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ON SOME INEQUALITIES FOR NUMERICAL RADIUS OF OPERATORS IN HILBERT SPACES

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Abstract. By the use of inequalities for nonnegative Hermitian forms some new inequalities for numerical radius of bounded linear operators in complex Hilbert spaces are established.

1. Introduction

Let $\mathbb{K}$ be the field of real or complex numbers, i.e., $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and X be a linear space over $\mathbb{K}$.

Definition 1. A functional $(\cdot,\cdot) : X \times X \to \mathbb{K}$ is said to be a Hermitian form on X if

(H1) $(ax + by, z) = a\,(x, z) + b\,(y, z)$ for $a, b \in \mathbb{K}$ and $x, y, z \in X$;
(H2) $(x, y) = \overline{(y, x)}$ for all $x, y \in X$.

The functional $(\cdot,\cdot)$ is said to be *positive semi-definite* on a subspace Y of X if

(H3) $(y, y) \geq 0$ for every $y \in Y$,

and *positive definite* on Y if it is positive semi-definite on Y and

(H4) $(y, y) = 0$, $y \in Y$ implies $y = 0$.

The functional $(\cdot,\cdot)$ is said to be *definite* on Y provided that either $(\cdot,\cdot)$ or $-(\cdot,\cdot)$ is positive semi-definite on Y.

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When a Hermitian functional $(\cdot,\cdot)$ is positive-definite on the whole space X, then, as usual, we will call it an *inner product* on X and will denote it by $\langle\cdot,\cdot\rangle$.

We use the following notations related to a given Hermitian form $(\cdot,\cdot)$ on X :

$$X_0 := \{x \in X | (x,x) = 0\}, \ K := \{x \in X | (x,x) < 0\}$$

and, for a given $z \in X$,

$$X^{(z)} := \{x \in X | (x,z) = 0\} \quad \text{and} \quad L(z) := \{az | a \in \mathbb{K}\}.$$

The following fundamental facts concerning Hermitian forms hold:

Theorem 1 (Kurepa, 1968 [28]). *Let X and $(\cdot,\cdot)$ be as above.*

1. *If $e \in X$ is such that $(e,e) \neq 0$, then we have the decomposition*

$$X = L(e) \bigoplus X^{(e)}, \tag{1.1}$$

   where $\bigoplus$ denotes the direct sum of the linear subspaces $X^{(e)}$ and $L(e)$;
2. *If the functional $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ for at least one $e \in K$, then $(\cdot,\cdot)$ is positive semi-definite on $X^{(f)}$ for each $f \in K$;*
3. *The functional $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ with $e \in K$ if and only if the inequality*

$$|(x,y)|^2 \geq (x,x)(y,y) \tag{1.2}$$

   holds for all $x \in K$ and all $y \in X$;
4. *The functional $(\cdot,\cdot)$ is semi-definite on X if and only if the Schwarz's inequality*

$$|(x,y)|^2 \leq (x,x)(y,y) \tag{1.3}$$

   holds for all $x, y \in X$;
5. *The case of equality holds in (1.3) for $x, y \in X$ and in (1.2), for $x \in K$, $y \in X$, respectively; if and only if there exists a scalar $a \in \mathbb{K}$ such that*

$$y - ax \in X_0^{(x)} := X_0 \cap X^{(x)}.$$

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, *nonnegative* forms on X.

If $(\cdot,\cdot)\in\mathcal{H}(X)$, then the functional $\|\cdot\|=(\cdot,\cdot)^{\frac{1}{2}}$ is a *semi-norm* on X and the following equivalent versions of Schwarz's inequality hold:

$$\|x\|^2\|y\|^2\geq|(x,y)|^2 \quad \text{or} \quad \|x\|\|y\|\geq|(x,y)| \tag{1.4}$$

for any $x,y\in X$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$, i.e.,

(e) $(\cdot,\cdot)_1,(\cdot,\cdot)_2\in\mathcal{H}(X)$ implies that $(\cdot,\cdot)_1+(\cdot,\cdot)_2\in\mathcal{H}(X)$;
(ee) $\alpha\geq0$ and $(\cdot,\cdot)\in\mathcal{H}(X)$ implies that $\alpha(\cdot,\cdot)\in\mathcal{H}(X)$.

The following simple result is of interest in itself as well:

Lemma 1. *Let X be a linear space over the real or complex number field $\mathbb{K}$ and $(\cdot,\cdot)$ a nonnegative Hermitian form on X. If $y\in X$ is such that $(y,y)\neq0$, then*

$$p_y:H\times H\to\mathbb{K},\ p_y(x,z)=(x,z)\|y\|^2-(x,y)(y,z) \tag{1.5}$$

is also a nonnegative Hermitian form on X.

We have the inequalities

$$\begin{aligned}&\left(\|x\|^2\|y\|^2-|(x,y)|^2\right)\left(\|y\|^2\|z\|^2-|(y,z)|^2\right)\\&\geq\left|(x,z)\|y\|^2-(x,y)(y,z)\right|^2\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}&\left(\|x+z\|^2\|y\|^2-|(x+z,y)|^2\right)^{\frac{1}{2}}\\&\leq\left(\|x\|^2\|y\|^2-|(x,y)|^2\right)^{\frac{1}{2}}+\left(\|y\|^2\|z\|^2-|(y,z)|^2\right)^{\frac{1}{2}}\end{aligned} \tag{1.7}$$

for any $x,y,z\in X$.

Remark 1. *The case when $(\cdot,\cdot)$ is an inner product in Lemma 1 was obtained in 1985 by S. S. Dragomir, [2].*

Remark 2. *Putting $z=\lambda y$ in (1.7), we get:*

$$0\leq\|x+\lambda y\|^2\|y\|^2-|(x+\lambda y,y)|^2\leq\|x\|^2\|y\|^2-|(x,y)|^2 \tag{1.8}$$

and, in particular,

$$0\leq\|x\pm y\|^2\|y\|^2-|(x\pm y,y)|^2\leq\|x\|^2\|y\|^2-|(x,y)|^2 \tag{1.9}$$

for every $x,y\in H$.

We note here that the inequality (1.8) is in fact equivalent to the following statement

$$\sup_{\lambda\in\mathbb{K}}\left[\|x+\lambda y\|^2\|y\|^2-|(x+\lambda y,y)|^2\right]=\|x\|^2\|y\|^2-|(x,y)|^2 \tag{1.10}$$

for each $x,y\in H$.

The following result holds (see [11, p. 38] for the case of inner product):

Theorem 2. *Let X be a linear space over the real or complex number field $\mathbb{K}$ and $(\cdot,\cdot)$ a nonnegative Hermitian form on X. For any $x,y,z\in X$, the following refinement of the Schwarz inequality holds:*

$$\begin{aligned}\|x\|\,\|z\|\,\|y\|^2&\geq\left|(x,z)\,\|y\|^2-(x,y)\,(y,z)\right|+|(x,y)\,(y,z)|\\&\geq|(x,z)|\,\|y\|^2.\end{aligned} \tag{1.11}$$

Corollary 1. *For any $x,y,z\in X$ we have*

$$\frac{1}{2}\left[\|x\|\,\|z\|+|(x,z)|\right]\|y\|^2\geq|(x,y)\,(y,z)|. \tag{1.12}$$

The inequality (1.12) follows from the first inequality in (1.11) and the triangle inequality for modulus

$$\left|(x,z)\,\|y\|^2-(x,y)\,(y,z)\right|\geq|(x,y)\,(y,z)|-\|y\|^2\,|(x,z)|$$

for any $x,y,z\in X$.

Remark 3. *We observe that if $(\cdot,\cdot)$ is an inner product, then (1.12) reduces to Buzano's inequality obtained in 1974 [1] in a different way.*

For some inequalities in inner product spaces and operators on Hilbert spaces see [3]- [26] and the references therein.

The *numerical radius* $w(T)$ of an operator T on H is given by [27, p. 8]:

$$w(T)=\sup\{|\lambda|,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.13}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators $T:H\to H$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [27, p. 9]:

Theorem 3 (Equivalent norm). *For any $T\in\mathcal{B}(H)$ one has*

$$w(T)\leq\|T\|\leq2w(T). \tag{1.14}$$

Utilising Buzano's inequality we obtained the following inequality for the numerical radius [12] or [13]:

Theorem 4. *Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then*

$$w^2(T) \le \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{1.15}$$

The constant $\frac{1}{2}$ is best possible in (1.15).

The following general result for the product of two operators holds [27, p. 37]:

Theorem 5. *If U, V are two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then $w(UV) \le 4w(U)w(V)$. In the case that $UV = VU$, then $w(UV) \le 2w(U)w(V)$. The constant 2 is best possible here.*

The following results are also well known [27, p. 38].

Theorem 6. *If U is a unitary operator that commutes with another operator V, then*

$$w(UV) \le w(V). \tag{1.16}$$

If U is an isometry and $UV = VU$, then (1.16) also holds true.

We say that U and V *double commute* if $UV = VU$ and $UV^* = V^*U$. The following result holds [27, p. 38].

Theorem 7. *If the operators U and V double commute, then*

$$w(UV) \le w(V)\|U\|. \tag{1.17}$$

As a consequence of the above, we have [27, p. 39]:

Corollary 2. *Let U be a normal operator commuting with V. Then*

$$w(UV) \le w(U)w(V). \tag{1.18}$$

For a recent survey of inequalities for numerical radius, see [21] and the references therein.

Motivated by the above facts we establish in this paper some new numerical radius inequalities concerning four operators A, B, C and P on a Hilbert space with P nonnegative in the operator order. Some particular cases of interest that generalize and improve an earlier result are also provided.

2. Main Results

The following result holds for $(H, \langle ., . \rangle)$ a Hilbert space over the real or complex numbers field $\mathbb{K}$.

THEOREM 8. *Let P be a nonnegative operator on H and A, B, C three bounded operators on H. Then for any $e \in H$ we have the inequalities*

$$\|A^* PCe\| \|B^* PCe\| \le \frac{1}{2} \left\|P^{1/2} Ce\right\|^2 \left[\left\|P^{1/2} A\right\| \left\|P^{1/2} B\right\| + \|B^* PA\|\right]. \tag{2.1}$$

Moreover, we have

$$w\left(C^* PAB^* PC\right) \le \frac{1}{2} \left\|P^{1/2} C\right\|^2 \left[\left\|P^{1/2} A\right\| \left\|P^{1/2} B\right\| + \|B^* PA\|\right]. \tag{2.2}$$

Proof. We observe that if $P \ge 0$, then the mapping $(.,.) : H \times H \to \mathbb{K}$ defined by

$$(x, y)_P := \langle Px, y \rangle$$

is a hermitian form on H and by (1.12) we have the inequality

$$\frac{1}{2} \left[\|x\|_P \|y\|_P + |(x, y)_P|\right] \|e\|_P^2 \ge |(x, e)_P (y, e)_P| \tag{2.3}$$

for any $x, y, e \in H$.

This can be written as

$$\frac{1}{2} \left[\langle Px, x \rangle^{1/2} \langle Py, y \rangle^{1/2} + |\langle Px, y \rangle|\right] \langle Pe, e \rangle \ge |\langle Px, e \rangle \langle Py, e \rangle| \tag{2.4}$$

for any $x, y, e \in H$.

Now if we replace x by Ax, y by By and e by Ce we get

$$\begin{aligned} &\frac{1}{2} \left[\langle PAx, Ax \rangle^{1/2} \langle PBy, By \rangle^{1/2} + |\langle PAx, By \rangle|\right] \langle PCe, Ce \rangle \\ &\ge |\langle PAx, Ce \rangle \langle PBy, Ce \rangle| \end{aligned} \tag{2.5}$$

for any $x, y, e \in H$, which is equivalent to

$$\begin{aligned} &\frac{1}{2} \left[\langle A^* PAx, x \rangle^{1/2} \langle B^* PBy, y \rangle^{1/2} + |\langle B^* PAx, y \rangle|\right] \langle C^* PCe, e \rangle \\ &\ge |\langle x, A^* PCe \rangle \langle y, B^* PCe \rangle| \end{aligned} \tag{2.6}$$

for any $x, y, e \in H$.

Taking the supremum over $x, y \in H$ with $\|x\| = \|y\| = 1$ we have

(2.7)

$$\begin{aligned}
&\|A^*PCe\| \, \|B^*PCe\| \\
&= \sup_{\|x\|=1} |\langle x, A^*PCe\rangle| \sup_{\|y\|=1} |\langle y, B^*PCe\rangle| \\
&= \sup_{\|x\|=\|y\|=1} \{|\langle x, A^*PCe\rangle \langle y, B^*PCe\rangle|\} \\
&\le \frac{1}{2} \langle C^*PCe, e\rangle \\
&\times \sup_{\|x\|=\|y\|=1} \left[\langle A^*PAx, x\rangle^{1/2} \langle B^*PBy, y\rangle^{1/2} + |\langle B^*PAx, y\rangle|\right] \\
&\le \frac{1}{2} \langle C^*PCe, e\rangle \\
&\times \left[\sup_{\|x\|=1} \langle A^*PAx, x\rangle^{1/2} \sup_{\|y\|=1} \langle B^*PBy, y\rangle^{1/2} + \sup_{\|x\|=\|y\|=1} |\langle B^*PAx, y\rangle|\right] \\
&= \frac{1}{2} \langle C^*PCe, e\rangle \left[\|A^*PA\|^{1/2} \|B^*PB\|^{1/2} + \|B^*PA\|\right]
\end{aligned}$$

for any $e \in H$.

Since

$$A^*PA = \left|P^{1/2}A\right|^2, \; B^*PB = \left|P^{1/2}B\right|^2$$

and

$$C^*PC = \left|P^{1/2}C\right|^2$$

then by (2.7) we get the desired inequality in (2.1).

By Schwarz inequality we have

$$|\langle C^*PBA^*PCe, e\rangle| \le \|A^*PCe\| \, \|B^*PCe\| \tag{2.8}$$

for any $e \in H$.

Using inequality (2.1) we then have

(2.9)

$$|\langle C^*PBA^*PCe, e\rangle| \le \frac{1}{2} \left\|P^{1/2}Ce\right\|^2 \left[\left\|P^{1/2}A\right\| \left\|P^{1/2}B\right\| + \|B^*PA\|\right]$$

for any $e \in H$.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (2.9) we get

$$w\left(C^*PBA^*PC\right) \le \frac{1}{2} \left\|P^{1/2}C\right\|^2 \left[\left\|P^{1/2}A\right\| \left\|P^{1/2}B\right\| + \|B^*PA\|\right] \tag{2.10}$$

and since

$$w\left(C^{*}PBA^{*}PC\right)=w\left(C^{*}PAB^{*}PC\right)$$

then by (2.10) we get the desired result (2.2). □

The following result also holds.

Theorem 9. *Let P be a nonnegative operator on H and A, B, C three bounded operators on H such that $B^{*}PC=C^{*}PA$, then*

$$w^{2}\left(C^{*}PA\right)\leq\frac{1}{2}\left\|P^{1/2}C\right\|^{2}\left[\left\|P^{1/2}A\right\|\left\|P^{1/2}B\right\|+w\left(B^{*}PA\right)\right] \tag{2.11}$$

and

$$w^{2}\left(C^{*}PA\right)\leq\frac{1}{2}\left\|P^{1/2}C\right\|^{2}\left[\left\|\frac{\left|P^{1/2}A\right|^{2}+\left|P^{1/2}B\right|^{2}}{2}\right\|+w\left(B^{*}PA\right)\right]. \tag{2.12}$$

Proof. From the inequality (2.6) we have

$$\begin{aligned}&\frac{1}{2}\left[\left\langle A^{*}PAe,e\right\rangle^{1/2}\left\langle B^{*}PBe,e\right\rangle^{1/2}+\left|\left\langle B^{*}PAe,e\right\rangle\right|\right]\left\langle C^{*}PCe,e\right\rangle\\&\geq\left|\left\langle e,A^{*}PCe\right\rangle\left\langle e,B^{*}PCe\right\rangle\right|\end{aligned} \tag{2.13}$$

for any $e\in H$.

Since

$$B^{*}PC=C^{*}PA=\left(A^{*}PC\right)^{*}$$

then

$$\begin{aligned}\left|\left\langle e,A^{*}PCe\right\rangle\left\langle e,B^{*}PCe\right\rangle\right|&=\left|\left\langle e,A^{*}PCe\right\rangle\left\langle e,\left(A^{*}PC\right)^{*}e\right\rangle\right|\\&=\left|\left\langle A^{*}PCe,e\right\rangle\right|^{2}=\left|\left\langle C^{*}PAe,e\right\rangle\right|^{2}\end{aligned} \tag{2.14}$$

for any $e\in H$.

By (2.13) and (2.14) we then have

$$\begin{aligned}&\left|\left\langle C^{*}PAe,e\right\rangle\right|^{2}\\&\leq\frac{1}{2}\left[\left\langle A^{*}PAe,e\right\rangle^{1/2}\left\langle B^{*}PBe,e\right\rangle^{1/2}+\left|\left\langle B^{*}PAe,e\right\rangle\right|\right]\left\langle C^{*}PCe,e\right\rangle\end{aligned} \tag{2.15}$$

for any $e\in H$. This inequality is of interest in itself.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (2.15) we have

$$\begin{aligned}
& w^2 (C^* PA) \\
& = \sup_{\|e\|=1} |\langle C^* PAe, e\rangle|^2 \\
& \le \frac{1}{2} \sup_{\|e\|=1} \left\{ \left[\langle A^* PAe, e\rangle^{1/2} \langle B^* PBe, e\rangle^{1/2} + |\langle B^* PAe, e\rangle| \right] \langle C^* PCe, e\rangle \right\} \\
& \le \frac{1}{2} \sup_{\|e\|=1} \left[\langle A^* PAe, e\rangle^{1/2} \langle B^* PBe, e\rangle^{1/2} + |\langle B^* PAe, e\rangle| \right] \sup_{\|e\|=1} \langle C^* PCe, e\rangle \\
& \le \frac{1}{2} \left[\sup_{\|e\|=1} \langle A^* PAe, e\rangle^{1/2} \sup_{\|e\|=1} \langle B^* PBe, e\rangle^{1/2} + \sup_{\|e\|=1} |\langle B^* PAe, e\rangle| \right] \\
& \times \sup_{\|e\|=1} \langle C^* PCe, e\rangle \\
& = \frac{1}{2} \left[\|A^* PA\|^{1/2} \|B^* PB\|^{1/2} + w (B^* PA) \right] \|C^* PC\| ,
\end{aligned}$$

which proves the inequality (2.11).

Using the *arithmetic mean - geometric mean* inequality we also have

$$\begin{aligned}
\langle A^* PAe, e\rangle^{1/2} \langle B^* PBe, e\rangle^{1/2} & \le \frac{1}{2} [\langle A^* PAe, e\rangle + \langle B^* PBe, e\rangle] \\
& = \left\langle \frac{A^* PA + B^* PB}{2} e, e \right\rangle
\end{aligned}$$

for any $e \in H$.

By (2.15) we then have

(2.16)
$$|\langle C^* PAe, e\rangle|^2 \le \frac{1}{2} \left[\left\langle \frac{A^* PA + B^* PB}{2} e, e \right\rangle + |\langle B^* PAe, e\rangle| \right] \langle C^* PCe, e\rangle$$

for any $e \in H$.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (2.16) we obtain the desired result (2.12). □

3. Some Particular Inequalities

In this section we explore some particular inequalities of interest that can be obtained from the main results stated above.

If we take in (2.1) and (2.2) $B = A^*$, then we get

$$\|A^* P C e\| \|A P C e\| \leq \frac{1}{2} \left\|P^{1/2} C e\right\|^2 \left[\left\|P^{1/2} A\right\| \left\|A P^{1/2}\right\| + \|A P A\|\right] \tag{3.1}$$

for any $e \in H$ and

$$w\left(C^* P A^2 P C\right) \leq \frac{1}{2} \left\|P^{1/2} C\right\|^2 \left[\left\|P^{1/2} A\right\| \left\|A P^{1/2}\right\| + \|A P A\|\right], \tag{3.2}$$

where A, C are bounded operators on H and P is a nonnegative operator on H.

If we put in (2.1) and (2.2) $P = 1_H$, then we have

$$\|A^* C e\| \|B^* C e\| \leq \frac{1}{2} \|C e\|^2 [\|A\| \|B\| + \|B^* A\|] \tag{3.3}$$

for any $e \in H$ and

$$w(C^* A B^* C) \leq \frac{1}{2} \|C\|^2 [\|A\| \|B\| + \|B^* A\|] \tag{3.4}$$

where A, B, C are bounded operators on H.

Choosing $B = A^*$ in (3.3) and (3.4), we get

$$\|A^* C e\| \|A C e\| \leq \frac{1}{2} \|C e\|^2 \left[\|A\|^2 + \left\|A^2\right\|\right] \tag{3.5}$$

for any $e \in H$ and

$$w\left(C^* A^2 C\right) \leq \frac{1}{2} \|C\|^2 \left[\|A\|^2 + \left\|A^2\right\|\right]. \tag{3.6}$$

If we take in (2.1) and (2.2) $C = 1_H$, then we get

$$\|A^* P e\| \|B^* P e\| \leq \frac{1}{2} \left\|P^{1/2} e\right\|^2 \left[\left\|P^{1/2} A\right\| \left\|P^{1/2} B\right\| + \|B^* P A\|\right] \tag{3.7}$$

for any $e \in H$ and

$$w(P A B^* P) \leq \frac{1}{2} \|P\| \left[\left\|P^{1/2} A\right\| \left\|P^{1/2} B\right\| + \|B^* P A\|\right], \tag{3.8}$$

where A, B are bounded operators on H and P is a nonnegative operator on H. Moreover, if in (3.7) and (3.8) we take $B = A^*$, then we get the inequalities

$$\|A^* P e\| \|A P e\| \leq \frac{1}{2} \left\|P^{1/2} e\right\|^2 \left[\left\|P^{1/2} A\right\| \left\|A P^{1/2}\right\| + \|A P A\|\right] \tag{3.9}$$

for any $e \in H$ and

$$w\left(P A^2 P\right) \leq \frac{1}{2} \|P\| \left[\left\|P^{1/2} A\right\| \left\|A P^{1/2}\right\| + \|A P A\|\right]. \tag{3.10}$$

Further, if we assume that $APC = C^*PA$, then by taking $B = A^*$ in (2.11) and (2.12) we get

$$w^2(APC) \le \frac{1}{2}\left\|P^{1/2}C\right\|^2\left[\left\|P^{1/2}A\right\|\left\|AP^{1/2}\right\| + w(APA)\right] \tag{3.11}$$

and

$$w^2(APC) \le \frac{1}{2}\left\|P^{1/2}C\right\|^2\left[\left\|\frac{\left|P^{1/2}A\right|^2 + \left|P^{1/2}A^*\right|^2}{2}\right\| + w(APA)\right]. \tag{3.12}$$

If $AC = C^*A$, then by taking $P = 1_H$ in (3.11) and (3.12) we have

$$w^2(AC) \le \frac{1}{2}\|C\|^2\left[\|A\|^2 + w\left(A^2\right)\right] \tag{3.13}$$

and

$$w^2(AC) \le \frac{1}{2}\|C\|^2\left[\left\|\frac{|A|^2 + |A^*|^2}{2}\right\| + w\left(A^2\right)\right]. \tag{3.14}$$

Since

$$\left\|\frac{|A|^2 + |A^*|^2}{2}\right\| \le \frac{1}{2}\left[\left\||A|^2\right\| + \left\||A^*|^2\right\|\right] = \|A\|^2,$$

then the inequality (3.14) is better than (3.13).

If $AP = PA$, then by taking $C = 1_H$ in (3.11) and (3.12) we also have

$$w^2(AP) \le \frac{1}{2}\|P\|\left[\left\|P^{1/2}A\right\|\left\|AP^{1/2}\right\| + w\left(PA^2\right)\right] \tag{3.15}$$

and

$$w^2(AP) \le \frac{1}{2}\|P\|\left[\left\|\frac{\left|P^{1/2}A\right|^2 + \left|P^{1/2}A^*\right|^2}{2}\right\| + w\left(PA^2\right)\right]. \tag{3.16}$$

Taking into account the above results, we can state the following two inequalities for an operator T, namely

$$w^2(T) \le \frac{1}{2}\left[\|T\|^2 + w\left(T^2\right)\right], \text{ see (1.15)}, \tag{3.17}$$

and

$$w^2(T) \le \frac{1}{2}\left[\left\|\frac{|T|^2 + |T^*|^2}{2}\right\| + w\left(T^2\right)\right]. \tag{3.18}$$

The inequality (3.18) is better than (3.17).

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