COLORED PERMUTATIONS WITH NO MONOCHROMATIC CYCLES

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Abstract. An \((n_1, n_2, \ldots, n_k)\)-colored permutation is a permutation of \(n_1 + n_2 + \cdots + n_k\) in which 1, 2, \ldots, \(n_1\) have color 1, and \(n_1 + 1, n_1 + 2, \ldots, n_1 + n_2\) have color 2, and so on. We give a bijective proof of Steinhardt's result: the number of colored permutations with no monochromatic cycles is equal to the number of permutations with no fixed points after reordering the first \(n_1\) elements, the next \(n_2\) element, and so on, in ascending order. We then find the generating function for colored permutations with no monochromatic cycles. As an application we give a new proof of the well known generating function for colored permutations with no fixed colors, also known as multi-derangements.

1. Introduction

Let \(S_n\) denote the set of permutations of \([n] := \{1, 2, \ldots, n\}\). Let \(\pi = \pi_1 \pi_2 \cdots \pi_n\) be a permutation in \(S_n\). An integer \(i \in [n]\) is called a fixed point of \(\pi\) if \(\pi_i = i\). A derangement is a permutation with no fixed points. An integer \(i \in [n-1]\) is called a descent of \(\pi\) if \(\pi_i > \pi_{i+1}\), and an ascent of \(\pi\) if \(\pi_i < \pi_{i+1}\).

If the set of descents of \(\pi\) is equal to \(\{1, 3, 5, \ldots\} \cap [n-1]\), then \(\pi\) is called an alternating permutation. There are many interesting properties of alternating permutations, see [10].

More generally, if \(B = \{b_1, b_2, \ldots, b_n\}\) is an \(n\)-set with \(b_1 < b_2 < \cdots < b_n\), a rearrangement \(\sigma = s_1 s_2 \cdots s_n\) of elements of \(B\) is called a permutation of \(B\). Let \(S_B\) denote the set of all permutations of \(B\). The statistics ascent in \(S_B\) can be defined as in \(S_n\), i.e., \(i\) is an ascent of \(\sigma\) if \(s_i < s_{i+1}\).

In [9, Conjecture 6.3] Stanley conjectured that for \(n \geq 2\), the number of alternating permutations of \([2n]\) with maximum number of fixed points, which is \(n\), is equal to the number of derangements of \([n]\). This conjecture was proved by Chapman and Williams [2]. Han and Xin [6, Theorem 1] generalized Stanley’s conjecture by enumerating the number of permutations \(\pi \in S_n\) such that the set of descents is \(J\) and the number of fixed points is \(n - |J|\), which is the largest possible, for any set \(J \subseteq [n-1]\). They showed that this number is equal to the
number of derangements of \( n \) order, the next \( n \) \( \pi \) (respectively in descending order), then \( i \) points stands for \( N \), and so on. A cycle of an \( \pi \) Theorem 2.1. In fact we will show a more general formula using permutation statistics, see the next \( n \) \( \pi \) permutation obtained from \( NMCy(\pi) \) if the elements of the cycle have the same color. We denote by monochromatic \( \pi' \) has no fixed points. Here, \( \text{NFIA} \) stands for \( \text{No Fixed points in Ascending order} \) and \( \text{NFID} \) stands for \( \text{No Fixed points in Descending order} \). Note that \( |\text{NFId}(n_1, n_2, \ldots, n_k)|/n_1! \cdots n_k! \) is the number of derangements of \( n \) such that the first \( n_1 \) elements are in ascending order, the next \( n_2 \) elements are in ascending order, and so on.

Using symmetric functions, Han and Xin [6, Theorem 9] showed that

\[
\sum_{n_1, n_2, \ldots, n_k \geq 0} |\text{NFIA}(n_1, n_2, \ldots, n_k)| x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} n_1! n_2! \cdots n_k! = \frac{1}{(1 + x_1) \cdots (1 + x_k)(1 - x_1 - \cdots - x_k)}.
\]

Eriksen, Freij, and Wästlund [3, Section 2] found a combinatorial proof of (1). Steinhardt [12, Corollary 4.2] proved the following analogous result of (1):

\[
\sum_{n_1, n_2, \ldots, n_k \geq 0} |\text{NFID}(n_1, n_2, \ldots, n_k)| x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} n_1! n_2! \cdots n_k! = \frac{(1 - x_1) \cdots (1 - x_k)}{1 - x_1 - \cdots - x_k}.
\]

In this paper we show that the left hand side of (2) has a natural interpretation in terms of colored permutations defined below. The key idea is the compositional formula for multivariate exponential generating functions.

An \( (n_1, n_2, \ldots, n_k) \)-colored permutation is a permutation in \( S_{n_1+n_2+\cdots+n_k} \) such that 1, 2, \ldots, \( n_1 \) have color 1, and \( n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 \) have color 2, and so on. A cycle of an \( (n_1, n_2, \ldots, n_k) \)-colored permutation is called monochromatic if the elements of the cycle have the same color. We denote by \( \text{NMCy}(n_1, n_2, \ldots, n_k) \) the set of \( (n_1, n_2, \ldots, n_k) \)-colored permutations with no monochromatic cycles (\( \text{NMCy} \) stands for \( \text{No Monochromatic Cycles} \)).

In Section 2 we show that

\[
\sum_{n_1, n_2, \ldots, n_k \geq 0} |\text{NMCy}(n_1, n_2, \ldots, n_k)| x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} n_1! n_2! \cdots n_k! = \frac{(1 - x_1) \cdots (1 - x_k)}{1 - x_1 - \cdots - x_k}.
\]

In fact we will show a more general formula using permutation statistics, see Theorem 2.1.
For an application of (3) we consider the set \( \text{NFCo}(n_1, n_2, \ldots, n_k) \) of \((n_1, n_2, \ldots, n_k)\)-colored permutations \( \pi \) such that \( i \) and \( \pi_i \) have different colors for every \( i \). Here, NFCo stands for No Fixed Colors. Such permutations are also called multi-derangements. By finding a simple relation between the generating functions for \( |\text{NMcy}(n_1, n_2, \ldots, n_k)| \) and \( |\text{NFCo}(n_1, n_2, \ldots, n_k)| \), we obtain a new proof of the following well known formula

\[
\sum_{n_1, n_2, \ldots, n_k \geq 0} |\text{NFCo}(n_1, n_2, \ldots, n_k)| \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1! n_2! \cdots n_k!} = 1 - e_2 - 2e_3 - \cdots - (k-1)e_k,
\]

where \( e_i \) is the \( i \)-th elementary symmetric function on \( x_1, x_2, \ldots, x_k \), which is defined by

\[
e_i := \sum_{1 \leq j_1 < \cdots < j_i \leq k} x_{j_1} \cdots x_{j_i}.
\]

We will show a more general formula using permutation statistics, see Theorem 3.1.

Note that by (2) and (3) we have

\[
|\text{NFiA}(n_1, n_2, \ldots, n_k)| = |\text{NMcy}(n_1, n_2, \ldots, n_k)|.
\]

Steinhardt [12, Theorem 6.2] also proved (5) but his proof is not bijective, see Remark 1. In Section 4 we give a bijective proof of (5).

2. The generating function for \( \text{NMCy}(n_1, n_2, \ldots, n_k) \)

For a permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \) of \([n]\), an \textit{excedance} of \( \pi \) is an integer \( i \in \{1, 2, \ldots, n\} \) such that \( \pi_i > i \). We will denote by \( \text{exc}(\pi) \) and \( \text{cyc}(\pi) \) the number of excedances of \( \pi \) and the number of cycles of \( \pi \) respectively. Define a generating function for \( \text{NMCy}(n_1, n_2, \ldots, n_k) \) by

\[
f_{\text{NMCy}}(x_1, x_2, \ldots, x_k; y, z) := \sum_{n_1, n_2, \ldots, n_k \geq 0} \left( \sum_{\pi \in \text{NMcy}(n_1, n_2, \ldots, n_k)} y^{\text{exc}(\pi)} z^{\text{cyc}(\pi)} \right) \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1! n_2! \cdots n_k!}.
\]

In this section we show the following theorem.

Theorem 2.1. We have

\[
f_{\text{NMCy}}(x_1, x_2, \ldots, x_k; y, z) = \left( (1-y)^{1-k} \frac{(1-y^{(1-y)x_1}) \cdots (1-y^{(1-y)x_k})}{1-y^{(1-y)(x_1+\cdots+x_k)}} \right) z.
\]

Note that if \( y \to 1 \) and \( z \to 1 \) in Theorem 2.1, we obtain (3).

Recall that for a permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \), an \textit{ascent} of \( \pi \) is an integer \( i \in \{1, 2, \ldots, n-1\} \) such that \( \pi_i < \pi_{i+1} \). Let \( \text{asc}(\pi) \) denote the number of
ascent of \( \pi \). It is well known that the two statistics \( \text{exc}(\pi) \) and \( \text{asc}(\pi) \) are equidistributed in \( S_n \), see [11, Proposition 1.4.3]. Let \( A_n(y) \) be the Eulerian polynomial defined by

\[
A_n(y) := \sum_{\pi \in S_n} y^{\text{exc}(\pi)} = \sum_{\pi \in S_n} y^{\text{asc}(\pi)}.
\]

We denote by \( C_n \) the set of \( n \)-cycles formed with \( 1, 2, \ldots, n \).

**Lemma 2.2.** We have

\[
(6) \quad \sum_{n \geq 0} A_n(y) \frac{x^n}{n!} = \frac{(1 - y)e^{(1-y)x}}{1 - ye^{(1-y)x}},
\]

\[
(7) \quad \sum_{n \geq 1} \left( \sum_{\pi \in C_n} y^{\text{exc}(\pi)} \right) \frac{x^n}{n!} = \log \frac{1 - y}{1 - ye^{(1-y)x}}.
\]

**Proof.** Equation (6) is well known, see [11, Proposition 1.4.5]. For (7), observe that if we write an \( n \)-cycle \( \pi \in C_n \) as \( \pi = (n, a_1, a_2, \ldots, a_{n-1}) \), then \( \text{exc}(\pi) = 1 + \text{asc}(a_1a_2\cdots a_{n-1}) \). Thus we have

\[
\sum_{\pi \in C_n} y^{\text{exc}(\pi)} = \sum_{\sigma \in S_{n-1}} y^{1 + \text{asc}(\sigma)} = yA_{n-1}(y).
\]

Integrating both sides of (6) with respect to \( x \), we obtain

\[
\sum_{n \geq 1} A_{n-1}(y) \frac{x^n}{n!} = \frac{1}{y} \log \frac{1 - y}{1 - ye^{(1-y)x}},
\]

which finishes the proof of (7). \( \square \)

We now prove Theorem 2.1.

**Proof of Theorem 2.1.** We claim that

\[
(8) \quad \sum_{n \geq 0} \frac{X^n}{n!} \sum_{n_1 + \cdots + n_k = n} \binom{n}{n_1, \ldots, n_k} x_1^{n_1} \cdots x_k^{n_k} \sum_{\pi \in \text{NMCD}_n(n_1, n_2, \ldots, n_k)} y^{\text{exc}(\pi)} z^{\text{cyc}(\pi)}
\]

A \( k \)-colored permutation is a permutation in which every integer has color \( i \) for some \( i = 1, 2, \ldots, k \). Then the left hand side of (8) is equal to

\[
(9) \quad \sum_{n \geq 0} \frac{X^n}{n!} \sum_{\pi: \text{a } k\text{-colored permutation of } [n]} \text{wt}(\pi),
\]

where

\[
\text{wt}(\pi) = \prod_{i=1}^{k} x_i^{\# \text{elements of color } i \text{ in } \pi} y^{\text{exc}(\pi)} z^{\text{cyc}(\pi)}.
\]
Since a \( k \)-colored permutation \( \pi \) is divided into cycles, by the exponential formula [8, Corollary 5.1.6], (9) is equal to

\[
\exp \left( \sum_{n \geq 1} \frac{X^n}{n!} \sum_{\pi, \text{ a } k\text{-colored cycle of } [n] \text{ with at least two colors}} \text{wt}(\pi) \right),
\]

which is equal to the right hand side of (8).

Setting \( X = 1 \) in (8) and using (7), we get the desired formula. \( \square \)

3. The generating function for \( \text{NFCo}(n_1, n_2, \ldots, n_k) \)

Define a generating function for \( \text{NFCo}(n_1, n_2, \ldots, n_k) \) by

\[
f_{\text{NFCo}}(x_1, x_2, \ldots, x_k; y, z) := \sum_{n_1, n_2, \ldots, n_k \geq 0} \left( \sum_{\pi \in \text{NFCo}(n_1, n_2, \ldots, n_k)} y^{\text{exc}(\pi)} z^{\text{cyc}(\pi)} \right) \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1! n_2! \cdots n_k!}.
\]

In this section we will prove the following theorem.

**Theorem 3.1.** We have

\[
f_{\text{NFCo}}(x_1, x_2, \ldots, x_k; y, z) = (1 - y e_{2} - (y + y^2) e_{3} - \cdots - (y + y^2 + \cdots + y^{k-1}) e_{k})^{-z}.
\]

Askey and Ismail [1] showed (10) when \( z = 1 \) using MacMahon’s master theorem. Foata and Zeilberger [4] showed (10) when \( y = 1 \) using the \( \beta \)-extension of MacMahon’s master theorem. Kim and Zeng [7] found a combinatorial proof of (10) when \( z = 1 \). Zeng [13] showed (10) without restriction using the \( \beta \)-extension of MacMahon’s master theorem. Zeng [14] proved (10) by decomposing multi-derangements into “wave segments”.

We will show (10) by finding a relation between \( f_{\text{NMCy}}(x_1, x_2, \ldots, x_k) \) and \( f_{\text{NFCo}}(x_1, x_2, \ldots, x_k) \). We need a multivariate analog of the compositional formula [8, Theorem 5.1.4].

Let \( \Pi(n) \) be the set of partitions of \( \{1, 2, \ldots, n\} \). For \( \mu \in \Pi(n) \), the number of blocks of \( \mu \) is denoted by \( |\mu| \). We use the convention that the empty product is 1. For instance, if \( S = \emptyset \), then \( \prod_{i \in S} g(i) = 1 \) for any function \( g \). Lemma 3.2 is a multivariate compositional formula. This can be shown by the same arguments as in the proof of [8, Theorem 5.1.4].

**Lemma 3.2 (A multivariate compositional formula).** Suppose that

\[
G(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k \geq 0} g(n_1, n_2, \ldots, n_k) \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1! n_2! \cdots n_k!}
\]

is a multivariate formal power series, and for \( i = 1, 2, \ldots, k \),

\[
F_i(x) = \sum_{n \geq 1} f_i(n) \frac{x^n}{n!}
\]

is a formal power series.

Then

\[
fG(x_1, x_2, \ldots, x_k) := \sum_{n \geq 1} f(n) \frac{x^n}{n!} \]

is a formal power series such that

\[
G(x_1, x_2, \ldots, x_k) = \prod_{i=1}^{k} (1 - x_i) + \sum_{n \geq 1} f(n) \frac{x^n}{n!} \prod_{i=1}^{k} \left(\sum_{j=1}^{n} x_i^j \right).
\]
is a formal power series. Let

\[ H(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k \geq 0} h(n_1, n_2, \ldots, n_k) \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1! n_2! \cdots n_k!} \]

be the multivariate formal power series, where

\[ h(n_1, n_2, \ldots, n_k) = \sum_{\mu \in B(n_i)} g(|\mu_1|, |\mu_2|, \ldots, |\mu_k|) \prod_{B \in \mu_i} f_i(\mu_i). \]

Then we have

\[ H(x_1, x_2, \ldots, x_k) = G(F_1(x_1), F_2(x_2), \ldots, F_k(x_k)). \]

**Proposition 3.3.** We have

\[ f_{\text{NMCy}}(x_1, x_2, \ldots, x_k; y, z) = f_{\text{FCo}} \left( \frac{e^{(1-y)x_1} - 1}{1 - ye^{(1-y)x_1}}, \ldots, \frac{e^{(1-y)x_k} - 1}{1 - ye^{(1-y)x_k}}; y, z \right), \]

\[ f_{\text{FCo}}(x_1, x_2, \ldots, x_k; y, z) = f_{\text{NMCy}} \left( \frac{1 + x_1}{1 - y}, \ldots, \frac{1 + x_k}{1 - y} ; y, z \right). \]

**Proof.** The second identity is obtained from the first one by substituting \( x'_i = \frac{e^{(1-y)x_i} - 1}{1 - ye^{(1-y)x_i}} \), which is equivalent to \( x_i = \frac{1}{1-y} \log \frac{1+x_i}{1+y x_i} \). Thus it suffices to show (11).

Let \( \pi \in \text{NMCy}(n_1, n_2, \ldots, n_k) \), and consider a cycle \( \gamma \) of \( \pi \). Since \( \pi \) has no monochromatic cycles, the cycle \( \gamma \) contains more than one colors. We split \( \gamma \) into intervals, \( \sigma_1, \sigma_2, \ldots, \sigma_r \), in such a way that \( \gamma \) is the concatenation of \( \sigma_1, \sigma_2, \ldots, \sigma_r \), and each \( \sigma_i \) is monochromatic, and for each \( i \) the color of \( \sigma_i \) differs from that of \( \sigma_{i+1} \) with convention \( \sigma_{r+1} = \sigma_1 \). We call each \( \sigma_i \) a maximal monochromatic interval in \( \gamma \), and regard it, being a sequence of distinct integers, as a permutation of its elements. Then \( \gamma \) can be regarded as an \( r \)-cycle (\( \sigma_1, \sigma_2, \ldots, \sigma_r \)) of permutations \( \sigma_1, \sigma_2, \ldots, \sigma_r \).

We now identify \( \gamma \) with the pair \( (T, \tau) \), where \( T = \{ \sigma_1, \sigma_2, \ldots, \sigma_r \} \) is the set of maximal monochromatic intervals defined above and \( \tau \) is the \( r \)-cycle (\( \sigma_1, \sigma_2, \ldots, \sigma_r \)). It is easy to see that

\[ \text{exc}(\gamma) = \text{exc}(\tau) + \sum_{i=1}^r \text{asc}(\sigma_i), \]

where \( \text{exc}(\tau) \) is defined based on the linear order on \( \sigma_1, \ldots, \sigma_r \) by \( \sigma_i \succ \sigma_j \) if the first element of \( \sigma_i \) is bigger than that of \( \sigma_j \).

Let \( \{ \gamma_1, \gamma_2, \ldots, \gamma_m \} \) be the set of disjoint cycles of \( \pi \in \text{NMCy}(n_1, n_2, \ldots, n_k) \), where each \( \gamma_i \) is identified with \( (T_i, \tau_i) \). Then \( \{ T_1, T_2, \ldots, T_m \} \), regarded as a disjoint cycle decomposition, is a permutation of \( T_1 \cup T_2 \cup \cdots \cup T_m \).

Thus we can identify \( \pi \) as a pair \( (U, \rho) \) satisfying the following:
• $U := T_1 \cup T_2 \cup \cdots \cup T_m$ is the set of all monochromatic permutations, i.e., maximal monochromatic intervals from disjoint cycles of $\pi$,
• every element $j \in [n_1 + \cdots + n_k]$ appears in exactly one $\sigma$ in $U$ and
• $\rho := \{\tau_1, \tau_2, \ldots, \tau_m\}$ is a permutation of $U$ such that $\sigma$ and $\rho(\sigma)$ have different colors for every $\sigma \in U$, i.e., $\rho$ is a permutation of no fixed color.
Clearly $\text{cyc}(\pi) = \text{cyc}(\rho)$. Also, from (13), we get
$$\text{exc}(\pi) = \text{exc}(\rho) + \sum_{\sigma \in U} \text{asc}(\sigma).$$

Thus we have
$$\sum_{\pi \in \text{NMCy}(n_1, n_2, \ldots, n_k)} y^{\text{exc}(\pi)} z^{\text{cyc}(\pi)} = \sum_{\mu \in \Pi(n_i)} \left( \sum_{\rho \in \text{NFCo}(|\mu|, |\mu_2|, \ldots, |\mu_k|)} y^{\text{exc}(\rho)} z^{\text{cyc}(\rho)} \right) \prod_{B \in \mu} \sum_{\sigma \in S_B} y^{\text{asc}(\sigma)}.$$
Since
$$\sum_{\sigma \in S_B} y^{\text{asc}(\sigma)} = \sum_{\sigma \in S_{|B|}} y^{\text{asc}(\sigma)},$$
by Lemma 3.2 and (6), we obtain (11).

We are ready to give a new proof of Theorem 3.1.

**Proof of Theorem 3.1.** By Proposition 3.3 and Theorem 2.1 we have
$$f_{\text{NFCo}}(x_1, x_2, \ldots, x_k; y, z)$$
$$= f_{\text{NMCy}} \left( \frac{1}{1-y} \log \frac{1+x_1}{1+yx_1}, \ldots, \frac{1}{1-y} \log \frac{1+x_k}{1+yx_k}, y, z \right)$$
$$= \left( (1-y)^{1-k} \prod_{i=1}^k \left( 1 - y \exp \left[ \frac{1-y}{1-y} \log \frac{1+x_i}{1+yx_i} \right] \right) \right)^z \prod_{i=1}^k \left( 1 + yx_i \right) - y \prod_{i=1}^k \left( 1 + x_i \right).$$

Using the fact
$$\prod_{i=1}^k (1 + x_i y) = \sum_{i=0}^k e_i y^i,$$
one can easily see that
$$\prod_{i=1}^k (1 + x_i y) - y \prod_{i=1}^k (1 + x_i)$$
$$= (1 - y) \left( 1 - ye_2 - (y + y^2)e_3 - \cdots - (y + y^2 + \cdots + y^{k-1})e_k \right).$$
ornaments in \( \Omega(\pi_{n}) \) of all its necklaces are primitive. Let \( \Omega_{0} \) where we put a bar ‘\( \bar{\cdot} \)’ between \( \pi_{n} \) and so on, is in ascending order. Note that we can consider \( NFiA_{n} \) as the set
\[
A(n_{1}, n_{2}, \ldots, n_{k}) \times S_{n_{1}} \times \cdots \times S_{n_{k}}.
\]

For example, let \((n_{1}, n_{2}, \ldots, n_{k}) = (8, 5, 1)\) and
\[
\pi = \begin{vmatrix}
8 & 7 & 9 & 12 & 6 & 5 & 11 & 10 & 2 & 3 & 4 & 1 & 14 & 13
\end{vmatrix} \in \text{NFiA}(n_{1}, n_{2}, \ldots, n_{k}),
\]
where we put a bar ‘\( \bar{\cdot} \)’ between \( \pi_{n_{1}} \) and \( \pi_{n_{2}} \) for each \( i = 1, 2, \ldots, k - 1 \), and at the beginning and at the end for visibility. Then \( \pi' \) is the permutation obtained from \( \pi \) by rearranging the integers between two consecutive bars in ascending order:

\[
\pi' = \begin{vmatrix}
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 14 & 13
\end{vmatrix} \in A(n_{1}, n_{2}, \ldots, n_{k}).
\]

We divide \( \pi \) into the \( k \) subwords of lengths \( n_{1}, n_{2}, \ldots, n_{k} \) and then consider them as permutations in \( S_{n_{1}}, S_{n_{2}}, \ldots, S_{n_{k}} \) to get \( \sigma_{1}, \sigma_{2}, \ldots, \sigma_{k} \):
\[
\begin{array}{cccccccc}
8 & 7 & 9 & 12 & 6 & 5 & 11 & 10 & 2 & 3 & 4 & 1 & 14 & 13 & 1 & 2 & 3 & 4 & 5 & = & \sigma_{1}, \\
2 & 3 & 4 & 1 & 14 & \cong & 2 & 3 & 4 & 1 & 5 & = & \sigma_{2}, \\
z & = & 13 & \cong & 1 & = & \sigma_{3}.
\end{array}
\]

Here, for two words \( u = u_{1} \cdots u_{n} \) and \( v = v_{1} \cdots v_{n} \) of integers, we write \( u \equiv v \) if \( u_{i} < v_{i} \) implies \( v_{i} < v_{j} \) and vice versa for all \( i, j \). Then we identify \( \pi \) with \( (\pi', \sigma_{1}, \sigma_{2}, \ldots, \sigma_{k}) \).

We now review Gessel and Reutenauer’s map [5].

A necklace is a cycle of integers with possible repetitions. An ornament is a multiset of necklaces. Let \( \Omega(n_{1}, n_{2}, \ldots, n_{k}) \) denote the set of ornaments \( \omega \) such that \( i \) appears \( n_{i} \) times in the necklaces of \( \omega \) for each \( i \). Let \( \eta = (b_{1}, b_{2}, \ldots, b_{m}) \) be a necklace. Define \( b_{i} \) for all integers \( i \) so that \( b_{i} = b_{j} \) if \( i \equiv j \mod m \). A period of \( \eta \) is an integer \( d \) such that \( b_{i+d} = b_{i} \) for all \( i \). We say that \( \eta \) is \( r \)-repeating if \( r = m/d \), where \( d \) is the smallest period of \( \eta \). A primitive necklace is a 1-repeating necklace. An ornament is called primitive if all of its necklaces are primitive. Let \( \Omega_{0}(n_{1}, n_{2}, \ldots, n_{k}) \) be the set of primitive ornaments in \( \Omega(n_{1}, n_{2}, \ldots, n_{k}) \) with no necklaces containing only one element.

Thus we get
\[
f_{NFC_{0}}(x_{1}, x_{2}, \ldots, x_{k}; y, z) = (1 - ye_{2} - (y + y^{2})e_{3} - \cdots - (y + y^{2} + \cdots + y^{k-1})e_{k})^{-z},
\]
which completes the proof. □

4. Bijections

In this section we give a bijective proof of (5). We will follow Steinhardt’s approach [12] using Gessel and Reutenauer’s map.

Let \( A(n_{1}, n_{2}, \ldots, n_{k}) \) be the set of derangements \( \pi = \pi_{1}\pi_{2} \cdots \pi_{n} \) of \( n = n_{1} + n_{2} + \cdots + n_{k} \) such that each of the \( k \) intervals
\[
\pi_{1}\pi_{2} \cdots \pi_{n_{1}}, \pi_{n_{1}+1}\pi_{n_{1}+2} \cdots \pi_{n_{1}+n_{2}}, \ldots, and so on,
\]
is in ascending order. Note that we can consider \( NFiA(n_{1}, n_{2}, \ldots, n_{k}) \) as the set
\[
A(n_{1}, n_{2}, \ldots, n_{k}) \times S_{n_{1}} \times \cdots \times S_{n_{k}}.
\]
For a permutation $\pi$, we define $\phi_{n_1,n_2,\ldots,n_k}(\pi) \in \Omega(n_1,n_2,\ldots,n_k)$ to be the ornament obtained from the cycles of $\pi$ by replacing $j$ with $i$ if
$$n_1 + \cdots + n_{i-1} + 1 \leq j \leq n_1 + \cdots + n_{i-1} + n_i$$

for all $j \in [n]$. In other words, $\phi_{n_1,n_2,\ldots,n_k}(\pi)$ is the ornament obtained from the cycles of $\pi$ by replacing each element with its color. For example, the permutation $\pi'$ in (14) has the cycles
$$(1,5,9), (2,6,10), (3,7,11), (4,8,12), (13,14).$$
Thus the image of $\pi'$ under this map is
$$\phi_{8,5,1}(\pi') = \{(1,1,2), (1,1,2), (1,1,2), (1,1,2), (2,3)\}.$$

**Proposition 4.1** ([5, Lemma 3.4]). The map $\phi_{n_1,n_2,\ldots,n_k}$ is a bijection between $A(n_1,n_2,\ldots,n_k)$ and $\Omega_0(n_1,n_2,\ldots,n_k)$.

By Proposition 4.1, (5) is equivalent to
$$n_1!n_2!\cdots n_k!|\Omega_0(n_1,n_2,\ldots,n_k)| = |\text{NMCy}(n_1,n_2,\ldots,n_k)|.$$

**Remark 1.** In the sketch of proof of [12, Theorem 6.2] Steinhardt states (16) without explanation. However, (16) is nontrivial since $\text{NMCy}(n_1,n_2,\ldots,n_k)$ has no obvious symmetries giving the factor $n_1!n_2!\cdots n_k!$.

We will give a bijective proof of (16). We define the map
$$\psi : \Omega_0(n_1,n_2,\ldots,n_k) \times S_{n_1} \times \cdots \times S_{n_k} \to \text{NMCy}(n_1,n_2,\ldots,n_k)$$
as follows.

1. Let $(\omega,\sigma_1,\ldots,\sigma_k) \in \Omega_0(n_1,n_2,\ldots,n_k) \times S_{n_1} \times \cdots \times S_{n_k}$. Any necklace in $\omega$ can be represented by the word that is the smallest in lexicographic order among the words read from it. Let $\gamma_1,\ldots,\gamma_m$ be the sequence of words obtained by reading the necklaces in $\omega$ such that each $\gamma_i$ is the smallest word which makes the corresponding necklace and $\gamma_1 \leq \cdots \leq \gamma_m$ in lexicographic order.
2. For a permutation $\sigma$ and an integer $j$, let $\sigma+j$ denote the word obtained from $\sigma$ by increasing each integer by $j$. For $1 \leq i \leq k$, let $\sigma'_i = \sigma_i + (n_1 + \cdots + n_{i-1})$, where $n_0 = 0$.
3. Note that, for each $i$, the integer $i$ appears $n_i$ times in $\gamma_1,\ldots,\gamma_m$.
4. Let $S \subset [n]$ be a maximal set subject to $\gamma_i = \gamma_j$ for all $i,j \in S$. Then $S = \{s+1, s+2, \ldots, s+r\}$ for some integers $s$ and $r$. Let $\tau = \tau_1 \cdots \tau_r \in S_r$ be the permutation such that $\tau_i < \tau_j$ if and only if $\rho_{s+i} < \rho_{s+j}$ in lexicographic order. In this case we say that $\tau$ and $\rho_{s+1},\ldots,\rho_{s+r}$ are *order-isomorphic*. Let $G_S$ be the set of cycles obtained from the cycles of $\tau$ by replacing $\tau_i$ with $\rho_{s+i}$ for all $i$. We define $\psi(\omega,\sigma_1,\ldots,\sigma_k)$ to be

$$\psi(\omega,\sigma_1,\ldots,\sigma_k) = \psi((\omega,\sigma_1,\ldots,\sigma_k), \text{NMCy}(n_1,n_2,\ldots,n_k)).$$
the permutation whose cycles are the elements of the union of $C_S$ for all $S$.

**Example 1.** Let $(n_1, n_2, \ldots, n_k) = (8, 5, 1)$. Let

$$\omega = \{(1, 1, 2), (1, 1, 2), (1, 1, 2), (1, 1, 2), (2, 3)\}$$

be the ornament in (15) and $\sigma_1 = 43582176$, $\sigma_2 = 23415$ and $\sigma_3 = 1$ as before. Note that

$$\gamma_1, \ldots, \gamma_5 = 112, 112, 112, 23,$$

and

$$\sigma'_1 = \sigma_1 = 4\ 3\ 5\ 8\ 2\ 1\ 7\ 6,$$

$$\sigma'_2 = \sigma_2 + n_1 = 10\ 11\ 12\ 9\ 13,$$

$$\sigma'_3 = \sigma_3 + (n_1 + n_2) = 14.$$

By replacing the eight 1’s with $\sigma'_1$, the five 2’s with $\sigma'_2$, and the one 3 with $\sigma'_3$ in (17), we have

$$\rho_1, \ldots, \rho_5 = 4\ 3\ 10,\ 5\ 8\ 11,\ 2\ 1\ 12,\ 7\ 6\ 9,\ 13\ 14,$$

where the elements of $\sigma'_1$ are written in bold face. Since $\gamma_1 = \cdots = \gamma_4$, we consider $\rho_1, \ldots, \rho_4$ which is order-isomorphic to $2314 = (123)(4) \in S_4$. Thus we construct the cycles

$$(\rho_1, \rho_2, \rho_3) = (4, 3, 10, 5, 8, 11, 2, 1, 12),\quad (\rho_4) = (7, 6, 9).$$

Thus,

$$\psi(\omega, \sigma_1, \sigma_2, \sigma_3) = (4, 3, 10, 5, 8, 11, 2, 1, 12)(7, 6, 9)(13, 14).$$

**Theorem 4.2.** The map

$$\psi : \Omega_0(n_1, n_2, \ldots, n_k) \times S_{n_1} \times \cdots \times S_{n_k} \to \text{NMCy}(n_1, n_2, \ldots, n_k)$$

is a bijection.

**Proof.** We will show this theorem by constructing the inverse map of $\psi$.

Let $\pi \in \text{NMCy}(n_1, n_2, \ldots, n_k)$. We define a map $\pi \mapsto (\omega, \sigma_1, \ldots, \sigma_k)$ as follows.

1. Let $H$ be the set of words $\gamma$ on $\{1, 2, \ldots, k\}$ such that

   - $\phi_{n_1, n_2, \ldots, n_k}(\pi)$ contains the necklace $(\overbrace{\gamma, \ldots, \gamma}^{j})$ for some integer $j \geq 1$,
   - $(\gamma)$ is primitive and $\gamma$ is the smallest word among all of its cyclic shifts in lexicographic order,

   where we regard a word $\gamma$ as a sequence of integers in the natural way.
(2) For $\gamma \in H$, we define $T_\gamma$ to be the set of all words $\rho$ satisfying that $\rho$ is a consecutive subsequence in some cycle of $\pi$ and $\phi_{n_1, n_2, \ldots, n_k}(\rho) = \gamma$. Here, $\phi_{n_1, n_2, \ldots, n_k}(\rho)$ denotes the word obtained from $\rho$ by replacing each number in $\rho$, say $j$, with $i$ if
\[ n_1 + \cdots + n_{i-1} + 1 \leq j \leq n_1 + \cdots + n_{i-1} + n_i. \]

(3) For $\gamma \in H$, let $\rho_1^\gamma < \rho_2^\gamma < \cdots < \rho_{m_\gamma}^\gamma$ be the elements of $T_\gamma$ ordered by lexicographic order. Consider the cycles of $\pi$ containing the words in $T_\gamma$ as consecutive subsequences. In these cycles, if we replace the consecutive subsequence which for $\rho_i^{\gamma}$ by $i$ for each $i$, we obtain cycles consisting of $1, 2, \ldots, m_\gamma$. The resulting cycles form a permutation, which we denote by $\tau_\gamma = \tau_1^\gamma \tau_2^\gamma \cdots \tau_{m_\gamma}^\gamma$.

Then we define $W_\gamma$ to be the sequence of the elements in $T_\gamma$ according to the permutation $\tau_\gamma$, that is,
\[ W_\gamma = \rho_1^\gamma, \rho_2^\gamma, \ldots, \rho_{m_\gamma}^\gamma. \]

(4) Let
\[ W = \rho_1, \rho_2, \ldots, \rho_m \]
be the concatenation of the sequence $W_\gamma$ for all $\gamma \in H$ where we start with the lexicographically smallest $\gamma$ and proceed with the next smallest one, and so on.

(5) We now define $\omega$ to be the ornament $\{(\gamma_1), \ldots, (\gamma_m)\}$ where $\gamma_i = \phi_{n_1, n_2, \ldots, n_k}(\rho_i)$. Here, we consider $\gamma_i$ as a sequence of integers as before.

(6) For $1 \leq i \leq k$, we define $\sigma_i$ to be the permutation in $S_{n_i}$ which is order-isomorphic to the word obtained from $W$ by taking the integers from $n_1 + \cdots + n_{i-1} + 1$ to $n_1 + \cdots + n_{i-1} + n_i$.

It is easy to see that $\pi \mapsto (\omega, \sigma_1, \ldots, \sigma_k)$ is the inverse map of $\psi$. \hfill \Box

Combining $\phi_{n_1, n_2, \ldots, n_k}$ and $\psi$, we obtain a bijective proof of (5).

**Example 2.** Let $(n_1, n_2, \ldots, n_k) = (8, 5, 1)$ and consider $\pi = (4, 3, 10, 5, 8, 11, 2, 1, 12)(7, 6, 9)(13, 14) \in \text{NMCy}(n_1, n_2, \ldots, n_k)$.

The map $\pi \mapsto (\omega, \sigma_1, \ldots, \sigma_k)$ in the proof of Theorem 4.2 is constructed as follows. Since $\phi_{n_1, n_2, \ldots, n_k}(\pi) = (1, 1, 2, 1, 1, 1, 2, 1, 1, 2)(1, 1, 2)(2, 3) \in \text{NMCy}(n_1, n_2, \ldots, n_k)$, we have $H = \{112, 23\}$,
\[ T_{112} = \{\rho_1^{112} = 2 \ 1 \ 12, \ \rho_2^{112} = 4 \ 3 \ 10, \ \rho_3^{112} = 5 \ 8 \ 11, \ \rho_4^{112} = 7 \ 6 \ 9\}, \]
\[ T_{23} = \{13 \ 14\}. \]
The cycles of $\pi$ containing the elements in $T_{112}$ are

$$(4, 3, 10, 5, 8, 11, 2, 12), \quad (7, 6, 9).$$

If we replace the consecutive subsequences “2,1,12”, “4,3,10”, “5,8,11”, “7,6,9” with 1, 2, 3, 4 respectively in these cycles, we obtain $(2, 3, 1) = (1, 2, 3)$ and $(4)$. Thus

$$\tau_{112} = (1, 2, 3)(4) = 2314,$$

and

$$W_{112} = \rho_{112}^{12}, \rho_{3}^{112}, \rho_{1}^{112}, \rho_{4}^{112} = 4 \ 3 \ 10, \quad 5 \ 8 \ 11, \quad 2 \ 1 \ 12, \quad 7 \ 6 \ 9.$$  

Similarly, we have $\tau_{23} = (1) = 1$ and $W_{23} = 13 \ 14$. Thus,

$$W = W_{112}, W_{23} = 4 \ 3 \ 10, \quad 5 \ 8 \ 11, \quad 2 \ 1 \ 12, \quad 7 \ 6 \ 9, \quad 13 \ 14.$$

Finally we obtain that

$$\omega = \{(1, 1, 2), (1, 1, 2), (1, 1, 2), (1, 1, 2), (2, 3)\}$$

and $\sigma_{1} = 43582176, \sigma_{2} = 23415$ and $\sigma_{3} = 1$.

5. Final remarks

As NFiA$(n_{1}, n_{2}, \ldots, n_{k})$ has a counterpart NMCy$(n_{1}, n_{2}, \ldots, n_{k})$, the set NFiD$(n_{1}, n_{2}, \ldots, n_{k})$ has a combinatorial counterpart as follows.

Let EMCy$(n_{1}, n_{2}, \ldots, n_{k})$ be the set of $(n_{1}, n_{2}, \ldots, n_{k})$-colored permutations in which the sum of the lengths of the monochromatic cycles of each color is even (EMCy stands for Evenly Monochromatic Cycles). Using the exponential formula, one can show that

$$\sum_{n_{1}, n_{2}, \ldots, n_{k} \geq 0} |\text{EMCy}(n_{1}, n_{2}, \ldots, n_{k})| \frac{x_{1}^{n_{1}} x_{2}^{n_{2}} \cdots x_{k}^{n_{k}}}{n_{1}! n_{2}! \cdots n_{k}!} = 1$$

(18)

$$= \frac{1}{(1 + x_{1}) \cdots (1 + x_{k})(1 - x_{1} \cdots - x_{k})}.$$  

Thus from (1) and (18) we get

$$|\text{NFiD}(n_{1}, n_{2}, \ldots, n_{k})| = |\text{EMCy}(n_{1}, n_{2}, \ldots, n_{k})|.$$  

(19)

We can also prove (19) bijectively, by using the same idea as in Theorem 4.2.

It will be interesting to find a refinement of (18) which is analogous to Theorem 2.1.

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