FACTORIZATION OF CERTAIN SELF-MAPS
OF PRODUCT SPACES

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Abstract. In this paper, we show that, under some conditions, self-maps of product spaces can be represented by the composition of two specific self-maps if their induced homomorphism on the $i$-th homotopy group is an automorphism for all $i$ in some section of positive integers. As an application, we obtain closeness numbers of several product spaces.

1. Introduction

For a connected pointed topological space $X$, let $\mathcal{E}(X)$ denote the set of homotopy classes of pointed self-maps of $X$ that are homotopy equivalences. Then, $\mathcal{E}(X)$ is a group with a group operation given by a composition of homotopy classes. Let $[X, X]$ be the set of all based homotopy classes of self-maps of $X$. When $[X, X]$ is given by a composition of homotopy classes, the set is a monoid. Choi and Lee [5] studied certain submonoid of $[X, X]$ containing $\mathcal{E}(X)$ as a set. If $A^\#_k(X)$ denotes the set of homotopy classes of self-maps of $X$ that induce an automorphism of $\pi_i(X)$ for $0 \leq i \leq k$, then $A^\#_k(X)$ is a submonoid of $[X, X]$ with an operation given by a composition of homotopy classes for any nonnegative integer $k$. If $k = \infty$, we simply denote $A^\#_\infty(X)$ as $A^\#(X)$. By definition, $A^\#_n(X) \subseteq A^\#_m(X)$ if $n \geq m$. Therefore, we have the following descending series:

$$\mathcal{E}(X) \subseteq A^\#_n(X) \subseteq \cdots \subseteq A^\#_1(X) \subseteq A^\#_0(X) = [X, X].$$

For any connected CW-complex $X$, $A^\#(X) = \mathcal{E}(X)$ according to the Whitehead theorem.

The group $\mathcal{E}(X \times Y)$ has been studied extensively by several authors, for instance, Booth and Heath [3], Heath [6], Lee [7], Pavešić [8–10] and Sieradski [11]. In particular, Pavešić [9] demonstrated that the group of self-homotopy equivalences $\mathcal{E}(X \times Y)$ can be represented as a product of two subgroups under...
the assumption that the self-equivalences of $X \times Y$ can be diagonalized (or are reducible). In this study, we examine the sufficient conditions under which all elements of the submonoid $\mathcal{A}^k_Y(X \times Y)$ of $[X \times Y, X \times Y]$ can be factorized by two specific self-maps for a non-negative integer $k$. In Section 2, we introduce the concept of $k$-reducibility and find several conditions for the factorization of $\mathcal{A}^k_Y(X \times Y)$. In Section 3, we study the split short exact sequences of several monoids. In Section 4, we discuss an alternative idea of the $k$-reducibility in the category of CW-complexes and their relationships with self-closeness numbers [5] of product spaces.

Let $i_X$ and $i_Y$ denote the inclusions (as slices determined by the base-points) of $X$ and $Y$ in $X \times Y$, respectively, and $p_X$ and $p_Y$ be the projections of $X \times Y$ onto $X$ and $Y$, respectively. Given a self-map $f : X \times Y \to X \times Y$ and $I, J \in \{X, Y\}$, write $f_I : X \times Y \to I$ for the composition $f_I := p_I \circ f$ so that $f$ is represented componentwise as $f = (f_X, f_Y)$ and $f_{IJ} : J \to I$ for the composition $f_{IJ} := p_I \circ f \circ i_J$. The self-homotopy equivalence $f$ of $X \times Y$ can be diagonalized (or is reducible) if $f_{XX}$ and $f_{YY}$ are self-homotopy equivalences of $X$ and $Y$, respectively [8]. Now, we recall that the isomorphism $\Psi : \pi_n(X \times Y) \to \pi_n(X) \times \pi_n(Y)$ is given by $\Psi = (p_X \#_*, p_Y \#_*)$ with the inverse $\Phi$, where $\Phi(\alpha, \beta) = i_{X \#}(\alpha) + i_{Y \#}(\beta)$ for $(\alpha, \beta) \in \pi_n(X) \times \pi_n(Y)$. Therefore, for given self-map $f : X \times Y \to X \times Y$, the induced homomorphism $\pi_n(f)$ can be identified with the $2 \times 2$ matrix

$$\pi_n(f) = \begin{pmatrix} \pi_1(f_{XX}) & \pi_1(f_{XY}) \\ \pi_1(f_{YX}) & \pi_1(f_{YY}) \end{pmatrix}. \tag{4}$$

We refer to this $2 \times 2$-matrix as the matrix representation of the homomorphism $\pi_n(f)$ throughout this paper. Given two self-maps $f, g : X \times Y \to X \times Y$, the induced homomorphism $\pi_n(f \circ g)$ of the composition $f \circ g$ can be identified with the multiplication of their matrix representations.

Throughout this paper, all spaces are pointed, connected and have the homotopy type of a CW-complex with an abelian fundamental group. Moreover, all maps and homotopies preserve the base points and we do not distinguish between the notation of a map $f : X \to Y$ and that of its homotopy class in $[X, Y]$.

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2. **Internal direct product of $\mathcal{A}^k_{X, \#}(X \times Y)$ and $\mathcal{A}^k_{Y, \#}(X \times Y)$**

In this section, we discuss the factorization of $\mathcal{A}^k_Y(X \times Y)$ into two submonoids. We begin by introducing the following definition.

**Definition 1.** The self map $f : X \times Y \to X \times Y$ is said to be $k$-reducible if $f_{XX} \in \mathcal{A}^k_{X, \#}(X)$ and $f_{YY} \in \mathcal{A}^k_{Y, \#}(Y)$.

According to the definition, if a self-map $f : X \times Y \to X \times Y$ is reducible, then $f$ is $k$-reducible for each non-negative integer $k$. However, the converse
does not hold. On the other hand, it is easy to show that if a self-homotopy equivalence \( f : X \times Y \to X \times Y \) is \( \infty \)-reducible, \( f \) is reducible on \( \mathcal{E}(X \times Y) \) according to the Whitehead theorem.

**Example 1.** For \( 2 \leq m < n \), let \( f : S^m \to S^n \) and \( g : S^n \to S^n \) be maps with \( \deg(f) = 2 \). Because \( S^m \) is \((m - 1)\)-connected, \( \pi_k(f) \in \text{Aut}(\pi_k(S^m)) \) and \( \pi_k(g) \in \text{Aut}(\pi_k(S^n)) \) for \( 0 \leq k \leq m - 1 \). However, \( \pi_m(f) \) is not surjective because \( \deg(f) = 2 \). Therefore, \( f \times g \) is \((m - 1)\)-reducible but not reducible on \( \mathcal{E}(X \times Y) \).

Given two abelian groups \( G \) and \( H \), an homomorphism \( \lambda : H \to G \) is said to be \( R \)-quasi-regular if for any homomorphism \( \mu : G \to H \), the function \( id_G - \lambda \mu \) given by \( (id_G - \lambda \mu)(g) = g - \lambda(\mu(g)) \) is an automorphism of \( G \). Similarly, \( \lambda \) is said to be \( L \)-quasi-regular if \( id_H - \mu \lambda \) is an automorphism of \( H \). Moreover, an homomorphism \( \lambda : H \to G \) is said to be \( RL \)-quasi-regular if it is \( R \)-quasi-regular and \( L \)-quasi-regular. Clearly, if \( \text{Hom}(G,H) \) or \( \text{Hom}(H,G) \) is trivial, then each homomorphism in \( \text{Hom}(H,G) \) is \( RL \)-quasi-regular.

**Lemma 1.** If \( f \) is an element of \( \mathcal{A}_{X,Y}^i(X \times Y) \) such that \( \pi_i(f_{XY}) \) is \( RL \)-quasi-regular for \( 0 \leq i \leq k \), then \( f \) is \( k \)-reducible.

**Proof.** For each \( f = (f_X, f_Y) \in \mathcal{A}_{X,Y}^i(X \times Y) \), the induced homomorphism \( \pi_i(f) \) belongs to \( \text{Aut}(\pi_i(X \times Y)) \) for \( 0 \leq i \leq k \). For \( 0 \leq i \leq k \), the homomorphism

\[
\pi_i(f) = \begin{pmatrix} \pi_i(f_{XX}) & \pi_i(f_{XY}) \\ \pi_i(f_{YX}) & \pi_i(f_{YY}) \end{pmatrix}
\]

has an inverse homomorphism \( \Phi_i \) of \( \pi_i(f) \). Let

\[
\Phi_i = \begin{pmatrix} \varphi_{XX} & \varphi_{XY} \\ \varphi_{YX} & \varphi_{YY} \end{pmatrix}
\]

be the matrix representation. Then, \( \pi_i(f) \circ \Phi_i = id_{\pi_i(X \times Y)} \) implies that \( \pi_i(f_{XX})\varphi_{XX} + \pi_i(f_{XY})\varphi_{YX} = id_{\pi_i(X)} \). Since \( \pi_i(f_{XY}) \) is \( R \)-quasi-regular, \( \pi_i(f_{XX}) \) is an isomorphism for \( 0 \leq i \leq k \). Similarly, \( \Phi_i \circ \pi_i(f) = id_{\pi_i(X \times Y)} \) implies that \( \varphi_{XX}\pi_i(f_{XX}) + \varphi_{YY}\pi_i(f_{YY}) = id_{\pi_i(Y)} \). Since \( \pi_i(f_{YY}) \) is \( L \)-quasi-regular, \( \pi_i(f_{YY}) \) is an isomorphism for \( 0 \leq i \leq k \).

We define the subset \( \mathcal{A}_{X,Y}^k(X \times Y) \) as the set of all maps in \( \mathcal{A}_{X,Y}^k(X \times Y) \) with the form \( f = (p_X, p_Y) : X \times Y \to X \times Y \). Similarly, we define the subset \( \mathcal{A}_{Y,X}^k(X \times Y) \) as the set of all maps in \( \mathcal{A}_{Y,X}^k(X \times Y) \) with the form \( f = (f_X, p_Y) : X \times Y \to X \times Y \).

**Lemma 2.** (a) \( \mathcal{A}_{X,Y}^k(X \times Y) \) and \( \mathcal{A}_{Y,X}^k(X \times Y) \) are submonoids of \( \mathcal{A}_{X,Y}^k(X \times Y) \) for any nonnegative integer \( k \).

(b) \( (p_X, g) \in \mathcal{A}_{X,Y}^k(X \times Y) \) if and only if \( g \circ i_Y \in \mathcal{A}_{X,Y}^k(Y) \).

(c) \( (f, p_Y) \in \mathcal{A}_{Y,X}^k(X \times Y) \) if and only if \( f \circ i_X \in \mathcal{A}_{Y,X}^k(X) \).
Proof. (a) For the given elements \((p_X, f_Y)\) and \((p_X, g_Y)\) in \(A^k_X(X \times Y)\), we have \((p_X, f_Y) \circ (p_X, g_Y) = (p_X, f_Y \circ (p_X, g_Y))\). Moreover, the induced homomorphisms \(\pi_i(p_X, f_Y)\) and \(\pi_i(p_X, g_Y)\) are in \(\text{Aut}(\pi_i(X \times Y))\) for \(0 \leq i \leq k\). Therefore, \(\pi_i(p_X, f_Y) \circ \pi_i(p_X, g_Y) = \pi_i((p_X, f_Y) \circ (p_X, g_Y)) \in \text{Aut}(\pi_i(X \times Y))\) for \(0 \leq i \leq k\). It follows that \((p_X, f_Y) \circ (p_X, g_Y) \in A^k_X(X \times Y)\). Clearly, \((p_X, f_Y) \circ ((p_X, g_Y) \circ (p_X, h_Y)) = ((p_X, f_Y) \circ (p_X, g_Y)) \circ (p_X, h_Y)\) and \((p_X, p_Y)\) is an identity of \(A^k_X(X \times Y)\).

(b) Suppose \((p_X, g) \in A^k_X(X \times Y)\). Then, the matrix representation of isomorphism \(\pi_i(p_X, g)\) for \(0 \leq i \leq k\) is given by

\[
\begin{pmatrix}
\text{id}_{\pi_i(X)} & 0 \\
\pi_i(g \circ i_X) & \pi_i(g \circ i_Y)
\end{pmatrix}.
\]

Therefore, \(\pi_i(g \circ i_Y)\) is an isomorphism on \(\pi_i(Y)\) for \(0 \leq i \leq k\).

Conversely, suppose that \(g \circ i_Y \in A^k_Y(Y)\). Then the inverse of \(\pi_i(p_X, g)\) is represented by

\[
\begin{pmatrix}
\text{id}_{\pi_i(X)} & 0 \\
\pi_i(g \circ i_Y)^{-1} \circ \pi_i(g \circ i_X)^{-1} & \pi_i(g \circ i_Y)^{-1}
\end{pmatrix},
\]

where \(-\pi_i(g \circ i_X) : \pi_i(X) \to \pi_i(Y)\) is the homomorphism given by \(-\pi_i(g \circ i_X)(\alpha) = -\pi_i(g \circ i_X)(\alpha)\) in \(\pi_i(Y)\) for each \(\alpha \in \pi_i(X)\).

(c) This can be proved in a similar method to that of (b). \(\Box\)

Corollary 1. If \(f = (f_X, f_Y) \in A^k_{#}(X \times Y)\) is \(k\)-reducible, then \((p_X, f_Y) \in A^k_X(X \times Y)\) and \((f_X, p_Y) \in A^k_Y(X \times Y)\).

Let \(U\) be a monoid and \(S\) and \(T\) be submonoids of \(U\). Then \(U\) is called the \textit{internal direct product} of \(S\) and \(T\) if

1. \(U\) is uniquely factorizable with factors \(S\) and \(T\);
2. for all \(s \in S\) and for all \(t \in T\), \(st = ts\).

On the other hand, the monoid \(S \times T = \{(s, t) \mid s \in S, t \in T\}\) is called the \textit{external direct product of the two monoids} \(S\) and \(T\) if the binary operation is given by \((s, t)(s', t') = (ss', tt')\) on \(S \times T\) with the identity \((1_S, 1_T)\).

In [8], Pavešič showed that if \(X\) and \(Y\) are connected CW-complexes and all self-homotopy equivalences of \(X \times Y\) are reducible, then \(\text{Aut}(X \times Y) = \text{Aut}_X(X \times Y) \text{Aut}_Y(X \times Y)\). Here, we discuss the factorization of \(A^k_{#}(X \times Y)\) into \(A^k_{#}(X \times Y)\) and \(A^k_{#}(X \times Y)\). However, we cannot apply the method in [8] to \(A^k_{#}(X \times Y)\) directly because not all elements of \(A^k_{#}(X \times Y)\) are always self-homotopy equivalences.

Theorem 1. Suppose that each \(f = (f_X, f_Y) \in A^k_{#}(X \times Y)\) is \(k\)-reducible and \(f_Y \simeq f_{YY} \circ p_Y\). Then

\[
A^k_{#}(X \times Y) = A^k_{#}(X \times Y)A^k_{#}(X \times Y).
\]

Furthermore, if \(f_X \simeq f_{XX} \circ p_X\), then \((p_X, f_Y) \circ (f_X, p_Y) = (f_X, p_Y) \circ (p_X, f_Y)\).
Proof. According to Corollary 1, \((p_X, f_Y) \in A^k_{X,Y}(X \times Y)\) and \((f_X, p_Y) \in A^k_{X,Y}(X \times Y)\), and therefore, they are contained in \(A^k_{X,Y}(X \times Y)\). Moreover, because \(A^k_{X,Y}(X \times Y) \cap A^k_{X,Y}(X \times Y) = \{(p_X, p_Y)\}\), it is sufficient to show that each element \(f = (f_X, f_Y) \in A^k_{X,Y}(X \times Y)\) can be factored as \(f = g \circ h\), where \(g \in A^k_{X,Y}(X \times Y)\) and \(h \in A^k_{X,Y}(X \times Y)\). Via a the direct computation, we have

\[
(px, fy) \circ (fx, py) = (fx, fy \circ (fx, py)) \\
\simeq (fx, fy \circ py \circ (fx, py)) \\
= (fx, fy \circ py) \\
\simeq (fx, fy).
\]

From Theorem 1, we arrive at the following corollary.

**Corollary 2.** If \(f = (f_X, f_Y) \in A^k_{X,Y}(X \times Y)\) is \(k\)-reducible and \(f_X \simeq f_{XX} \circ p_X\) and \(f_Y \simeq f_{YY} \circ p_Y\), then \(A^k_{X,Y}(X \times Y)\) is the internal direct product of \(A^k_{X,Y}(X \times Y)\) and \(A^k_{X,Y}(X \times Y)\).

Consider the inclusion map \(j : X \cup Y \to X \times Y\), where \(X \cup Y\) is the wedge product of \(X\) and \(Y\). Then we arrive at the following lemma.

**Lemma 3.** \(j^2 : [X \times Y, X] \to [X \cup Y, X]\) is injective and \([Y, X] = 0\) if and only if for each map \(f : X \times Y \to X \times Y\), \(f_X \simeq f_{XX} \circ p_X\).

**Proof.** Suppose that \(j^2\) is injective. It suffices to show that \(j^2(f_X) = f_X \circ j = f_{XX} \circ p_X \circ j = j^2(f_{XX} \circ p_X)\). This is true because

\[
f_{XX} \circ p_X \circ j \circ i_1 = f_{XX} \circ p_X \circ i_X = f_X \circ i_X = f_X \circ j \circ i_1
\]

and

\[
f_{XX} \circ p_X \circ j \circ i_2 = f_{XX} \circ p_X \circ i_Y \simeq * \simeq f_{XY} = f_X \circ i_Y = f_X \circ j \circ i_2,
\]

where \(i_1 : X \to X \cup Y\) and \(i_2 : Y \to X \cup Y\) are injective maps defined by \(i_1(x) = (x, *)\) and \(i_2(y) = (*, y)\), respectively.

Conversely, suppose that for each map \(f : X \times Y \to X \times Y\), \(f_X \simeq f_{XX} \circ p_X\). For \(u, v \in [X \times Y, X]\), define \(g : X \times Y \to X \times Y\) and \(h : X \times Y \to X \times Y\) by \(g = (u, py)\) and \(h = (v, py)\), respectively. Then \(g_X = u\) and \(h_X = v\). If \(j^2(u) = j^2(v)\), then

\[
u \simeq u \circ i_X \circ p_X = u \circ j \circ i_1 \circ p_X \simeq v \circ j \circ i_1 \circ p_X = v \circ i_X \circ p_X \simeq v
\]

according to the hypothesis. Therefore \(j^2\) is injective. Moreover, \([Y, X] = 0\).

In fact, if we define \(f : X \times Y \to X \times Y\) by \(f(x, y) = (w(y), y)\) for each map \(w : Y \to X\), then \(w = f_{XY} \ast f_X \circ i_Y \simeq f_{XX} \circ p_X \circ i_Y \simeq *\). □

Consider the following cofibre sequence:

\[
X \cup Y \xrightarrow{j} X \times Y \xrightarrow{q} X \wedge Y.
\]
This gives rise to the following Barrat-Puppe sequence:

\[ \cdots \rightarrow [\Sigma(X \vee Y), X] \rightarrow [X \wedge Y, X] \stackrel{q}{\rightarrow} [X \times Y, X] \stackrel{p}{\rightarrow} [X \vee Y, X]. \]

From this sequence and Lemma 3, we arrive at the following corollary.

**Corollary 3.** If \( [X \wedge Y, X] = 0 \) and \([Y, X] = 0\), then \( f_X \simeq f_{XX} \circ p_X \) for each map \( f : X \times Y \rightarrow X \times Y \).

According to Lemma 1, Theorem 1 and Corollary 3, we arrive at the following corollary.

**Corollary 4.** \( A_{#}^k(S^1 \times S^n) = A_{#}^k(S^1 \times S^n)A_{#}^k(S^1 \times S^n) \) for each pair of integers \( k \) and \( n \) such that \( 1 \leq k < n \).

### 3. Short exact sequences of monoids

In this section, we derive certain short exact sequences related to \( A_{#}^k(X \times Y) \). Pavšić [9, Lemma 1.3, Proposition 1.4 and Theorem 1.5] introduced the monoid homomorphism from \( \text{Aut}_Y(X \times Y) \) to \( \text{Aut}(X) \) and several split short exact sequences. First, we introduce a similar monoid homomorphism.

**Lemma 4.** If \( \Phi_X : A_{#}^k(X \times Y) \rightarrow A_{#}^k(X) \) is a map defined by \( \Phi_X(f_X, p_Y) = f_{XX} \), then \( \Phi_X \) is a monoid epimorphism.

**Proof.** Clearly, the function \( \Phi_X \) is surjective according to Lemma 2(b).

Since

\[ (f_X, p_Y) \circ i_X = (f_X \circ i_X, p_Y \circ i_X) = (f_{XX}, *) = i_X \circ f_{XX}, \]

we have

\[ \Phi_X((f_X, p_Y) \circ (f'_X, p_Y)) = p_X \circ (f_X, p_Y) \circ (f'_X, p_Y) \circ i_X = f_X \circ i_X = f_{XX} \circ \Phi_X(f'_X, p_Y) \]

for \((f_X, p_Y), (f'_X, p_Y) \in A_{#}^k(X \times Y)\). Furthermore, because the induced map

\[ \pi_i(f_{XX}) = \begin{pmatrix} \pi_i(f_{XX}) & \pi_i(f_{XY}) \\ 0 & \text{id}_{\pi_i(Y)} \end{pmatrix} \]

is an isomorphism for \( 0 \leq i \leq k \), \( \pi_i(f_{XX}) \) is an isomorphism for \( 0 \leq i \leq k \). □

Let \( A_{#}^{X,k}(X \times Y) \) denote the submonoid of \( A_{#}^{X,Y}(X \times Y) \), which consists of \((f_X, p_Y) \in A_{#}^{X,Y}(X \times Y) \) such that \( (f_X, p_Y) \circ i_X = i_X \). Similarly, let \( A_{#}^{X,k}(X \times Y) \) denote the submonoid of \( A_{#}^{X,Y}(X \times Y) \) which consists of \((p_X, f_Y) \in A_{#}^{X,Y}(X \times Y) \) such that \( (p_X, f_Y) \circ i_Y = i_Y \). If \((g_X, p_Y) \in \text{Ker}\Phi_X \), then \( g_X \circ i_X = \text{id}_X \) (that is, \((g_X, p_Y) \circ i_X = i_X \)). Therefore, \( \text{Ker}\Phi_X = A_{#}^{X,k}(X \times Y) \). Consequently, we have the following lemma.
Lemma 5. There exists a split short exact sequence of monoids

\[ 1 \longrightarrow A_{X_{#}}(X \times Y) \longrightarrow A_{Y_{#}}(X \times Y) \xrightarrow{\Phi_{X}} A_{#}(X) \longrightarrow 1, \]

where $\Phi_{X}(f_{X}, p_{Y}) = f_{XX}$. 

Proof. Define $\sigma_{X} : A_{#}(X) \rightarrow A_{X_{#}}(X \times Y)$ by $\sigma_{X}(f) = f \times id_{Y}$. Then $\sigma_{X}$ is the section of $\Phi_{X}$. \hfill \Box

Lemma 6. $A_{X_{#}}(X \times Y)$ is trivial if and only if $f_{X} \simeq f_{XX} \circ p_{X}$ for each $(f_{X}, f_{Y}) \in A_{#}(X \times Y)$. 

Proof. Since $A_{X_{#}}(X \times Y)$ is trivial, $\Phi_{X}$ is an isomorphism. Moreover,

\[ \Phi_{X}(f_{XX} \circ p_{X}, p_{Y}) = f_{XX} \circ p_{X} \circ i_{X} = f_{XX} = \Phi_{X}(f_{X}, p_{Y}). \]

Therefore, $f_{X} \simeq f_{XX} \circ p_{X}$. Conversely, suppose that $f_{X} \simeq f_{XX} \circ p_{X}$. Then $\Phi_{X}$ is a monoid monomorphism because $\Phi_{X}(f_{X}, p_{Y}) = id_{X}$ implies $f_{X} = p_{X}$. According to Lemma 5, $A_{X_{#}}(X \times Y)$ is trivial. \hfill \Box

Theorem 2. Assume that $A_{X_{#}}(X \times Y)$ is trivial and that all elements of $A_{#}(X \times Y)$ are $k$-reducible. Then, there is a split short exact sequence of monoids

\[ 1 \longrightarrow A_{X_{#}}(X \times Y) \longrightarrow A_{#}(X \times Y) \xrightarrow{\Phi} A_{#}(X) \times A_{#}(Y) \longrightarrow 1, \]

where $\Phi$ is given by $\Phi(f) = (f_{XX}, f_{YY})$ for each $f \in A_{#}(X \times Y)$. 

Proof. Because each $f \in A_{#}(X \times Y)$ is $k$-reducible, the function $\Phi$ is well-defined. Moreover, because $f_{XX} \circ p_{X} \simeq f_{X}$ for $(f_{X}, f_{Y}) \in A_{#}(X \times Y)$ according to Lemma 6 and $p_{X} \circ i_{X} = id_{X}$, we have $\Phi((f_{X}, f_{Y}) \circ (f_{X}', f_{Y}')) = \Phi((f_{XX} \circ p_{X}, f_{YY}) \circ (f_{XX}', f_{YY}')) = \Phi((f_{XX} \circ p_{X} \circ i_{X} \circ (f_{XX}', f_{YY})) = \Phi((f_{XX}, f_{YY}) \circ (f_{XX}', f_{YY})) = (f_{XX} \circ f_{XX}', f_{YY} \circ f_{YY}')$. Therefore, $\Phi$ is a homomorphism.

Clearly, $\text{Ker} \Phi_{Y} = A_{X_{#}}(X \times Y)$. Furthermore, if we define $\sigma : A_{#}(X) \times A_{#}(Y) \rightarrow A_{#}(X \times Y)$ by $\sigma(g, g') = g \times g'$, $\sigma$ is clearly a homomorphism and the section of $\Phi$. \hfill \Box

From Theorem 2 and Lemma 5, we arrive at the following corollary.

Corollary 5. If both $A_{X_{#}}(X \times Y)$ and $A_{Y_{#}}(X \times Y)$ are trivial and all elements of $A_{#}(X \times Y)$ are $k$-reducible, then $A_{#}(X \times Y)$ is isomorphic to the external direct product $A_{X_{#}}(X \times Y) \times A_{Y_{#}}(X \times Y)$. 

4. Self-closeness number of product spaces

In this section, we discuss the relationship between the $k$-reducibility and the self-closeness number introduced by Choi and Lee [5].

**Lemma 7.** Let $(f_X, f_Y) \in A^k_\#(X \times Y)$. If $f_X \simeq f_X X \circ p_X (f_Y \simeq f_Y Y \circ p_Y)$, then $f_{XY} (f_{YX})$ is null homotopic.

**Proof.** Clearly, $f_{XY} \simeq f_X \circ i_Y \simeq f_X X \circ p_X \circ i_Y = \ast$. \hfill \Box

**Theorem 3.** Let $f = (f_X, f_Y) \in A^k_\#(X \times Y)$. If $f_{XY} \simeq \ast$ and $f_{YX} \simeq \ast$, then $f$ is $k$-reducible.

**Proof.** According to the hypothesis, the induced homomorphisms $\pi_i(f_{XY})$ and $\pi_i(f_{YX})$ are trivial. If $\Phi = (\varphi_{XX}, \varphi_{XY}, \varphi_{YX}, \varphi_{YY})$ is the inverse homomorphism of $\pi_i(f)$ for $0 \leq i \leq k$, then the homomorphisms $\varphi_{XX}$ and $\varphi_{YY}$ are inverse homomorphisms of $\pi_i(f_{XX})$ and $\pi_i(f_{YY})$, respectively. Therefore, $f$ is $k$-reducible. \hfill \Box

From Theorem 3, we arrive at the following corollary.

**Corollary 6.** If for all $f = (f_X, f_Y) \in A^k_\#(X \times Y)$, $f_X \simeq f_X X \circ p_X$ and $f_Y \simeq f_Y Y \circ p_Y$, then $A^k_\#(X \times Y) \cong A^k_{Y,\#}(X \times Y) \times A^k_{X,\#}(X \times Y) \cong A^k_\#(X) \times A^k_\#(Y)$; moreover, $A^k_{\#}(X \times Y)$ is the internal direct product of $A^k_{X,\#}(X \times Y)$ and $A^k_{Y,\#}(X \times Y)$.

**Proof.** From Corollary 5, $A^k_{\#}(X \times Y) \cong A^k_{Y,\#}(X \times Y) \times A^k_{X,\#}(X \times Y)$. Moreover, $A^k_{X,\#}(X \times Y)$ and $A^k_{Y,\#}(X \times Y)$ are trivial according to Lemma 6. Therefore, $A^k_\#(X \times Y) \cong A^k_\#(X) \times A^k_\#(Y)$ in agreement with Lemma 5. \hfill \Box

For given spaces $X$ and $Y$, let $f : X \times Y \to X \times Y$ be a map such that $f_X \simeq f_{XX} \circ p_X$. Because the projection map $p_X : X \times Y \to X$ is a fibration, we obtain the following commutative diagram of fibrations:

\[
\begin{array}{ccc}
Y & \xrightarrow{f_{YY}} & Y \\
\downarrow{i_Y} & & \downarrow{i_Y} \\
X \times Y & \xrightarrow{f} & X \times Y \\
\downarrow{p_X} & & \downarrow{p_X} \\
X & \xrightarrow{f_{XX}} & X \\
\end{array}
\]

In fact, $f \circ i_Y = (f_{XY}, f_{YY}) \simeq (\ast, f_{YY}) = i_Y \circ f_{YY}$. 
Conversely, let \( g : X \to X \) and \( h : Y \to Y \) be maps such that \( g \circ p_X \simeq p_X \circ f \) and \( i_Y \circ h \simeq f \circ i_Y \). Because \( p_X \circ i_X = id_X \), \( g \simeq p_X \circ f \circ i_X = f_{XX} \). Similarly, \( h \simeq f_{YY} \). Therefore, \( f_{XX} : X \to X \) and \( f_{YY} : Y \to Y \) are representatives such that the above diagram is homotopy commutative for any \( f : X \times Y \to X \times Y \).

Consider the commutative ladder of homotopy groups induced from the above diagram:

\[
\begin{array}{cccccccc}
\vdots & \pi_{k+1}(X) & \pi_k(Y) & \pi_k(X \times Y) & \pi_k(X) & \pi_{k-1}(Y) & \vdots \\
\downarrow & & & \downarrow & & & \downarrow \\
\pi_{k+1}(f_{XX}) & \pi_k(f_{YY}) & \pi_k(f) & \pi_k(f_{XX}) & \pi_k(f_{YY}) & \pi_{k-1}(f_{YY}) & \vdots
\end{array}
\]

Using this commutative ladder, we will prove Theorem 4.

First, we recall the closeness number introduced by Choi and Lee [5]. The self-closeness number of \( X \) denoted by \( NE(X) \) is the least nonnegative integer \( k \) such that \( \mathcal{E}(X) = \mathcal{A}_k(X) \). That is,

\[
NE(X) = \min\{k \mid \mathcal{E}(X) = \mathcal{A}_k(X) \text{ for } k \geq 0\}.
\]

**Lemma 8** ([5, Theorem 2]). If \( X \) is a CW-complex with dimension \( n \), then

\[
NE(X) \leq n.
\]

**Lemma 9** ([5, Theorem 3]). Let \( X \) and \( Y \) be CW-complexes. Then, we have

\[
NE(X \times Y) \geq \max\{NE(X), NE(Y)\}.
\]

**Theorem 4.** Let \( X \) and \( Y \) be CW-complexes. If each map \( f : X \times Y \to X \times Y \) satisfies the conditions \( f_X \simeq f_{XX} \circ p_X \) and \( f_Y \simeq f_{YY} \), then

\[
NE(X \times Y) = \max\{NE(X), NE(Y)\}.
\]

**Proof.** Let \( NE(X) = m \) and \( NE(Y) = n \). We assume \( m \geq n \). For each \( l \geq m \), let \( f \in \mathcal{A}_m(X \times Y) \). Then, we have the commutative ladder mentioned above. According to Lemma 7 and Theorem 3, \( f \) is \( l \)-reducible. Therefore, \( f_{XX} \in \mathcal{A}_m(X) \subset \mathcal{A}_l(X) \) and \( f_{YY} \in \mathcal{A}_m(Y) \subset \mathcal{A}_l(Y) \). According to the definition of the self-closeness number, \( \mathcal{A}_l(X) = \mathcal{E}(X) \) and \( \mathcal{A}_l(Y) = \mathcal{E}(Y) \). Therefore, \( \pi_k(f_{XX}) \) and \( \pi_k(f_{YY}) \) are automorphisms for all \( k \geq 0 \). By the Five Lemma, \( \pi_k(f) \) is also an automorphism for all \( k \geq 0 \) in the homotopy commutative ladder. Therefore, \( f \) is a homotopy equivalence according to the Whitehead theorem. This implies that \( f \in \mathcal{A}_m(X \times Y) = \mathcal{E}(X \times Y) \) for each \( l \geq m \). Therefore, \( NE(X \times Y) = m = \max\{NE(X), NE(Y)\} \) in accordance with Lemma 9 and the minimality of the self-closeness number.

From Lemma 3, Corollary 3, Theorem 4, and [5, Corollary 2], we obtain the following corollaries.

**Corollary 7.** Let \( X \) and \( Y \) be CW-complexes with \( [X \wedge Y, X] = 0 \). If \([X, Y] = 0\) and \([Y, X] = 0\), then \( NE(X \times Y) = \max\{NE(X), NE(Y)\} \).

From Corollary 7 and [5, Corollary 2], we obtain the following corollary.
Corollary 8. Let \( m \neq n \). Then, \( N\mathcal{E}(S^m \times S^n) = \max\{m, n\} \) provided that \( \pi_{m+n}(S^{\min\{m,n\}}) = 0 \) and \( \pi_{\max\{m,n\}}(S^{\min\{m,n\}}) = 0 \).

Therefore, if \( 1 < n \), then \( N\mathcal{E}(S^1 \times S^n) = n \). Furthermore, \( N\mathcal{E}(S^{12} \times S^7) = 12 \) because \( \pi_9(S^7) = 0 \) and \( \pi_{12}(S^7) = 0 \). Similarly, \( N\mathcal{E}(S^8 \times S^{12}) = 12 \).

Suppose that \( X \) and \( Y \) are group-like spaces. Consider the cofibration
\[
X \vee Y \xrightarrow{j} X \times Y \xrightarrow{q} X \wedge Y
\]
and the short exact sequence of additive groups of homotopy classes obtained from the cofibration:
\[
0 \xrightarrow{} [X \wedge Y, X \times Y] \xrightarrow{q^*} [X \times Y, X \times Y] \xrightarrow{j^*} [X \vee Y, X \times Y] \xrightarrow{} 0.
\]

All elements of \([X \vee Y, X \times Y]\) can be identified with the \( 2 \times 2 \) matrix
\[
(f_{IJ}) = \begin{pmatrix} f_{XX} & f_{XY} \\ f_{YX} & f_{YY} \end{pmatrix}
\]
with entries \( f_{IJ} \) in the homotopy sets \([I, J]\) for \( I, J = X, Y \). In [11, Corollary 7], it was shown that if \([X \wedge Y, X \times Y] = 0\), the group of self-homotopy equivalences of \( X \times Y \) is \( GL(2, \Lambda_{IJ}) \) contained in \([X \vee Y, X \times Y]\), the group of invertible matrices with entries \( f_{IJ} \in \Lambda_{IJ} = [I, J] \) for \( I, J = X, Y \).

Theorem 5. Let \( X \) and \( Y \) be group-like spaces such that \([X \wedge Y, X \times Y] = 0\) and \([Y, X] = 0\). If \( f \) is a self-map of \( X \times Y \) such that \( f_{XX} \in \mathcal{E}(X) \), \( f_{YY} \in \mathcal{E}(Y) \) and \((f_{XX})^{-1}\) are \( H \)-maps, then \( f \) is a self-homotopy equivalence.

Proof. Let \( f \) be a self-map of \( X \times Y \) such that \( f_{XX} \in \mathcal{E}(X) \), \( f_{YY} \in \mathcal{E}(Y) \) and \((f_{XX})^{-1}\) are \( H \)-maps. Under the condition \([Y, X] = 0\), each element \((f_{IJ})\) in \([X \vee Y, X \times Y]\) has a left inverse and a right inverse
\[
\begin{pmatrix} (f_{XX})^{-1} & -(f_{XX})^{-1} \circ f_{XY} \circ (f_{YY})^{-1} \\ 0 & (f_{YY})^{-1} \end{pmatrix}
\]
and
\[
\begin{pmatrix} (f_{XX})^{-1} & (f_{XX})^{-1} \circ (-f_{XY}) \circ (f_{YY})^{-1} \\ 0 & (f_{YY})^{-1} \end{pmatrix},
\]
respectively. Therefore, if \(-(f_{XX})^{-1} \circ f_{XY} \circ (f_{YY})^{-1} = (f_{XX})^{-1} \circ (-f_{XY}) \circ (f_{YY})^{-1}\), \([X \vee Y, X \times Y] = GL(2, \Lambda_{IJ})\). Let \( m \) and \( a \) be the multiplication and the homotopy inverse of \( X \), respectively. Then,
\[
* = (f_{XX})^{-1} \circ \ast \circ f_{XY}
\]
\[
= (f_{XX})^{-1} \circ m(id \circ a) \circ (f_{XY} \times f_{XY}) \Delta
\]
\[
= (f_{XX})^{-1} \circ m(f_{XY} \times (a \circ f_{XY})) \Delta
\]
\[
= m((f_{XX})^{-1} \times (f_{XX})^{-1})(f_{XY} \times ((a \circ f_{XY}))) \Delta
\]
\[
= m(((f_{XX})^{-1} \circ f_{XY}) \times ((f_{XX})^{-1} \circ ((a \circ f_{XY}))) \Delta,
\]
where \( \Delta : Y \to Y \times Y \) is the diagonal map. Therefore, we have

\[
(f_{XX})^{-1} \circ f_{XY} + (f_{XX})^{-1} \circ (-f_{YY})(f_{YY})^{-1} = (f_{XX})^{-1} \circ f_{XY} \circ (f_{YY})^{-1} + (f_{XX})^{-1} \circ (-f_{XY}) \circ (f_{YY})^{-1} = 0,
\]

and further,

\[-(f_{XX})^{-1} \circ f_{XY} \circ (f_{YY})^{-1} = (f_{XX})^{-1} \circ (-f_{XY}) \circ (f_{YY})^{-1} = 0.
\]

Consequently, there is a unique homotopy inverse for each \((f_{ij})\) in \([X \vee Y, X \times Y]\).

In accordance with [11, Corollary 7], \(f\) is a self-homotopy equivalence. \(\square\)

From Theorem 5, we obtain the following corollary.

**Corollary 9.** For each pair of integers \(m\) and \(n\) such that \(1 \leq m < n\) and the abelian groups \(G\) and \(H\),

\[N \mathcal{E}(K(G, m) \times K(H, n)) = n,
\]

where \(K(G, m)\) and \(K(H, n)\) are Eilenberg-MacLane spaces.

**Proof.** Let \(X = K(G, m)\) and \(Y = K(H, n)\). Then, \(X\) and \(Y\) are group-like spaces and \([X \wedge Y, X \times Y] = 0\). For every map \(f_{XX} \in [X, X]\), \(f_{XX}\) is an H-map because \(X = K(G, m) = \Omega K(G, m + 1)\). Since \(m < n\), \([Y, X] = 0\). According to Lemma 1, every element of \(A^k_{\#}(X \times Y)\) is \(k\)-reducible. Moreover, \(A^m_{\#}(X) = \mathcal{E}(X)\), \(A^m_{\#}(Y) = \mathcal{E}(Y)\), and \(N \mathcal{E}(X \times Y) \geq \max\{N \mathcal{E}(X), N \mathcal{E}(Y)\} = n\) because \(N \mathcal{E}(K(G, m)) = m < n = N \mathcal{E}(K(H, n))\). Therefore, \(A^m_{\#}(X \times Y) = \mathcal{E}(X \times Y)\) in accordance with Theorem 5. \(\square\)

**References**


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