LINEAR PRESERVERS OF SYMMETRIC ARCTIC RANK
OVER THE BINARY BOOLEAN SEMIRING

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Abstract. A Boolean rank one matrix can be factored as $uv^t$ for vectors $u$ and $v$ of appropriate orders. The perimeter of this Boolean rank one matrix is the number of nonzero entries in $u$ plus the number of nonzero entries in $v$. A Boolean matrix of Boolean rank $k$ is the sum of $k$ Boolean rank one matrices, a rank one decomposition. The perimeter of a Boolean matrix $A$ of Boolean rank $k$ is the minimum over all Boolean rank one decompositions of $A$ of the sums of perimeters of the Boolean rank one matrices. The arctic rank of a Boolean matrix is one half the perimeter. In this article we characterize the linear operators that preserve the symmetric arctic rank of symmetric Boolean matrices.

1. Introduction and preliminaries

The binary Boolean algebra consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

Let $\mathcal{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ Boolean matrices with entries in $\mathbb{B}$. If $m = n$, we use the notation $\mathcal{M}_n(\mathbb{B})$ instead of $\mathcal{M}_{n,n}(\mathbb{B})$. The matrix $I_n$ is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix, and we write $J_n$ for $J_{n,n}$ and $O_n$ for $O_{n,n}$. We omit the subscripts when the order is obvious from the context and we write $I$, $J$ and $O$, respectively. For matrices $A$ and $B$, $A \oplus B$ is the direct sum of $A$ and $B$ so that $A \oplus B = \begin{bmatrix} A & O \\ O & B \end{bmatrix}$.

Note that any subset of matrices $\mathcal{L}$ in $\mathcal{M}_{m,n}(\mathbb{B})$ that is closed under addition (and scalar multiplication) has a unique “basis”, that is a set of matrices such that any member $A \in \mathcal{L}$ is a linear combination of elements of that set and no member of that set is a linear combination of the remaining members of that set.
Further, the linear combination of elements of a basis that equals $A$ is unique, no other different linear combination of elements of the basis equals $A$ (See [4] or [8]). The elements of a basis are called base elements.

A matrix in $\mathcal{M}_{m,n}(\mathbb{B})$ is called a cell if it has exactly one nonzero entry, that being a 1. We denote the cell whose nonzero entry is in the $(i,j)^{th}$ position by $E_{i,j}$. Let $\mathcal{E} = \{E_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Then, $\mathcal{E}$ is the unique basis for $\mathcal{M}_{m,n}(\mathbb{B})$.

We let $S_n(\mathbb{B})$ denote the set of all $n \times n$ symmetric Boolean matrices. For $1 \leq i < j \leq n$ let $D_{i,j} = E_{i,j} + E_{j,i}$. The matrix $D_{i,j}$ is called a digon. In $\mathcal{M}_{m,n}(\mathbb{B})$ base elements are all the cells in $\mathcal{E}$, whereas in $S_n(\mathbb{B})$ the base elements are digons and diagonal cells. For convenience of notation, we occasionally use $D_{i,i}$ to represent $E_{i,i}$.

For $A \in \mathcal{M}_{m,n}(\mathbb{B})$ let $|A|$ denote the number of nonzero entries in $A$. That is $| \cdot | : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathbb{Z}_+$ is the function such that $|A|$ is the number of nonzero entries in $A$, where $\mathbb{Z}_+$ is the set of nonnegative integers.

For $X, Y \in \mathcal{M}_{m,n}(\mathbb{B})$, $X$ dominates $Y$, written $X \sqsupseteq Y$, if $y_{i,j} \neq 0$ implies $x_{i,j} \neq 0$ for all $i$ and $j$. Thus, for $A \in \mathcal{M}_{m,n}(\mathbb{B})$, $|A|$ is the number of base elements of $\mathcal{M}_{m,n}(\mathbb{B})$ that are dominated by $A$. If $X \sqsupseteq Y$, then $X \setminus Y = Z$ is the matrix such that $z_{i,j} = x_{i,j}$ if $x_{i,j} = 0$ and is 0 otherwise.

For $A \in S_n(\mathbb{B})$, we let $\#(A)$ denote the number of base elements of $S_n(\mathbb{B})$ that $A$ dominates. That is $\# : S_n(\mathbb{B}) \to \mathbb{Z}_+$ is the function such that $\#(A)$ is the number of base elements of $S_n(\mathbb{B})$ that $A$ dominates. Note that if $A$ is not symmetric, then $\#(A)$ is undefined.

**Example 1.1.** Since $D_{i,j} \in S_n(\mathbb{B})$ and $D_{i,j}$ is also in $\mathcal{M}_{n}(\mathbb{B})$, we have that for $i \neq j \mid D_{i,j} \mid = 2$ while $\#(D_{i,j}) = 1$. For another example, let $K_{v,v} = \begin{bmatrix} I_v & J_v \\ J_v & I_v \end{bmatrix}$. Then, $|K_{v,v}| = 2v^2 + 2v$ and $\#(K_{v,v}) = v^2 + 2v$.

For $A, B \in \mathcal{M}_{m,n}(\mathbb{B})$ the Hadamard or Schur product of $A$ and $B$ is the matrix $C$ if the $(i,j)^{th}$ entry of $C$ is $c_{i,j} = a_{i,j}b_{i,j}$, and we write $A \circ B = C$.

The (Boolean) rank or factor rank, $b(A)$, of a nonzero $A \in \mathcal{M}_{m,n}(\mathbb{B})$ is defined as the least integer $k$ for which there exist $B \in \mathcal{M}_{m,k}(\mathbb{B})$ and $C \in \mathcal{M}_{k,n}(\mathbb{B})$ such that $A = BC$. The rank of the zero matrix is zero.

The rank of $A \in \mathcal{M}_{m,n}(\mathbb{B})$ is 1 if and only if there exist nonzero (Boolean) vectors $b \in \mathbb{B}^n = \mathcal{M}_{n,1}(\mathbb{B})$ and $c \in \mathbb{B}^n = \mathcal{M}_{1,n}(\mathbb{B})$ such that $A = bc^T$. It is easy to verify that these vectors $b$ and $c$ are uniquely determined by $A$. It is well known ([4]) that $b(A)$ is the least $k$ such that $A$ is the sum of $k$ matrices of rank 1. It follows that $0 \leq b(A) \leq m$ for all nonzero $A \in \mathcal{M}_{m,n}(\mathbb{B})$.

The perimeter ([4]) of the rank-1 matrix $A = bc^T \in \mathcal{M}_{m,n}(\mathbb{B})$, $\text{per}(A)$, is $|b| + |c|$ the sum of the number of nonzero entries in $b$ plus the number of nonzero entries in $c$.

For $A \in \mathcal{M}_{m,n}(\mathbb{B})$, let $\mathcal{F}(A)$ be the set of ordered pairs of matrices that factor $A$. That is, $\mathcal{F}(A) = \{(B,C) \mid B \in \mathcal{M}_{m,k}(\mathbb{B}), C \in \mathcal{M}_{k,n}(\mathbb{B}) \text{ for some } k \text{ such that } A = BC\}$. 
Then $\text{per}(A) = \min_{(B,C) \in \mathcal{F}(A)} \{ |B| + |C| \}$. That is, the perimeter of a matrix is a measure of the minimum number of nonzero entries in a rank $k$ factorization, for any $k$ for which there is such a factorization. An easy observation is that every matrix in $\mathcal{M}_{m,n}(\mathbb{B})$ whose perimeter is either 2 or 3 has rank 1. The arctic rank of $A$, $\text{Arc}(A)$ is one half the perimeter. Note that $\text{Arc}(A)$ may not be an integer. (See [3]).

Let $\mathcal{F}_{\text{sym}}$ denote the matrices in $\mathcal{S}_n(\mathbb{B})$ that have a symmetric factorization, that is if $A \in \mathcal{F}_{\text{sym}}$, then for some $k$, there exists $B \in \mathcal{M}_{n,k}(\mathbb{B})$ such that $A = BB^t$. In this case, $A = b_1b_1^t + b_2b_2^t + \cdots + b_kb_k^t$ where $b_i$ is the $i$th column of $B$. Then, the symmetric perimeter of $A$, $\rho_{\text{sym}}(A)$, is the minimum number, $|B| + |B^t| = 2|B|$, over all symmetric factorizations of $A = BB^t$. The symmetric arctic rank of $A$, $\rho_{\text{ar}}(A)$, is one half the symmetric perimeter of $A$, so $\rho_{\text{ar}}(A) = \frac{1}{2} \rho_{\text{sym}}(A)$, the minimum number of nonzero entries in $B$ for a symmetric factorization of $A = BB^t$. Note that this symmetric arctic rank may be much larger than the order of the given symmetric matrix. For example, consider a $3 \times 3$ matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$ 

Then $\rho_{\text{sym}}(A) = 4$ which is larger than 3.

Note that not all members of $\mathcal{S}_n(\mathbb{B})$ have a symmetric factorization, for example, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has rank two, but no two symmetric rank one matrices sum to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In fact, the product of a symmetric $2 \times 2$ matrix and its transpose can never equal $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, since if $B$ has a nonzero entry, $B^tB$ has a nonzero entry on the main diagonal. If $A \in \mathcal{S}_n(\mathbb{B}) \setminus \mathcal{F}_{\text{sym}}$, then we let $\rho_{\text{sym}}(A) = \rho_{\text{ar}}(A) = \infty$. In fact, we can characterize all symmetric Boolean matrices that have a symmetric factorization.

Recall that the product of two matrices can be calculated in two ways: First by considering each entry of the product as an inner product of rows of the first factor times columns of the second (inner products); and second by considering the product as a sum of rank one matrices (outer products of columns of the first and rows of the second). Let $x_i$ represent the $i$th column of $X$ and $x^j$ the $j$th row of $X$. If $A = BC$, with $B$ an $n \times k$ matrix and $C$ a $k \times m$ matrix, then $a_{i,j} = b^i c_j$, using the first method, and $A = \sum_{\ell=1}^k b_\ell c^\ell$ using the second method. The second method will be used in the proof that follows.

**Theorem 1.2.** Suppose $A \in \mathcal{S}_n(\mathbb{B})$. Then $A \in \mathcal{F}_{\text{sym}}$ if and only if either $A = O$ or for some $0 \leq s \leq n-1$ there exists a permutation matrix $P$ such that $PAP^t = B \oplus O_s$ where $b_{i,i} = 1$ for all $i = 1, \ldots, n-s$.

**Proof.** Suppose that $A \neq O$ and that $A = XX^t$ for $X \in \mathcal{M}_{n,q}(\mathbb{B})$. In this case, $A = \sum_{i=1}^n \sum_{j=1}^n y_i y_j^t$ where $y_i$ is the $i$th row of $X$. If $a_{i,j} = 1$, then since $a_{i,j} = y_i y_j^t$, we have that $y_j \neq 0$, and hence $a_{i,i} = y_i y_i^t = 1$. Thus any row of $A$ that contains a nonzero entry has a nonzero entry on the diagonal. Let $P$ be
a permutation matrix which permutes the nonzero rows of \( A \) to the first \( n - s \) rows, if there are \( n - s \) such rows. Then \( PAP^t \) has nonzero entries in the first \( n - s \) rows/columns and only zero entries in the last \( s \) rows/columns. That is \( PAP^t = B \oplus O_s \). Since any row of \( A \) that has a nonzero entry has a nonzero diagonal entry, \( b_{i,i} = 1 \) for all \( i = 1, \ldots, n - s \).

For the converse, if \( A = O \), then \( A = O_{n,1}O_{1,n}^t \) so that \( A \) has a symmetric factorization.

Now suppose that \( A \neq O \) and for some \( 0 \leq s \leq n - 1 \) and for some permutation matrix \( P \), \( PAP^t = B \oplus O_s \) such that \( b_{i,i} = 1 \) for \( 1 \leq i \leq n - s \). Then, \( A \) has a nonzero diagonal entry in every row which has a nonzero entry. Let \( A_{i,i} = a_{i,i}E_{i,i} \) and for \( 1 \leq i < j \leq n \) let \( A_{i,j} = a_{i,j}(E_{i,i} + E_{j,j} + D_{i,j}) \). Note that \( A_{i,i} = \sum_{i} a_{i,i}e_i e_i^t \) and if \( i < j \), \( A_{i,j} = a_{i,j}[e_i + e_j)(e_i + e_j)] \), so for \( 1 \leq i \leq j \leq n \), either \( A_{i,j} = O \) or \( A_{i,j} \) is a rank 1 matrix. Further, \( A = \sum_{i,j} A_{i,j} \) is a diagonal cell. That is, \( A = \sum_{i,j} A_{i,j} \) and for \( i < j \) let \( x_{i,j} = a_{i,j}(e_i + e_j) \). Observe that either \( x_{i,j} \) is the zero vector or a vector with one or two nonzero entries. Consider the \( n \times n \) matrix \( X = [x_{1,1}, x_{1,2}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}, \ldots, x_{i,i}, x_{i,i+1}, \ldots, x_{i,n}, \ldots, x_{n,n}] \). Then, \[ XX^t = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j} x_{i,j}^t = \sum_{i=1}^{n} \sum_{j=1}^{n} a^2_{i,j}(e_i + e_j)(e_i + e_j)^t = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \]

since \( a^2_{i,j} = a_{i,j} \) and \( e_i + e_i = e_i \) over the Boolean semiring \( \mathbb{B} \). That is, \( XX^t = A \). (Note that the fact that \( X \) may have many zero columns does not change the fact that \( A \) has a symmetric factorization.) 

Let \( SA_k \) denote the set of all matrices in \( M_n(\mathbb{B}) \) whose symmetric arctic rank is \( k \) and note that \( SA_\infty \) is the set of all matrices in \( S_n(\mathbb{B}) \) that do not have a symmetric factorization.

A matrix of special interest in our investigations is \( D^+_i = E_{i,i} + E_{i,j} + D_{i,j} \). Note that:

If \( \rho_{sa}(A) = 1 \), then \( A \) is a diagonal cell. That is, \( SA_1 \) is the set of all diagonal cells;

If \( \rho_{sa}(A) = 2 \), then for some \( i \neq j \), \( A = D^+_i = E_{i,i} + E_{i,j} + D_{i,j} \) or \( A = E_{i,i} + E_{j,j} \); and

If \( \rho_{sa}(A) = 3 \), then up to permutational similarity, \( A = J_3 \oplus O \). \( A = I_3 \oplus O \) or \( A = O \). Note that when \( \rho_{sa}(A) = 3 \), \#(A) = 3, 4 or 6 only.

A mapping \( T : S_n(\mathbb{B}) \to S_n(\mathbb{B}) \) is called a (Boolean) linear operator if \( T(A + B) = T(A) + T(B) \) for all \( A, B \in S_n(\mathbb{B}) \), and \( T(O) = O \). A linear operator \( T : S_n(\mathbb{B}) \to S_n(\mathbb{B}) \) is called a \((P, P^t)\)-operator if there exists a permutation matrix \( P \) such that \( T(X) = PXP^t \) for all \( X \in S_n(\mathbb{B}) \), where \( X^t \) denotes the transpose of \( X \). A linear operator \( T \) is said to preserve a set \( Q \) if \( A \in Q \) implies \( T(A) \in Q \). Also, \( T \) strongly preserves the set \( Q \) if \( A \in Q \) if and only if \( T(X) \in Q \).
Lately there have been many articles on linear preserver problems. For an excellent survey see [6, 7]. In [10], the linear operators that preserve perimeters of matrices over semirings were characterized. In this article we investigate the linear operators that preserve symmetric arctic rank of symmetric matrices over the binary Boolean semirings.

It is well known that the adjacency matrix of an undirected graph is a symmetric (Boolean) (0,1)-matrix and that the adjacency matrix of a clique (a graph whose nonisolated vertices induce a complete graph) is a symmetric rank one Boolean matrix. So the symmetric arctic rank is the minimum over all decompositions of the graph into a union of cliques of the sum of the orders of the cliques. Let $A$ be the adjacency matrix of the complete balanced bipartite graph on $n = 2\ell$ vertices, that is $A = A(K_{\ell,\ell})$, and let $Z = I + A$, the adjacency matrix of $K_{\ell,\ell}$ with loops added at each vertex. Then the factor rank of $Z$, $b(Z)$, is $n$, the symmetric factor rank of $Z$ (the smallest number of symmetric rank one matrices whose sum is $Z$) is $n^2$, and the symmetric arctic rank of $Z$, $\rho_{sa}(Z)$, is $n^2$. So the symmetric rank of a Boolean matrix may be much larger than any other rank. For more on the connection between clique covers of graphs and the ranks of Boolean matrices see [2].

2. Preservers of symmetric arctic ranks

We begin this section with an example to show that preservers of the three sets $SA_1$, $SA_2$, and $SA_3$ do not have the same characterization as, for example, strong preservers of $SA_4$.

**Example 2.1.** Let $T : S_n(\mathbb{B}) \to S_n(\mathbb{B})$ be defined by $T(E_{i,i}) = E_{i,i}, T(D_{i,j}) = E_{i,i} + E_{j,j}$ and extend linearly. Then $T$ strongly preserves $SA_1$, $T$ strongly preserves $SA_2$ and $T$ preserves (not strongly) $SA_3$. Further $T$ does not preserve any other $SA_k$, $4 \leq k \leq n$, since, for example, for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus I_{k-4} \oplus O_{n-k+1}$, $\rho_{sa}(A) = k$ while $\rho_{sa}(T(A)) = k - 1$, since $T(A) = I_{k-1} \oplus O_{n-k+1}$.

An easy observation is that if a linear operator $T : S_n(\mathbb{B}) \to S_n(\mathbb{B})$ preserves $SA_1$, then the image of a diagonal cell is a diagonal cell.

**Lemma 2.2.** If a linear operator $T : S_n(\mathbb{B}) \to S_n(\mathbb{B})$ preserves $SA_1$ and $SA_k$ for any $k, 2 \leq k \leq n$, then $T$ maps the set of diagonal cells bijectively onto the set of diagonal cells. That is $T$ is bijective on $SA_1$.

**Proof.** Since $T$ preserves $SA_1$, the image of a diagonal cell is a diagonal cell. Suppose that the image of two diagonal cells is the same diagonal cell. Without loss of generality assume that $T(E_{1,1}) = T(E_{2,2})$. Then $T(I_k \oplus O_{n-k})$ is the sum of at most $k - 1$ diagonal cells and hence $\rho_{sa}(T(I_k \oplus O_{n-k})) \leq k - 1$, a contradiction, since $T$ preserves $SA_k$. Since $SA_1$ is finite, $T$ is bijective on $SA_1$. $\square$
Theorem 2.3. Let \( L : S_n(\mathbb{B}) \to S_n(\mathbb{B}) \) be a linear operator. Then, \( L \) preserves \( SA_1 \) and \( SA_k \) for some \( 4 \leq k \leq n-1 \) if and only if there exist scalars \( \alpha_{i,j}, \beta_{i,j} \in \mathbb{B} \) for \( 1 \leq i < j \leq n \) and a permutation matrix \( P \) such that

\[
L(X) = P \left( X + \sum_{1 \leq i < j \leq n} x_{i,j} (\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j}) \right) P^t
\]

for all \( X \in S_n(\mathbb{B}) \).

Proof. Since \( L \) preserves both \( SA_1 \) and \( SA_k \) for some \( 4 \leq k \leq n-1 \), by Lemma 2.2, \( L \) is bijective on the set of diagonal cells. Permute by \( P \) so that \( T(X) = P^tL(X)P \) for all \( X \) and such that \( T(E_{i,i}) = E_{i,i} \) for all \( i \). Since permutational similarity preserves all symmetric arctic ranks we also have that \( T \) preserves \( SA_1 \) and \( SA_k \).

Now suppose that \( T(D_{i,j}^+) \) dominates at least three diagonal cells. Let \( I' \) be the sum of \( k-2 \) diagonal cells such that \( \rho_{sa}(D_{i,j}^+ + I') = k \) and such that \( T(D_{i,j}^+ + I') \) dominates \( k+1 \) diagonal cells. This is always possible since \( T \) is bijective on the set of diagonal cells. Then, \( \rho_{sa}(T(D_{i,j}^+ + I')) \geq k+1 \), a contradiction since \( \rho_{sa}(D_{i,j}^+ + I') = k \).

Now suppose that \( T(D_{i,j}^+) \) dominates two digons. Then there is some \( k \) such that \( T(D_{i,j}^+) \) dominates \( D_{k,t} \) for some \( \ell \), but does not dominate \( E_{k,k} \). Let \( I' \) be the sum of \( k-2 \) diagonal cells whose image does not dominate \( E_{k,k} \) and such that \( \rho_{sa}(D_{i,j}^+ + I') = k \). We then have arrived at a contradiction since \( \rho_{sa}(T(D_{i,j}^+ + I')) = \infty \), since \( T(D_{i,j}^+ + I') \notin F_{sym} \). Thus, for each \( (i,j) \) there is some \( (p,q) \) such that \( T(D_{i,j}^+) \subseteq D_{p,q}^+ \).

Now suppose that \( T(D_{i,j}^+) \subseteq D_{p,q}^+ \) and \( T(D_{k,t}^+) \subseteq D_{p,q}^+ \) for \( (i,j) \neq (k,\ell) \). Let \( I' \) be the sum of either \( k-3 \) or \( k-4 \) diagonal cells such that \( \rho_{sa}(D_{i,j}^+ + D_{k,t}^+ + I') = k \). Then, \( T(D_{i,j}^+ + D_{k,t}^+ + I') \subseteq D_{p,q}^+ + T(I') \) which has symmetric arctic rank at most \( k-1 \), a contradiction.

We now have that \( T(D_{i,j}^+) \subseteq D_{i,j}^+ \) for all \( (i,j) \). Suppose that \( T(D_{i,j}^+) \neq D_{i,j}^+ \). Then, \( T(D_{i,j}^+) = E_{i,j} + E_{j,i} \). Without loss of generality, assume that \( T(D_{i,j}^+) = E_{1,1} + E_{2,2} \). Then, \( \rho_{sa}(D_{1,2}^+ + D_{1,3}^+ + E_{4,4} + \cdots + E_{k-1,k-1}) = k \) (see Example 2.1), while \( \rho_{sa}(E_{2,2} + D_{1,3}^+ + E_{4,4} + \cdots + E_{k-1,k-1}) = k-1 \), but \( T(D_{1,2}^+ + D_{1,3}^+ + E_{4,4} + \cdots + E_{k-1,k-1}) = T(E_{2,2} + D_{1,3}^+ + E_{4,4} + \cdots + E_{k-1,k-1}) \), a contradiction. Thus, \( T(D_{i,j}^+) = D_{i,j}^+ \) for all \( (i,j) \).

Further, \( T(D_{i,j}^+) \subseteq D_{i,j}^+ \), so let \( \alpha_{i,j} \) and \( \beta_{i,j} \) be defined by \( T(D_{i,j}) = D_{i,j}^+ + \alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j} \). Since to get \( T \) we permuted by \( P \), it follows that \( L(X) = P \left( X + \sum_{1 \leq i < j \leq n} x_{i,j} (\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j}) \right) P^t \) for all \( X \in S_n(\mathbb{B}) \).

For the converse, if \( L(X) = P \left( X + \sum_{1 \leq i < j \leq n} x_{i,j} (\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j}) \right) P^t \) for all \( X \in \mathbb{B} \), then \( L \) preserves all symmetric arctic ranks except \( \rho_{sa}(A) = \infty \). \( \square \)
Theorem 2.4. Let $L : S_n(\mathbb{B}) \to S_n(\mathbb{B})$ be a linear operator. Then, $L$ preserves $SA_2$ and $SA_k$ for some $4 \leq k \leq n-1$ if and only if there exist scalars $\alpha_{i,j}, \beta_{i,j} \in \mathbb{B}$ for $1 \leq i < j \leq n$ and a permutation matrix $P$ such that

$$T(X) = P \left( X + \sum_{1 \leq i < j \leq n} x_{i,j} (\alpha_{i,j} E_{i,i} + \beta_{i,j} E_{j,j}) \right) P^T$$

for all $X \in S_n(\mathbb{B})$.

Proof. We first show that $T$ preserves $SA_1$.

Suppose that for some $i$, $\rho_{sa}(T(E_{i,i})) \neq 1$, and assume without loss of generality that $i = 1$. Either $T(E_{1,1}) \supseteq D_{1,j}$ for some $j \neq 1$ or $T(E_{1,1}) \subseteq I_1$ and $\#(T(E_{1,1})) \geq 2$.

If $T(E_{1,1}) \supseteq D_{1,j}$, then $T(E_{1,1}) \subseteq D_{1,j}^+$, and for any $q, 1 \leq q \leq n$, $T(E_{q,q}) \subseteq T(E_{11} + E_{q,q}) \subseteq D_{1,j}^+$ since $E_{11} + E_{j,j}$ is in $SA_2$ and the only member of $SA_2$ that dominates $D_{1,j}$ is $D_{1,j}^+$. But then, $T(I_k \oplus O_{n-k}) \subseteq D_{1,j}^+$, a contradiction since $\rho_{sa}(I_k \oplus O_{n-k}) = k$ and $\rho_{sa}(D_{1,j}^+) = 2$. Thus, $T(E_{1,i}) \subseteq I_n$ for all $i$, and for some $i \#(T(E_{1,i})) \geq 2$. But then without loss of generality we may assume that $E_{11} + E_{22} \subseteq D_{1,2}^+$, and hence, we must have that $T(E_{1,1}) \supseteq T(I_2 \oplus O_{n-2})$ so that we have that $T(E_{1,1}) = E_{11} + E_{22}$. But then, $T(E_{j,j}) \subseteq T(E_{11} + E_{j,j}) \subseteq I_2 \oplus O_{n-2}$, for all $j$ since $\rho_{sa}(E_{1,1} + E_{j,j}) = 2$. We now must have that $T(I_k) \subseteq I_2 \oplus O_{n-2}$, and as above we have a contradiction. That is, $T$ preserves $SA_1$.

By Theorem 2.3 the theorem follows. \qed

3. Bijective preservers of symmetric arctic ranks

In this section we shall classify all bijective linear operators on $S_n(\mathbb{B})$ that preserve the set of matrices of symmetric arctic rank $k$. We begin with some lemmas.

Lemma 3.1. If $A \in \mathcal{F}_{sym}$ and $|A \circ I| \geq k$, then $\rho_{sa}(A) \geq k$. Further if $|A \circ I| > k$, then $\rho_{sa}(A) > k$.

Proof. Let $A = A_1 + A_2 + \cdots + A_\ell$ be a rank one decomposition of $A$ such that $\rho_{sa}(A) = \sum_{i=1}^\ell \rho_{sa}(A_i)$ and let $\alpha_i = |A_i \circ I|$ so that $\alpha_i = \rho_{sa}(A_i)$. Then $|A \circ I| = |(\sum_{i=1}^\ell A_i) \circ I| \leq \sum_{i=1}^\ell |A_i \circ I| = \sum_{i=1}^\ell \alpha_i = \rho_{sa}(A)$. That is, $k \leq \rho_{sa}(A)$, and if $|A \circ I| > k$, then $\rho_{sa}(A) > k$. \qed

Lemma 3.2. If $A \in \mathcal{F}_{sym}$ and $\#(A) > \frac{k^2 + k}{2}$, then $\rho_{sa}(A) > k$.

Proof. Suppose $A \in \mathcal{F}_{sym}$ and $\#(A) > \frac{k^2 + k}{2}$. If $|A \circ I| \leq k$, then all the nonzero entries of $A$ lie in the intersection of $|A \circ I|$ columns and the same $|A \circ I|$ rows of $A$, so that $\#(A) \leq \frac{k^2 + k}{2}$, a contradiction. Thus $|A \circ I| > k$, so by Lemma 3.1, $\rho_{sa}(A) > k$. \qed
Lemma 3.3. If $A \in \mathcal{F}_{\text{sym}}$, $\rho_{sa}(A) = k$ and $\#(A) = \frac{k^2 + k}{2}$, then there exists a permutation matrix $P$ such that $A = P^t(J_k \oplus O_{n-k})P$.

Proof. Let $P$ be a permutation matrix such that $P(A \circ I)P^t = I_{\ell} \oplus O_{n-\ell}$. Since $\rho_{sa}(A) = k$, $\ell \leq k$ by Lemma 3.1. Thus, since $A \in \mathcal{F}_{\text{sym}}$, $A \subseteq P^t(J_k \oplus O_{n-k})P$. But $\#(J_k \oplus O_{n-k}) = \frac{k^2 + k}{2}$, so we must have that $A = P^t(J_k \oplus O_{n-k})P$. □

Let $Z \in S_n(\mathbb{B})$. In the following lemma we use the notation $Z[1,\ldots,k][1,\ldots,k]$ to denote the submatrix of $Z$ consisting of the intersection of the first $k$ rows and the first $k$ columns. We use $Z[1,\ldots,k][1,\ldots,k]$ to denote the submatrix of $Z$ consisting of the intersection of the last $n-k$ rows and the last $n-k$ columns.

Lemma 3.4. Let $T : S_n(\mathbb{B}) \to S_n(\mathbb{B})$ be a linear operator. If $T$ is bijective and preserves $SA_k$ for some $3 \leq k \leq n$, then $T$ preserves $SA_1$.

Proof. Since $T$ is bijective on $S_n(\mathbb{B})$, $T$ is bijective on the set of base elements. Suppose that $T(E_{i,i}) = D_{r,s}$ for some $r \neq s$. We may assume that $r = 1$.

Consider $T(J_k \oplus O_{n-k})$. Since $T$ is bijective on the set of base elements, $\#(T(J_k \oplus O_{n-k})) = \#(J_k \oplus O_{n-k}) = \frac{k^2 + k}{2}$ and since $\rho_{sa}(T(J_k \oplus O_{n-k})) = k$, by Lemma 3.3 there is some permutation matrix $P$ such that $T(J_k \oplus O_{n-k}) = P^t(J_k \oplus O_{n-k})P$. Without loss of generality, by permuting, we may assume that $T(J_k \oplus O_{n-k}) = (J_k \oplus O_{n-k})$.

As in the paragraph above, there is some permutation matrix $Q$ such that $Q^tT(O_1 \oplus J_k \oplus O_{n-k-1})Q = J_k \oplus O_{n-k}$ so that we have that $T(O_1 \oplus J_k \oplus O_{n-k-1})$ is a rank one matrix.

Let $Z = T(O_1 \oplus J_k \oplus O_{n-k-1})$ and let $P_1$ be a permutation matrix such that $P_1Z[1,\ldots,k][1,\ldots,k]P_1^t = O_{k-\ell} \oplus J_\ell$. Let $Q_1$ be a permutation matrix such that $Q_1Z[1,\ldots,k][1,\ldots,k]Q_1^t = J_{k-\ell} \oplus O_{n-2k+\ell}$. Then,

$$ (P_1 \oplus Q_1)Z(P_1 \oplus Q_1)^t = \begin{bmatrix} O_{k-\ell} & O & A & B \\ O & J_\ell & C & D \\ A^t & C^t & J_{k-\ell} & O \\ B^t & D^t & O & O_{n-2k+\ell} \end{bmatrix}. $$

Note that if $X \in \mathcal{F}_{\text{sym}}$ and the $i$th row or column of $X$ has a nonzero entry, then $x_{i,i} \neq 0$. Thus, $A$, $B$, and $D$ are all zero matrices. Since $Z$ is rank one, we must have that $C = J_k \oplus O_{n-k}$.

Further, $(P_1 \oplus Q_1)(J_k \oplus O_{n-k})P_1 \oplus Q_1)^t = P_1J_kP_1^t \oplus Q_1O_{n-k}Q_1^t = J_k \oplus O_{n-k}$ so that $(P_1 \oplus Q_1)^tT(J_k \oplus O_{n-k})(P_1 \oplus Q_1)^t = J_k \oplus O_{n-k}$.

Let $L : S_n(\mathbb{B}) \to S_n(\mathbb{B})$ be defined by $L(X) = (P_1 \oplus Q_1)T(X)(P_1 \oplus Q_1)^t$. Then, $L$ is bijective on the set of base elements, $L(E_{1,1}) = D_{p,q}$ for some $p \neq q$, $L(J_k \oplus O_{n-k}) = J_k \oplus O_{n-k}$, and $L(J_k \oplus O_{n-k}) = O_{k-\ell} \oplus J_k \oplus O_{n-2k+\ell}$.

Since $O_1 \oplus J_{k-1} \oplus O_{n-k} \subseteq J_k \oplus O_{n-k}$ and $O_1 \oplus J_{k-1} \oplus O_{n-k} \subseteq O_1 \oplus J_k \oplus O_{n-k-1}$, we must have that $L(O_1 \oplus J_{k-1} \oplus O_{n-k}) \subseteq L(J_k \oplus O_{n-k}) = J_k \oplus O_{n-k}$ and $L(O_1 \oplus J_{k-1} \oplus O_{n-k}) \subseteq L(O_1 \oplus J_k \oplus O_{n-k-1}) = O_{k-\ell} \oplus J_k \oplus O_{n-2k+\ell}$. Since
is bijective on the base elements. This is a contradiction since \( \rho \).

We now have that \( L(E_{1,1} + (O_1 \oplus J_{k-1} \oplus O_{n-k})) = D_{p,q} + (O_1 \oplus J_{k-1} \oplus O_{n-k}) \), and that \( D_{p,q} = L(E_{1,1}) \nsubseteq L(O_1 \oplus J_{k-1} \oplus O_{n-k}) = O_1 \oplus J_{k-1} \oplus O_{n-k} \) since \( L \) is bijective on the base elements. This is a contradiction since \( \rho_{\text{sa}}(E_{1,1} + (O_1 \oplus J_{k-1} \oplus O_{n-k})) = k \) and \( \rho_{\text{sa}}(D_{p,q} + (O_1 \oplus J_{k-1} \oplus O_{n-k})) = \infty \).

Thus \( L \), and hence \( T \), preserves \( S\text{A}_1 \).

**Corollary 3.5.** Let \( T : S_n(\mathbb{B}) \to S_n(\mathbb{B}) \) be a linear operator. Then \( T \) is bijective and preserves \( S\text{A}_k \) for some \( 4 \leq k \leq n-1 \) if and only if \( T \) is a \((P, P^t)\)-operator.

**Proof.** Suppose \( T \) is bijective and preserves \( S\text{A}_k \) for some \( 4 \leq k \leq n-1 \).

By Lemma 3.4 \( T \) preserves \( S\text{A}_1 \). Now, by Theorem 2.3 there exist scalars \( \alpha_{i,j}, \beta_{i,j} \in \mathbb{B} \) for \( 1 \leq i < j \leq n \) and a permutation matrix \( P \) such that

\[
T(X) = P \left( X + \sum_{1 \leq i < j \leq n} x_{i,j} (\alpha_{i,j} E_{i,i} + \beta_{i,j} E_{j,j}) \right) P^t
\]

for all \( X \in S_n(\mathbb{B}) \). If \( \alpha_{i,j} = 1 \), then \( T(D_{i,j}) = T(D_{i,j} + E_{i,j}) \), contradicting that \( T \) is bijective. Thus, \( \alpha_{i,j} = 0 \) for all \( i \neq j \). Similarly \( \beta_{i,j} = 0 \) for all \( i \neq j \).

That is \( T(X) = PXP^t \) for all \( X \in S_n(\mathbb{B}) \).

The converse is routinely established. \( \square \)

**Lemma 3.6.** Let \( T : S_n(\mathbb{B}) \to S_n(\mathbb{B}) \) be a linear operator. If \( T \) is bijective and preserves \( S\text{A}_2 \), then \( T \) preserves \( S\text{A}_1 \).

**Proof.** Suppose \( T \) is bijective. Then \( T \) is bijective on the set of base elements since \( \mathbb{B} \) is antinegative and each member of \( S_n(\mathbb{B}) \) is a unique combination of base elements.

Suppose that \( T(E_{i,i}) = D_{p,q} \) for some \( i \) and \( p \neq q \). Let \( T(E_{j,j}) = F \), a base element. Then \( T(E_{i,j} + E_{j,i}) = D_{p,q} + F \). However \( \rho_{\text{sa}}(E_{i,j} + E_{j,i}) = 2 \) while the sum of \( D_{p,q} \) plus any base element is never in \( S\text{A}_2 \). Thus we have a contradiction and hence, \( T \) preserves \( S\text{A}_1 \). \( \square \)

**Theorem 3.7.** Let \( T : S_n(\mathbb{B}) \to S_n(\mathbb{B}) \) be a linear operator. If \( T \) is bijective and preserves \( S\text{A}_2 \), then \( T \) is a \((P, P^t)\)-operator.

**Proof.** Suppose that \( T \) is bijective and preserves \( S\text{A}_2 \). By Lemma 3.6, \( T \) preserves \( S\text{A}_1 \) and hence \( T \) is bijective on \( S\text{A}_1 \). Let \( P \) be the permutation matrix such that \( T(E_{i,i}) = PE_{i,i}P^t \) for all \( i = 1, \ldots, n \), and let \( T(D_{i,i}) = F_{i,i} \), a base element. Then \( T(D_{i,j}) = T(E_{i,i} + E_{j,j} + D_{i,i}) = T(E_{i,i}) + T(E_{j,j}) + T(D_{i,i}) = PE_{i,i}P^t + PE_{j,j}P^t + F_{i,i} \). Since the only base element that can be added to \( PE_{i,i}P^t + PE_{j,j}P^t \) to get an element of \( S\text{A}_2 \) is \( PD_{i,j}P^t \), we must have that \( F = PD_{i,j}P^t \). That is, \( T(X) = PXP^t \) for all \( X \in S_n(\mathbb{B}) \). \( \square \)
Lemma 3.8. Let $T : S_n(\mathbb{B}) \rightarrow S_n(\mathbb{B})$ be a linear operator. If $T$ is bijective and preserves $SA_3$, then $T$ preserves $SA_2$.

Proof. If $T$ is bijective and preserves $SA_3$, then by Lemma 3.4, $T$ preserves $SA_1$. It now follows that $T$ is bijective on $SA_1$ and $T$ is bijective on the set of digons.

Let $P$ be a permutation matrix such that $L(X) = P^t T(X) P$ for all $X \in S_n(\mathbb{B})$ and such that $L(E_{i,i}) = E_{i,i}$ for all $i$. Further, $L$ is bijective on the set of digons and preserves sets $SA_1$ and $SA_3$. Suppose that $L$ does not preserve $SA_2$. Then since $L$ is the identity on the $SA_1$, we must have that $L(D_{i,j}) \neq D_{i,j}$ for some $i \neq j$. Say $L(D_{i,j}) = D_{r,s}$ with $s \neq i, j$. Let $k$ be any integer, $k \neq i, j, s$. Then $\rho_{n}(E_{i,i} + E_{j,j} + E_{k,k} + D_{i,j} + D_{i,k} + D_{j,k}) = 3$ and $L(E_{i,i} + E_{j,j} + E_{k,k} + D_{i,j} + D_{i,k} + D_{j,k}) = E_{i,j} + E_{j,i} + E_{k,k} + D_{r,s} + L(D_{i,k} + D_{j,k})$. Since $E_{s,s} \not\subseteq E_{i,i} + E_{j,j} + E_{k,k} + D_{r,s} + L(D_{i,k} + D_{j,k})$, $\rho_{n}(L(E_{i,i} + E_{j,j} + E_{k,k} + D_{i,j} + D_{i,k} + D_{j,k})) = \infty$, a contradiction. Thus $T$ preserves $SA_2$. \hfill \Box

Theorem 3.9. Let $T : S_n(\mathbb{B}) \rightarrow S_n(\mathbb{B})$ be a linear operator. If $T$ is bijective and preserves $SA_3$, then $T$ is a $(P, P^t)$-operator.

Proof. By Lemma 3.8, $T$ preserves $SA_2$, and then by Theorem 3.7 the theorem follows. \hfill \Box

Lemma 3.10. Let $T : S_n(\mathbb{B}) \rightarrow S_n(\mathbb{B})$ be a linear operator. If $T$ is bijective and preserves $SA_n$, then $T$ preserves $SA_2$.

Proof. If $T$ is bijective and preserves $SA_n$, then $T$ preserves $SA_1$ by Lemma 3.4. Thus, $T$ is bijective on the set of diagonal cells. Let $P$ be a permutation matrix such that $L(X) = P^t T(X) P$ for all $X \in S_n(\mathbb{B})$ and such that $L(E_{i,i}) = E_{i,i}$ for all $i$. Further, $L$ is bijective on the set of digons and preserves sets $SA_1$ and $SA_n$. By Lemma 3.8 we may assume that $n \geq 4$.

Suppose that $L(D_{i,j}) \neq D_{i,j}$ for some $i \neq j$. Without loss of generality we may assume that $(i, j) = (1, 2)$. Then $L(D_{1,2}) = D_{p,q}$ for some $q \geq 4$. Then, $L(E_{1,1} + E_{2,2} + \cdots + E_{q-1,q-1} + E_{q+1,q+1} + \cdots + E_{n,n} + D_{1,2} + D_{1,3}) = E_{1,1} + E_{2,2} + \cdots + E_{q-1,q-1} + E_{q+1,q+1} + \cdots + E_{n,n} + D_{p,q} + D_{r,s}$ for some $r < s$. But, $\rho_{n}(E_{1,1} + E_{2,2} + \cdots + E_{q-1,q-1} + E_{q+1,q+1} + \cdots + E_{n,n} + D_{1,2} + D_{1,3}) = n$ by Example 2.1, while $\rho_{n}(E_{1,1} + E_{2,2} + \cdots + E_{q-1,q-1} + E_{q+1,q+1} + \cdots + E_{n,n} + D_{p,q} + D_{r,s}) = \infty$ for any choice of $r \neq s$ since the $(q, q)$ entry is zero while the $(p, q)$ entry is nonzero, which is not possible in any member of $F_{sym}$. This contradicts that $L$ preserves $SA_n$. Thus, $L$ preserves $SA_2$. \hfill \Box

Theorem 3.11. Let $T : S_n(\mathbb{B}) \rightarrow S_n(\mathbb{B})$ be a linear operator. If $T$ is bijective and preserves $SA_n$, then $T$ is a $(P, P^t)$-operator.

Proof. By Lemma 3.10, $T$ preserves $SA_2$, and then by Theorem 3.7 the theorem follows. \hfill \Box
4. Strong preservers of symmetric arctic rank

In this section we shall classify all linear operators on $S_n(\mathbb{B})$ that strongly preserve the set of matrices of symmetric arctic rank $k$.

**Theorem 4.1.** Let $T : S_n(\mathbb{B}) \to S_n(\mathbb{B})$ be a linear operator. If $T$ strongly preserves $SA_k$ for some $3 \leq k \leq n$, then $T$ is bijective on the set of base elements.

**Proof.** Suppose that $T(X) = O$ for some $X \in S_n(\mathbb{B})$. Then, for some $i \leq j$ $T(D_{i,j}) = O$. Without loss of generality, we may assume that $1 \leq i, j \leq 2$. Then, $T((J_k \oplus O) \setminus D_{i,j}) = T(J_k \oplus O)$, contradicting that $T$ strongly preserves $SA_k$ since $\rho_{sa}(J_k \oplus O) = k$ while $\rho_{sa}((J_k \oplus O) \setminus D_{i,j})$ is $\infty$ if $i = j$ or is $2k - 2 \neq k$ if $i \neq j$ since $k \geq 3$. Thus $T$ is nonsingular. Note that this does not mean that $T$ is bijective, just that $T(X) = O$ only for $X = O$.

Now, suppose that $\#(T(D_{i,j})) \geq 2$ for some $i \leq j$. Again, without loss of generality we may assume that $1 \leq i, j \leq 2$. But then, there is a base element $E$ such that $T((J_k \oplus O) \setminus F) = T(J_k \oplus O)$, again a contradiction. Thus, $T$ maps base elements to base elements. Suppose that $E$ and $F$ are base elements, $E \neq F$, and $T(E) = T(F)$. If for some permutation matrix $P$, $(E + F) \subseteq P(J_k \oplus O)P$, then

$$T(P^t(J_k \oplus O)P) = T((P^t(J_k \oplus O)P \setminus E) + E)$$

$$= T(P^t(J_k \oplus O)P \setminus E) + T(E)$$

$$= T(P^t(J_k \oplus O)P \setminus E) + T(F)$$

$$= T((P^t(J_k \oplus O)P \setminus E) + F)$$

$$= T(P^t(J_k \oplus O)P \setminus E),$$

a contradiction since $\rho_{sa}(P^t(J_k \oplus O)P \setminus E) \neq k$ while $\rho_{sa}(P^t(J_k \oplus O)P) = k$. Note that if $k \geq 4$, then it is always possible to find such a permutation matrix $P$.

Thus, if there is no permutation matrix $P$ such that $P(E + F)P^t \subseteq J_k \oplus O$, then $k \leq 3$. However, there is a permutation $P$ such that $E \subseteq P(J_k \oplus O)P$. In this case, as above we have that

$$T(P^t(J_k \oplus O)P) = T((P^t(J_k \oplus O)P \setminus E) + E)$$

$$= T(P^t(J_k \oplus O)P \setminus E) + T(E)$$

$$= T(P^t(J_k \oplus O)P \setminus E) + T(F)$$

$$= T((P^t(J_k \oplus O)P \setminus E) + F),$$

which is a contradiction since $\rho_{sa}(P^t(J_k \oplus O)P) = k$ while $(P^t(J_k \oplus O)P \setminus E) + F \notin \mathcal{F}_{sym}$ if $F$ is a digon or $E$ is a diagonal cell, and is in $SA_{2k + 1}$ if $E$ is a digon and $F$ is a diagonal cell, so that $\rho_{sa}((P^t(J_k \oplus O)P \setminus E) + F) \neq k$.

We now have that $T$ is injective on the set of base elements. Since $S_n(\mathbb{B})$ is finite, $T$ is bijective on the set of base elements. \qed
Corollary 4.2. Let $T: S_n(\mathbb{B}) \to S_n(\mathbb{B})$ be a linear operator. If $T$ strongly preserves $SA_k$ for some $3 \leq k \leq n$, then $T$ is bijective on $S_n(\mathbb{B})$.

Proof. By the above theorem, $T$ is bijective on the set of base elements. Every member of $S_n(\mathbb{B})$ is a unique sum of base elements, so that $T$ is bijective on $S_n(\mathbb{B})$. 

Theorem 4.3. Let $T: S_n(\mathbb{B}) \to S_n(\mathbb{B})$ be a linear operator. Then $T$ strongly preserves $SA_k$ for some $4 \leq k \leq n-1$ if and only if $T$ is a $(P, P^t)$-operator.

Proof. Suppose that $T$ strongly preserves $SA_k$. By Corollary 4.2 we have that $T$ is bijective and by Lemma 3.4 that $T$ preserves $SA_1$. Now by Theorem 2.3, there exist scalars $\alpha_{i,j}, \beta_{i,j} \in \mathbb{B}$ for $1 \leq i < j \leq n$ and a permutation matrix $P$ such that

$$T(X) = P \left( X + \sum_{1 \leq i < j \leq n} x_{i,j}(\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j}) \right) P^t$$

for all $X \in S_n(\mathbb{B})$. Suppose that $\alpha_{i,j} \neq 0$ for some $i, j$. Without loss of generality we may assume that $(i, j) = (1, 2)$. Then $\rho_{sa}(D_{1,2} + (O_1 \oplus I_{k-1} \oplus O_{n-k})) = \infty$ but $T(D_{1,2} + (O_1 \oplus I_{k-1} \oplus O_{n-k})) = P(D_{1,2} + (I_k \oplus O_{n-k}))P^t$ which has symmetric arctic rank $k$, a contradiction. Thus, $\alpha_{i,j} = \beta_{i,j} = 0$ for all $i, j$. That is $T$ is a $(P, P^t)$-operator. Since every $(P, P^t)$-operator preserves all symmetric arctic ranks, the converse follows.

5. Summary

We summarize the above results below.

Theorem 5.1. Let $T: S_n(\mathbb{B}) \to S_n(\mathbb{B})$ be a linear operator. Then the following are equivalent:

1. $T$ preserves $SA_1$ and $SA_k$ for some $4 \leq k \leq n-1$;
2. $T$ preserves $SA_2$ and $SA_k$ for some $4 \leq k \leq n-1$;
3. there exist scalars $\alpha_{i,j}, \beta_{i,j} \in \mathbb{B}$ for $1 \leq i < j \leq n$ and a permutation matrix $P$ such that

$$T(X) = P \left( X + \sum_{1 \leq i < j \leq n} x_{i,j}(\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j}) \right) P^t$$

for all $X \in S_n(\mathbb{B})$.

Proof. (1) implies (3) by Theorem 2.3, and (2) implies (3) by Theorem 2.4. To show (3) implies (1) and (2), one observes that such a $T$ defined in (3) preserves all symmetric arctic ranks except $\infty$.

Theorem 5.2. Let $T: S_n(\mathbb{B}) \to S_n(\mathbb{B})$ be a linear operator. Then the following are equivalent:

1. $T$ is bijective and preserves $SA_k$ for some $2 \leq k \leq n$;
(2) \( T \) strongly preserves \( SA_k \) for some \( 4 \leq k \leq n - 1; \)

(3) \( T \) is a \((P, P^t)\)-operator.

Proof. (1) implies (3) by Theorem 3.7, Theorem 3.9, Corollary 3.5, and Theorem 3.11; (2) implies (3) by Theorem 4.3; and to show (3) implies (1) and (2) one observes that any \((P, P^t)\)-operator is bijective and strongly preserves all symmetric arctic ranks. \( \Box \)

Acknowledgements. The authors are highly grateful to the referee for his/her valuable comments and suggestions helpful in improving this paper.

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