_t}\right)^{\frac{2(\mu-\delta)}{\sigma^2}}\left(F_1\left(\frac{S_t}{B},\frac{K_t^2}{S_tX}\right)-F_2\left(\frac{S_t}{B},\frac{K_t}{S_t}\right)\right)\\
&\qquad+\left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\left(F_3\left(\frac{B}{S_t},\frac{B^2}{S_tX}\right)-F_4\left(\frac{B}{S_t},\frac{B^2}{S_tK_t}\right)\right)\\
&\qquad+\left(F_3\left(\frac{S_t}{B},\frac{S_t}{X}\right)-F_4\left(\frac{S_t}{B},\frac{S_t}{K_t}\right)\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{DS}(S,t;E_7)\\
&= S_te^{-q(T-t)}\Bigg[\left(\frac{K_t}{B}\right)^{\frac{2\mu}{\sigma^2}}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{\sigma^2}}\left(\overline{F}_1\left(\frac{B}{S_t},\frac{K_t^2S_t}{B^2X}\right)-\overline{F}_2\left(\frac{B}{S_t},\frac{K_tS_t}{B^2}\right)\right)\\
&\qquad+\left(\frac{K_t}{S_t}\right)^{\frac{2(\mu-\delta)}{\sigma^2}}\left(\overline{F}_1\left(\frac{S_t}{B},\frac{K_t^2}{S_tX}\right)-\overline{F}_2\left(\frac{S_t}{B},\frac{K_t}{S_t}\right)\right)\\
&\qquad+\left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\left(\overline{F}_3\left(\frac{B}{S_t},\frac{B^2}{S_tX}\right)-\overline{F}_4\left(\frac{B}{S_t},\frac{B^2}{S_tK_t}\right)\right)\\
&\qquad+\left(\overline{F}_3\left(\frac{S_t}{B},\frac{S_t}{X}\right)-\overline{F}_4\left(\frac{S_t}{B},\frac{S_t}{K_t}\right)\right)\Bigg],
\end{aligned}$$

where

$$F_1(x,y) = N_2\left(h_4(x), h_1(y); -\sqrt{\frac{T_1-t}{T-t}}\right),$$

$$F_2(x,y) = N_2\left(h_4(x), h_2(y); -\sqrt{\frac{T_1-t}{T-t}}\right),$$
$$F_3(x,y) = N_2\left(h_3(x), -h_1(y); -\sqrt{\frac{T_1-t}{T-t}}\right),$$
$$F_4(x,y) = N_2\left(h_4(x), -h_2(y); -\sqrt{\frac{T_1-t}{T-t}}\right),$$

and

$$h_3(z) = \frac{\ln z + \mu(T_1-t)}{\sigma\sqrt{T_1-t}},\ \ h_4(z) = \frac{\ln z - (\mu-2\delta)(T_1-t)}{\sigma\sqrt{T_1-t}}.$$

$\overline{F}_i(x,y)$ and $\overline{h}_j(z)$ are the same as $F_i(x,y)$, $h_j(z)$ except $\overline{\mu} = r - q + \frac{\sigma^2}{2}$ in replacement of μ for $i = 1,2,3,4$, $j = 3,4$ separately.

Proof. Apply Lemma 3.3 with letting $b = \frac{1}{\sigma}\ln\frac{B}{S_t}, k_t = \frac{1}{\sigma}\ln\frac{K_t}{S_t}(= \frac{1}{\sigma}\ln\frac{K_0e^{\delta t}}{S_t})$ and $x = \frac{1}{\sigma}\ln\frac{X}{S_t}$ to derive the risk-neutral probability of exercise at maturity.

Then

$$\begin{aligned}
&P(\tau_{B(T_1)} < T, \tau_{BK_t(T_1)} > T, S_T \le X|S_t)\\
&= \left(\frac{K_t}{B}\right)^{\frac{2\mu}{\sigma^2}}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{\sigma^2}}\left(N_2\left(h_4\left(\frac{B}{S_t}\right), h_1\left(\frac{K_t^2S_t}{B^2X}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)\right.\\
&\qquad\left. - N_2\left(h_4\left(\frac{B}{S_t}\right), h_2\left(\frac{K_tS_t}{B^2}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)\right)\\
&+ \left(\frac{K_t}{S_t}\right)^{\frac{2(\mu-\delta)}{\sigma^2}}\left(N_2\left(h_4\left(\frac{S_t}{B}\right), h_1\left(\frac{K_t^2}{S_tX}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)\right.\\
&\qquad\left. - N_2\left(h_4\left(\frac{S_t}{B}\right), h_2\left(\frac{K_t}{X}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)\right)\\
&+ \left(\frac{B}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\left(N_2\left(h_3\left(\frac{B}{S_t}\right), -h_1\left(\frac{B^2}{S_tX}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)\right.\\
&\qquad\left. - N_2\left(h_3\left(\frac{B}{S_t}\right), -h_2\left(\frac{B^2}{S_tK_t}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)\right)\\
&+ N_2\left(h_3\left(\frac{S_t}{B}\right), -h_1\left(\frac{S_t}{X}\right); -\sqrt{\frac{T_1-t}{T-t}}\right) - N_2\left(h_3\left(\frac{S_t}{B}\right), -h_2\left(\frac{S_t}{K_t}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)
\end{aligned}$$

Thus

$$\mathcal{D}(S,t;E_7) = e^{-r(T-t)} P(\tau_{B(T_1)} < T, \tau_{BK_t(T_1)} > T, S_T \le X | S_t)$$

is obtained as desired. The digital share $\mathcal{DS}(S,t;E_7)$ can be valued by changing μ to $\overline{\mu} = r - q + \frac{\sigma^2}{2}$ and replacing the discount factor $e^{-r(T-t)}$ to $S_t e^{-q(T-t)}$. □

THEOREM 3.7. *The value of the first-touch digital for the event* $E_8 = \{\tau_{BK_t(T_1)} < T\}$ *is*

$$\begin{aligned}
&\mathcal{E}(S,t,K_t;E_8)\\
=&X\Bigg[\left(\frac{S_tK_t}{BK_\tau}\right)^{q-p}\left(\frac{K_t}{B}\right)^{q+p}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{(q-p)\sigma^2}}H_1\left(\frac{B}{S_t},\frac{S_tK_t}{B^2}\right)\\
&\quad+\left(\frac{K_t}{K_\tau}\right)^{q-p}\left(\frac{K_t}{S_t}\right)^{q+p}\left(\frac{K_t}{S_t}\right)^{\frac{-2\delta}{(q-p)\sigma^2}}H_1\left(\frac{S_t}{B},\frac{K_t}{S_t}\right)\\
&\quad+\left(\frac{B}{K_\tau}\right)^{q-p}\left(\frac{B}{S_t}\right)^{q+p}H_2\left(\frac{B}{S_t},\frac{B^2}{K_tS_t}\right)+\left(\frac{S_t}{K_\tau}\right)^{q-p}H_2\left(\frac{S_t}{B},\frac{S_t}{K_t}\right)\Bigg]\\
&-K_t\Bigg[\left(\frac{S_tK_t}{BK_\tau}\right)^{q_1-p_1}\left(\frac{K_t}{B}\right)^{q_1+p_1}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{(q_1-p_1)\sigma^2}}\overline{H}_1\left(\frac{B}{S_t},\frac{S_tK_t}{B^2}\right)\\
&\quad+\left(\frac{K_t}{K_\tau}\right)^{q_1-p_1}\left(\frac{K_t}{S_t}\right)^{q_1+p_1}\left(\frac{K_t}{S_t}\right)^{\frac{-2\delta}{(q_1-p_1)\sigma^2}}\overline{H}_1\left(\frac{S_t}{B},\frac{K_t}{S_t}\right)\\
&+\left(\frac{B}{K_\tau}\right)^{q_1-p_1}\left(\frac{B}{S_t}\right)^{q_1+p_1}\overline{H}_2\left(\frac{B}{S_t},\frac{B^2}{K_tS_t}\right)+\left(\frac{S_t}{K_\tau}\right)^{q_1-p_1}\overline{H}_2\left(\frac{S_t}{B},\frac{S_t}{K_t}\right)\Bigg],
\end{aligned}$$

where

$$H_1(x,y) = N_2\left(g_2(x), g_1(y); -\sqrt{\frac{T_1-t}{T-t}}\right),$$
$$H_2(x,y) = N_2\left(g_3(x), -g_1(y); -\sqrt{\frac{T_1-t}{T-t}}\right),$$

and

$$g_2(z) = \frac{\ln z - \left(q\sigma^2 - \frac{2\delta}{q-p}\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \ \overline{g}_2(z) = \frac{\ln z - \left(q_1\sigma^2 - \frac{2\delta}{q_1-p_1}\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}},$$

$$g_3(z) = \frac{\ln z + q\sigma^2(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \ \overline{g}_3(z) = \frac{\ln z + q_1\sigma^2(T_1 - t)}{\sigma\sqrt{T_1 - t}}.$$

$\overline{H}_i(x, y)$ is the same as $H_i(x, y)$ except $r - \delta$ in replacement of r for $i = 1, 2$.

Proof. By Lemma 3.5, we note that $\mathcal{E}(S, t, K_t; E_8) = \frac{X-K_\tau}{K_\tau}\mathcal{DS}(S, t; E_8)$ for $0 < t < T_1$ and $T_1 < \tau < T$. i.e.

$$\begin{aligned}
&\mathcal{E}(S, t, K_t; E_8) \\
&= X\frac{S_t}{K_\tau}\Bigg[\left(\frac{K_t}{B}\right)^{\frac{2r}{\sigma^2}+1}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{\sigma^2}} N_2\left(\widetilde{h}_4\left(\frac{B}{S_t}\right), \widetilde{h}_2\left(\frac{S_tK_t}{B^2}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right) \\
&\quad + \left(\frac{K_t}{S_t}\right)^{\frac{2(r-\delta)}{\sigma^2}+1} N_2\left(\widetilde{h}_4\left(\frac{S_t}{B}\right), \widetilde{h}_2\left(\frac{K_t}{S_t}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right) \\
&\quad + \left(\frac{B}{S_t}\right)^{\frac{2r}{\sigma^2}+1} N_2\left(\widetilde{h}_3\left(\frac{B}{S_t}\right), -\widetilde{h}_2\left(\frac{B^2}{S_tK_t}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right) \\
&\quad + N_2\left(\widetilde{h}_3\left(\frac{S_t}{B}\right), -\widetilde{h}_2\left(\frac{S_t}{K_t}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right)\Bigg] \\
&- K_\tau\frac{S_t}{K_\tau}\Bigg[\left(\frac{K_t}{B}\right)^{\frac{2r}{\sigma^2}+1}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{\sigma^2}} N_2\left(\widetilde{h}_4\left(\frac{B}{S_t}\right), \widetilde{h}_2\left(\frac{S_tK_t}{B^2}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right) \\
&\quad + \left(\frac{K_t}{S_t}\right)^{\frac{2(r-\delta)}{\sigma^2}+1} N_2\left(\widetilde{h}_4\left(\frac{S_t}{B}\right), \widetilde{h}_2\left(\frac{K_t}{S_t}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right) \\
&\quad + \left(\frac{B}{S_t}\right)^{\frac{2r}{\sigma^2}+1} N_2\left(\widetilde{h}_3\left(\frac{B}{S_t}\right), -\widetilde{h}_2\left(\frac{B^2}{S_tK_t}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right) \\
&\quad + N_2\left(\widetilde{h}_3\left(\frac{S_t}{B}\right), -\widetilde{h}_2\left(\frac{S_t}{K_t}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right)\Bigg].
\end{aligned}$$

When the stock price pays dividends, the asset price follow the continuous diffusion process $dS_t = (r - q)S_tdt + \sigma S_tdW$. In order to calculate the first term with constant payment X, we set

$$V_t = S_t^{q-p}$$

where

$$p = \frac{\mu}{\sigma^2} \text{ and } q = \sqrt{p^2 + \frac{2r}{\sigma^2}}.$$

Then, by Ito's lemma,

$$dV_t = rV_t dt + (q-p)\sigma V_t dW_t \tag{1}$$

We may apply Lemma 3.5 to the process V_t since (1) does not contain the dividend term. The barriers for V_t corresponding to B and K_t are B^{q-p} and K_t^{q-p}. Furthermore, the volatility σ is replaced by $(q-p)\sigma$. Then $\frac{X}{K_\tau}\mathcal{DS}(S,t,K_t;A_8)$ is

$$\begin{aligned}
&X\frac{V_t}{K_\tau^{p-q}}\Bigg[\left(\frac{K_t^{(q-p)}}{B^{(q-p)}}\right)^{\frac{2r}{(q-p)^2\sigma^2}+1}\left(\frac{B^{2(q-p)}}{K_t^{(q-p)}V_t}\right)^{\frac{2\delta}{(q-p)^2\sigma^2}} \\
&\quad\times N_2\left(\widetilde{h}_4\left(\frac{B^{(q-p)}}{V_t}\right),\widetilde{h}_2\left(\frac{V_tK_t^{(q-p)}}{B^{2(q-p)}}\right);-\sqrt{\frac{T_1-t}{T-t}}\right) \\
&\quad+\left(\frac{K_t^{(q-p)}}{V_t}\right)^{\frac{2(r-\delta)}{(q-p)^2\sigma^2}+1}N_2\left(\widetilde{h}_4\left(\frac{V_t}{B^{(q-p)}}\right),\widetilde{h}_2\left(\frac{K_t^{(q-p)}}{V_t}\right);-\sqrt{\frac{T_1-t}{T-t}}\right) \\
&\quad+\left(\frac{B^{(q-p)}}{V_t}\right)^{\frac{2r}{(q-p)^2\sigma^2}+1}N_2\left(\widetilde{h}_3\left(\frac{B^{(q-p)}}{V_t}\right),-\widetilde{h}_2\left(\frac{B^{2(q-p)}}{V_tK_t^{(q-p)}}\right);-\sqrt{\frac{T_1-t}{T-t}}\right) \\
&\quad+N_2\left(\widetilde{h}_3\left(\frac{V_t}{B^{(q-p)}}\right),-\widetilde{h}_2\left(\frac{V_t}{K_t^{(q-p)}}\right);-\sqrt{\frac{T_1-t}{T-t}}\right)\Bigg] \\
&=X\Bigg[\left(\frac{S_tK_t}{BK_\tau}\right)^{(q-p)}\left(\frac{K_t}{B}\right)^{q+p}\left(\frac{B^2}{K_tS_t}\right)^{\frac{2\delta}{(q-p)\sigma^2}}H_1\left(\frac{B}{S_t},\frac{S_tK_t}{B^2}\right) \\
&\quad+\left(\frac{K_t}{K_\tau}\right)^{q-p}\left(\frac{K_t}{S_t}\right)^{q+p}\left(\frac{K_t}{S_t}\right)^{\frac{-2\delta}{(q-p)\sigma^2}}H_1\left(\frac{S_t}{B},\frac{K_t}{S_t}\right) \\
&\quad+\left(\frac{B}{K_\tau}\right)^{q-p}\left(\frac{B}{S_t}\right)^{q+p}H_2\left(\frac{B}{S_t},\frac{B^2}{K_tS_t}\right)+\left(\frac{S_t}{K_\tau}\right)^{q-p}H_2\left(\frac{S_t}{B},\frac{S_t}{K_t}\right)\Bigg].
\end{aligned}$$

For the second term with exponential payment K_τ, we note when a payment grows exponentially at rate δ, discounting the payment at the interest rate r is equivalent to discounting a constant payment at the rate $r-\delta$, therefore, set

$$V_t = S_t^{q_1-p_1}$$

where

$$p_1 = \frac{\mu-\delta}{\sigma^2} \text{ and } q_1 = \sqrt{p_1^2 + \frac{2(r-\delta)}{\sigma^2}}.$$

Then, by Ito's lemma,

$$dV_t = (r-\delta)V_t dt + (q_1 - p_1)\sigma V_t dW_t.$$

□

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