TWO MEROMORPHIC FUNCTIONS SHARING FOUR PAIRS OF SMALL FUNCTIONS

Van An Nguyen and Duc Quang Si

Abstract. The purpose of this paper is twofold. The first is to show that two meromorphic functions \( f \) and \( g \) must be linked by a quasi-Möbius transformation if they share a pair of small functions regardless of multiplicity and share other three pairs of small functions with multiplicities truncated to level 4. We also show a quasi-Möbius transformation between two meromorphic functions if they share four pairs of small functions with multiplicities truncated by 4, where all zeros with multiplicities at least \( k > 865 \) are omitted. Moreover the explicit Möbius-transformation between such \( f \) and \( g \) is given. Our results are improvement of some recent results.

1. Introduction

For a divisor \( \nu \) on \( \mathbb{C} \), we define its counting function by

\[
N(r, \nu) = \int_1^r \frac{n(t)}{t} dt \quad (1 < r < \infty), \quad \text{where } n(t) = \sum_{|z| \leq t} \nu(z).
\]

For two positive integers \( k, M \) (maybe \( M = \infty \)), we set

\[
\nu_{\leq k}^M(z) = \begin{cases} \min\{M, \nu(z)\} & \text{if } \nu(z) \leq k \\ 0 & \text{for otherwise,} \end{cases}
\]

and write \( N_{\leq k}^M(r, \nu) \) for \( N(r, \nu_{\leq k}^M) \). We will omit character \([M]\) (resp. \( \leq k \)) if \( M = +\infty \) (resp. \( k = +\infty \)). Similarly, we define \( N_{= k}^M(r, \nu) \) and \( N_{> k}^M(r, \nu) \).

For a discrete set \( S \subset \mathbb{C} \), we consider it as a reduced divisor and denote by

\[
N(r, S) \quad \text{its counting function.}
\]

Let \( f \) be a nonzero holomorphic function on \( \mathbb{C} \). For each \( z_0 \in \mathbb{C} \), expanding \( f \) as \( f(z) = \sum_{i=0}^{\infty} b_i (z - z_0)^i \) around \( z_0 \), then we define \( \nu_f^i(z_0) := \min\{i : b_i \neq 0\} \).

Let \( \varphi \) be a non-constant meromorphic function on \( \mathbb{C} \). Then there are two holomorphic functions \( \varphi_1, \varphi_2 \) without common zeros such that \( \varphi = \frac{\varphi_1}{\varphi_2} \). We

Received March 19, 2016; Revised October 4, 2016; Accepted March 10, 2017.

2010 Mathematics Subject Classification. Primary 32H30, 32A22; Secondary 30D35.

Key words and phrases. meromorphic function, small function, Möbius transformation.

©2017 Korean Mathematical Society

1159
define $\nu_{\varphi}^0 := \nu_{\varphi_1}^0$ and $\nu_{\varphi}^\infty := \nu_{\varphi_2}^0$. The proximity function of $\varphi$ is defined by:

$$m(r, \varphi) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\varphi(re^{i\theta})| d\theta \quad (r > 1),$$

here $\log^+ x = \max\{1, \log x\}$ for $x \in (0, \infty)$. The Nevanlinna characteristic function of $\varphi$ is defined by

$$T(r, \varphi) := m(r, \varphi) + N(r, \nu_{\varphi}^\infty).$$

We will denote by $S(r, \varphi)$ a quantitative equal to $o(T(r, \varphi))$ for all $r \in (1, \infty)$ outside a finite Borel measure set.

Let $f, a$ be two meromorphic functions on $\mathbb{C}$. The function $a$ is said to be small (with respect to $f$) if and only if $T(r, a) = S(r, f)$. We denote by $\mathcal{R}_f$ the field of all small (with respect to $f$) functions on $\mathbb{C}$.

Let $f$ and $g$ be two meromorphic functions on $\mathbb{C}$. Let $(a, b)$ be a pair of small (with respect to $f$ and $g$) meromorphic functions on $\mathbb{C}$ and let $n$ be a positive integer or $+\infty$.

**Definition 1.1.** We say that $f$ and $g$ share $(a, b)$ weakly with multiplicities truncated to level $n$, or share $(a, b) \text{ CM}_n^*$ in another word, if

$$\min\{n, \nu_{a-b}^0(z)\} = \min\{n, \nu_{a-b}^0(z)\}$$

for all $z \in \mathbb{C}$ outside a discrete set of counting function equal to $S(r, f) + S(r, g)$.

We will say that $f$ and $g$ share a $IM^*$ if $n = 1$ and say that $f$ and $g$ share a $CM^*$ if $n = \infty$ and write $CM^*$ for $CM^*_\infty$.

The function $f$ is said to be a quasi-Möbius transformation of $g$ if there exist small (with respect to $g$) functions $\alpha_i (1 \leq i \leq 4)$ with $\alpha_1 \alpha_4 - \alpha_2 \alpha_3 \neq 0$ such that $f = \frac{\alpha_1 g + \alpha_2}{\alpha_3 g + \alpha_4}$. If all functions $\alpha_i (1 \leq i \leq 4)$ are constants, then we say that the map $f$ is a Möbius transformation of $g$. An interesting question arises here: “Are there any quasi-Möbius transformation between $f$ and $g$ if they share some pairs of small functions $IM^*$ or $CM^*$?”

This problem has been studied by many authors, such as T. Czubiak-G. Gundersen [2], P. Li-C. C. Yang [3], P. Li-Y. Zhang [4], S. D. Quang-L. N. Quynh [7, 8], H. Z. Cao-T. B. Cao [1], L. Zhang-L. Yan [11] and others. We state here the recent result of P. Li and Y. Zhang, which is one of the best results available at present.

**Theorem A** (P. Li - Y. Zhang [4]). Let $f$ and $g$ be non-constant meromorphic functions and $a_i, b_i$ $(i = 1, 2, 3, 4)$ ($a_i \neq a_j, b_i \neq b_j, i \neq j$) be small functions (with respect to $f$ and $g$). If $f$ and $g$ share three pairs $(a_i, b_i)$, $(i = 1, 2, 3) \text{ CM}^*$, and share the fourth pair $(a_4, b_4) \text{ IM}^*$, then $f$ is a quasi-Möbius transformation of $g$.

In this paper, we will improve the above result to the following.

**Theorem 1.2.** Let $f$ and $g$ be non-constant meromorphic functions and $a_i, b_i$ $(i = 1, 2, 3, 4)$ ($a_i \neq a_j, b_i \neq b_j, i \neq j$) be small functions (with respect to $f$ and
g). If \( f \) and \( g \) share the pair \((a_1, b_1)\) \(IM^*\) and share three pairs \((a_i, b_i), (i = 2, 3, 4)\) \(CM_*\), then \( f \) is a quasi-Möbius transformation of \( g \). Moreover there is a permutation \((i_1, i_2, i_3, i_4)\) of \((1, 2, 3, 4)\) such that

\[
f - a_{i_1} a_{i_3} - a_{i_2} = g - b_{i_1} b_{i_3} - b_{i_2} \quad \text{or} \quad f - a_{i_1} a_{i_3} - a_{i_2} = g - b_{i_1} b_{i_3} - b_{i_2}
\]

\[
f - a_{i_2} a_{i_3} - a_{i_1} = g - b_{i_2} b_{i_3} - b_{i_1} \quad \text{or} \quad f - a_{i_2} a_{i_3} - a_{i_1} = g - b_{i_2} b_{i_3} - b_{i_1}.
\]

In the next theorem, we will consider the case where all zeros of functions \(f - a_i\) with multiplicities at least \(k > 865\) do not need to be counted. We prove the following.

**Theorem 1.3.** Let \( f \) and \( g \) be non-constant meromorphic functions and \(a_i, b_i\) \((i = 1, 2, 3, 4)\) \((a_i \neq a_j, b_i \neq b_j, i \neq j)\) be small functions (with respect to \( f \) and \( g \)). Assume that

\[
\min \{\nu_{f-a_i}^o(z), 4\} = \min \{\nu_{g-b_i}^o(z), 4\} \quad (1 \leq i \leq 4)
\]

for all \( z \) outside a discrete set \( S \) of counting function equal to \( S(r, f) + S(r, g) \). If \( k > 865 \), then there is a permutation \((i_1, i_2, i_3, i_4)\) of \((1, 2, 3, 4)\) such that

\[
f - a_{i_1} a_{i_3} - a_{i_2} = g - b_{i_1} b_{i_3} - b_{i_2} \quad \text{or} \quad f - a_{i_1} a_{i_3} - a_{i_2} = g - b_{i_1} b_{i_3} - b_{i_2}
\]

\[
f - a_{i_2} a_{i_3} - a_{i_1} = g - b_{i_2} b_{i_3} - b_{i_1} \quad \text{or} \quad f - a_{i_2} a_{i_3} - a_{i_1} = g - b_{i_2} b_{i_3} - b_{i_1}.
\]

2. Some lemmas and auxiliary results from Nevanlinna theory

**Theorem 2.1 ([10], Corollary 1).** Let \( f \) be a non-constant meromorphic function on \( \mathbb{C} \). Let \( a_1, \ldots, a_q \) \((q \geq 3)\) be \( q \) distinct small (with respect to \( f \)) meromorphic functions on \( \mathbb{C} \). Then, for each \( \epsilon > 0 \), the following holds

\[
(q - 2 - \epsilon)T(r, f) \leq \sum_{i=1}^{q} N^{[1]}(r, \nu_{f-a_i}^o) + S(r, f).
\]

**Theorem 2.2 ([9], Corollary 1).** Let \( f \) be a non-constant meromorphic function on \( \mathbb{C} \). Let \( a_1, \ldots, a_q \) \((q \geq 3)\) be \( q \) distinct small (with respect to \( f \)) meromorphic functions on \( \mathbb{C} \). Then the following holds

\[
\frac{q}{3}T(r, f) \leq \sum_{i=1}^{q} N^{[1]}(r, \nu_{f-a_i}^o) + o(T(r, f)).
\]

**Lemma 2.3 ([3], Lemma 7).** Let \( f_1 \) and \( f_2 \) be two non-constant meromorphic functions satisfying

\[
N^{[1]}(r, \nu_{f_1}^o) + N^{[1]}(r, \nu_{f_2}^o) = S(r, f_1) + S(r, f_2) \quad (i = 1, 2).
\]

If \((f_1 t f_2 - 1)\) is not identically zero for all integers \( s \) and \( t \) \(|s| + |t| > 0\), then for any positive number \( \epsilon \), we have

\[
N_0(r, 1; f_1, f_2) \leq \epsilon(T(r, f_1) + T(r, f_2)),
\]

where \(N_0(r, 1; f_1, f_2)\) denotes the reduced counting function of \( f_1 \) and \( f_2 \) related to the common 1-points.
Lemma 3.1. Let $f$ be a nonconstant meromorphic function and $a$ be a small function (with respect to $f$). Then for each positive integer $k$ ($k$ may be $+\infty$) we have

$$N^{[1]}(r, \nu_{f-a_i}^0) \leq \frac{k}{k+1} N^{[1]}(r, \nu_{f-a_i}^0 \leq k) + \frac{1}{k+1} T(r, f) + S(r, f).$$

Proof. We have

$$N^{[1]}(r, \nu_{f-a_i}^0) = N^{[1]}(r, \nu_{f-a_i}^0 \leq k) + N^{[1]}(r, \nu_{f-a_i}^0 > k) \leq (1 - \frac{1}{k+1}) N^{[1]}(r, \nu_{f-a_i}^0 \leq k) + \frac{1}{k+1} N(r, \nu_{f-a_i}^0) \leq \frac{k}{k+1} N^{[1]}(r, \nu_{f-a_i}^0 \leq k) + \frac{1}{k+1} T(r, f) + S(r, f).$$

The lemma is proved. $\square$

Let $\{H_i\}_{i=1}^q$ ($q \geq N + 2$) be a set of $q$ hyperplanes in $\mathbb{P}^N(\mathbb{C})$. We say that $\{H_i\}_{i=1}^q$ are in general position if for any $1 \leq i_1 < \cdots < i_{N+1} \leq q$ we have $\bigcap_{j=1}^{N+1} H_{i_j} = \emptyset$.

Theorem 2.5 ([5], Theorem 3.1). Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a linearly holomorphic mapping. Let $\{H_i\}_{i=1}^q$ ($q \geq N + 2$) be a set of $q$ hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. Then

$$(q - N - 1)T(r, f) \leq \sum_{i=1}^q N^{[N]}(r, f^* H_i) + S(r, f),$$

where $f^* H_i$ denotes the pull back divisor of $H_i$ by $f$.

3. Proof of the main theorems

Lemma 3.1. Let $f$ and $g$ be two meromorphic functions on $\mathbb{C}$. Let $\{a_i\}_{i=1}^3$ and $\{b_i\}_{i=1}^3$ be two sets of small (with respect to $f$) meromorphic functions on $\mathbb{C}$ with $a_i \neq a_j$ and $b_i \neq b_j$ for all $1 \leq i < j \leq 3$. Assume that

$$\min\{\nu_{f-a_i}^0(z), 1\} = \min\{\nu_{b_i}^0(z), 1\} \ (1 \leq i \leq 3)$$

for all $z$ outside a discrete subset $S$ of counting function equal to $S(r, f)$. If $k \geq 3$, then $T(r, f) = O(T(r, g))$ and $T(r, g) = O(T(r, f))$. In particular, $S(r, f) = S(r, g)$.

Proof. By the second main theorem (Theorem 2.2), we have

$$T(r, f) \leq \sum_{i=1}^3 N^{[1]}(r, \nu_{f-a_i}^0) + S(r, f) \leq \frac{k}{k+1} \sum_{i=1}^3 N^{[1]}(r, \nu_{f-a_i}^0) + \frac{3}{k+1} T(r, f) + S(r, f)$$
Lemma 3.2. \( z \in \mathbb{C} \setminus \mathbb{D}(\delta) \) and \( h = f - g \). \( \) \( z \) is a quasi-Möbius transformation of \( f \) such that

\[
|f - g| \leq \frac{4}{3} \frac{1}{k + 1} T(r, f) + S(r, f)
\]

This implies that \( \| T(r, f) = O(T(r, g)) \). Similarly, we have \( \| T(r, g) = O(T(r, f)) \). The lemma is proved.

**Lemma 3.2.** Let \( f \) and \( g \) be non-constant meromorphic functions and \( a_i, b_i \) \( (i = 1, 2, 3, 4) \) \( (a_i \neq a_j, b_i \neq b_j, i \neq j) \) be small functions \( \) (with respect to \( f \) and \( g \)) such that

\[
\min\{\nu^0_{f-a_i} \leq k(z), 1\} = \min\{\nu^0_{g-b_i} \leq k(z), 1\} \quad (1 \leq i \leq 4)
\]

for all \( z \) outside a discrete subset \( S \) of counting function equal to \( S(f, r) + S(g, r) \). Assume that \( f \) is a quasi-Möbius transformation of \( g \). If \( k \geq 3 \), then there is a permutation \( (i_1, i_2, i_3, i_4) \) of \( (1, 2, 3, 4) \) such that

\[
\frac{f - a_{i_1}}{f - a_{i_2}} = \frac{a_{i_2} - a_{i_1}}{a_{i_3} - a_{i_4}} \quad \text{or} \quad \frac{f - a_{i_1}}{f - a_{i_2}} = \frac{a_{i_2} - a_{i_1}}{a_{i_3} - a_{i_4}}
\]

\[
\text{Proof.} \quad \text{By Lemma 3.1 we have } S(r, f) = S(r, g). \quad \text{Suppose that there is only one index } i_0 \in \{1, 2, 3, 4\} \text{ such that } N^{|r|}_{\leq k}(r, \nu^0_{f-a_{i_0}}) \neq S(r, f). \then \text{by Theorem 2.1, we see that}
\]

\[
(2 - \epsilon)T(r, f) \leq N^{|r|}_{\leq k}(r, \nu^0_{f-a_{i_0}}) + \sum_{i \neq i_0} N^{|r|}_{\leq k}(r, \nu^0_{f-a_i}) + S(r, f)
\]

\[
\leq T(r, f) + \frac{k}{k + 1} \sum_{i \neq i_0} N^{|r|}_{\leq k}(r, \nu^0_{f-a_i}) + \frac{3}{k + 1} T(r, f) + S(r, f)
\]

\[
\leq (1 + \frac{3}{k + 1}) T(r, f) + S(r, f), \quad \forall \epsilon > 0.
\]

It implies that \( 2 \leq 1 + \frac{3}{k + 1} \), i.e., \( k \leq 2 \). This is a contradiction.

Therefore, there are at least two indices \( i_1, i_2 \) in \( \{1, 2, 3, 4\} \) so that

\[
N^{|r|}_{\leq k}(r, \nu^0_{f-a_{i_1}}) = N^{|r|}_{\leq k}(r, \nu^0_{f-a_{i_2}}) \neq S(r, f) \quad (1 \leq j \leq 2).
\]

Denote by \( i_3, i_4 \) the remaining indices, i.e., \( \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\} \). We set

\[
F = \frac{f - a_{i_1}}{f - a_{i_2}}, \quad \frac{a_{i_2} - a_{i_1}}{a_{i_3} - a_{i_4}}, \quad G = \frac{g - b_{i_1}}{g - b_{i_2}}, \quad \frac{b_{i_2} - b_{i_1}}{b_{i_3} - b_{i_4}}
\]

\[
A = \frac{a_{i_4} - a_{i_1}}{a_{i_4} - a_{i_2}}, \quad \frac{a_{i_3} - a_{i_2}}{a_{i_3} - a_{i_1}}, \quad B = \frac{b_{i_4} - b_{i_1}}{b_{i_4} - b_{i_2}}, \quad \frac{b_{i_3} - b_{i_2}}{b_{i_3} - b_{i_1}}
\]

Then we easily have the following assertions:

- \( T(r, F) = T(r, f) + S(r, f) \), \( T(r, G) = T(r, g) + S(r, f) \),
- \( T(r, A) = T(r, B) + S(r, f) = S(r, f) \),
• \( \min \{ \nu^0_{F-a}, \nu^0_{G-b} \} = \min \{ \nu^0_{G-b}, \nu^0_{G-b} \} \), \((a, b) \in \{(0,0), (1, 1), (\infty, \infty), (A, B)\} \) and for all \( z \) outside a discrete set of counting function equal to \( S(r, f) \).

Since \( f \) and \( g \) are quasi-Möbius transformations of each other, then so are \( F \) and \( G \). Hence, there exist four small functions (with respect to \( f \)) \( \alpha, \beta, \gamma, \lambda \) with \( \alpha \lambda - \beta \gamma \neq 0 \) such that

\[
G = \frac{\alpha F + \beta}{\gamma F + \lambda}.
\]

By the assumption, we have \( 0 = \frac{2(\gamma)}{\alpha(z)} \) for all \( z \in \text{Supp} (\nu^0_{-a}, \nu^0_{-b}) \) outside a discrete set of counting function equal to \( S(r, f) \). This implies that \( \frac{\alpha}{\gamma} \equiv 0 \) i.e., \( \beta \equiv 0 \). Similarly, we have \( \frac{2(\gamma)}{\alpha(z)} = 0 \) for all \( z \in \text{Supp} (\nu^0_{-a}, \nu^0_{-b}) \) outside a discrete set of counting function equal to \( S(r, f) \), and hence \( \frac{\alpha}{\gamma} \equiv 0 \), i.e., \( \gamma \equiv 0 \). Therefore, we obtain \( G = \frac{\alpha}{\gamma} F \).

We now suppose that \( \frac{\gamma}{\alpha} \notin \{1, B, \frac{1}{A}\} \). It is easy to see that:

- \( N^{|i|}_{\leq k}(r, \nu^0_{G-1}) \leq N^{|i|}_{\leq k}(r, \nu^0_{-1}) + S(r, f) = S(r, f) \),
- \( N^{|i|}_{\leq k}(r, \nu^0_{G-B}) \leq N^{|i|}_{\leq k}(r, \nu^0_{-B}) + S(r, f) = S(r, f) \),
- \( N^{|i|}_{\leq k}(r, \nu^0_{G-\frac{1}{B}}) \leq N^{|i|}_{\leq k}(r, \nu^0_{-\frac{1}{B}}) + S(r, f) \leq N^{|i|}_{\leq k}(r, \nu^0_{-1}) + S(r, f) = S(r, f) \).

Similarly to (3.3), we have

\[
(2 - \epsilon)T(r, G) \leq (1 + \frac{3}{k + 1})T(r, G) + S(r, G), \quad \forall \epsilon > 0.
\]

This implies that \( k \leq 2 \). It is a contradiction.

Then \( \frac{\gamma}{\alpha} \in \{1, B, \frac{1}{A}\} \). We have the following three cases:

**Case 1.** \( \frac{\gamma}{\alpha} \equiv 1 \). Then we have \( G = F \), i.e., \( \frac{f - a_{i_1}}{f - a_{i_2}} = \frac{b_{i_3} - b_{i_4}}{b_{i_3} - b_{i_4}} \).

This implies the desired conclusion of the lemma.

**Case 2.** \( \frac{\gamma}{\alpha} \equiv \frac{B}{A} \). Then we have \( G = BF \), i.e., \( \frac{f - a_{i_3}}{f - a_{i_4}} = \frac{b_{i_3} - b_{i_4}}{b_{i_3} - b_{i_4}} \).

After changing indices \( i_3 \) and \( i_4 \), we get the desired conclusion of the lemma.

**Case 3.** \( \frac{\gamma}{\alpha} \equiv B \). Then we have \( G = BF \), i.e., \( \frac{f - a_{i_3}}{f - a_{i_4}} = \frac{b_{i_3} - b_{i_4}}{b_{i_3} - b_{i_4}} \).

This implies the desired conclusion of the lemma.

Therefore, the above three cases complete the proof of the lemma. \( \square \)

**Proposition 3.5.** Let \( F \) and \( G \) be non-constant meromorphic functions and \( A_i, B_i \) \((i = 1, 2, 3) \). \( A_i \neq A_j, B_i \neq B_j, i \neq j \) be small functions (with respect to \( F \) and \( G \)). Assume that \( F \) is not a quasi-Möbius transformation of \( G \). Then for every positive integer \( n \) we have the following inequality

\[
N(r, \nu) \leq N^{|i|}(r, |\nu^0_{F-A_i} - \nu^0_{G-B_i}|) + N^{|i|}(r, |\nu^0_{F-A_i} - \nu^0_{G-B_i}|) + S(r),
\]

where \( S(r) = o(T(r, F) + T(r, G)) \) outside a finite Borel measure set of \( [1, +\infty) \) and \( \nu \) is the divisor defined by \( \nu(z) = \max\{0, \min\{\nu^0_{F-A_i}(z), \nu^0_{G-B_i}(z)\} - 1\} \).
Proof. By considering meromorphic functions $F - A_1, A_3 - A_2$ and $G - B_1, B_3 - B_2$ instead of $f$ and $g$, we may assume that $A_1 = B_1 = 0, A_2 = B_2 = \infty$ and $A_3 = B_3 = 1$.

Since $F$ is not a quasi-Möbius transformation of $G$, we have

$$H := \frac{F'}{F} - \frac{G'}{G} = \frac{(F/G)'}{(F/G)} \neq 0.$$ 

By the lemma on logarithmic derivatives, it follows that

$$m(r, H) \leq m(r, \frac{F'}{F}) + m(r, \frac{G'}{G}) = S(r).$$

We also see that $H$ has only simple poles and if $z$ is a pole of $H$, then it must be either $\nu_H^0(z) \neq \nu_F^0(z)$ or $\nu_H^\infty(z) \neq \nu_F^\infty(z)$. Then it follows that

$$N(r, \nu_H^0) \leq N^{[1]}(r, |\nu_F^0 - \nu_G^0|) + N^{[1]}(r, |\nu_F^\infty - \nu_G^\infty|).$$

On the other hand, if $z$ is a common zero of $(F - 1)$ and $(G - 1)$, then $z$ will be a zero of $H = \frac{(F - G)'}{(F/G)}$ with multiplicity at least $(\min\{\nu_{F-1}^0(z), \nu_{G-1}^0(z)\} - 1)$. This yields that

$$N(r, \nu_H^0) \geq N(r, \nu).$$

Thus

$$N(r, \nu) \leq N(r, \nu_H^0) \leq T(r, H) = m(r, H) + N(r, \nu_H^0) \leq N^{[1]}(r, |\nu_F^0 - \nu_G^0|) + N^{[1]}(r, |\nu_F^\infty - \nu_G^\infty|).$$

The proposition is proved. \hfill $\Box$

Proof of Theorem 1.2. Suppose that $f$ is not a quasi-Möbius transformation of $g$. By Lemma 3.1, we have $S(r, f) = S(r, g)$. For each $1 \leq i \leq 4$, we define a divisor $\nu_i$ by setting

$$\nu_i(z) = \max\{0, \min\{\nu_{f-a_i}^0(z), \nu_{g-a_i}^0(z)\} - 1\}.$$ 

By assumptions of the theorem and by Lemma 3.5, for a permutation $(i, j, s)$ of $(2, 3, 4)$ we have following estimates:

$$N_{a_2}^1(r, \nu_{f-a_2}^0) + 2N_{a_3}^1(r, \nu_{f-a_3}^0) + 3N_{a_4}^1(r, \nu_{f-a_4}^0) \leq N(r, \nu_i) + S(r, f) \leq N^{[1]}(r, |\nu_{f-a_j}^0 - \nu_{g-b_i}^0|) + N^{[1]}(r, |\nu_{f-a_s}^0 - \nu_{g-b_i}^0|) + S(r, f) \leq N_{a_2}^1(r, \nu_{f-a_2}^0) + N_{a_3}^1(r, \nu_{f-a_3}^0) + S(r, f).$$

From these inequalities, we easily obtain that

$$N_{a_2}^1(r, \nu_{f-a_2}^0) = N_{a_3}^1(r, \nu_{f-a_3}^0) + S(r, f) = N_{a_4}^1(r, \nu_{f-a_4}^0) + S(r, f) = S(r, f), \ i = 2, 3, 4.$$ 

This yields that

$$N_{a_i}^1(r, \nu_{f-a_i}^0) = S(r, f) \ (2 \leq i \leq 4).$$
Similarly, we also have
\[ N_{i,j}^{[1]}(r, \nu^{0}_{g-a_i}) \leq S(r, f) \quad (2 \leq i \leq 4). \]
We set \( f_1 = \frac{f-a_3}{f-a_1} \) and \( f_2 = \frac{f-a_2}{f-a_1} \). Then it is easy to see that
\[ N^{[1]}(r, \nu^{0}_{f_1}) + N^{[1]}(r, \nu^{0}_{f_2}) \leq N^{[1]}_{i,j}(r, \nu^{0}_{f-a_i}) + S(r, f). \]
This means the sets of multiple zeros of \( f - a_i \) and \( g - a_i \) are of counting functions equal to \( S(r, f) \). Therefore, for \( i = 2, 3, 4 \), we have
\[ \nu^{0}_{f-a_i}(z) = \nu^{0}_{g-a_i}(z) \in \{0, 1\} \]
for all \( z \) outside a discrete set of counting function equal to \( S(r, f) \). Hence, \( f \) and \( g \) share pairs \((a_i, b_i)\) weakly with counting multiplicities for \( i = 2, 3, 4 \). By Theorem A, we have that \( f \) is a quasi-Möbius transformation of \( g \). This is a contradiction.

Therefore, the supposition is untrue. Hence \( f \) is a quasi-Möbius transformation of \( g \). With the help of Lemma 3.2, we have the conclusion of the theorem. \(\square\)

**Proof of Theorem 1.3.** Suppose that \( f \) is not a quasi-Möbius transformation of \( g \). By Lemma 3.1, we have \( S(r, f) = S(r, g) \).

For each \( 1 \leq i \leq 4 \), we define a divisor \( \nu_i \) and \( \mu_i \) by setting
\[ \nu_i(z) = \max\{0, \min\{\nu_{f-a_i}(z), \nu_{g-a_i}(z)\} - 1\}, \]
\[ \mu_i(z) = \min\{1, |\nu_{f-a_i}(z) - \nu_{g-a_i}(z)|\}. \]
Take three indices \( i, j, t \in \{1, 2, 3, 4\} \). By Lemma 3.5 and by the assumptions of the theorem, we easily have the following
\[ 3N^{[1]}(r, \mu_i) \leq 3(N^{[1]}_{i,j}(r, \nu_{f-a_i}) + N^{[1]}_{i,j}(r, \nu_{g-a_i})) \]
\[ \leq N(r, \nu_i) + 3(N^{[1]}_{i,j}(r, \nu_{f-a_i}) + N^{[1]}_{i,j}(r, \nu_{g-a_i})) + S(r, f) \]
\[ \leq 3N^{[1]}_{i,j}(r, \nu_{f-a_i}) + N^{[1]}_{i,j}(r, \nu_{g-a_i}) + N(r, \mu_j) + N(r, \mu_i) + S(r, f). \]
Summing-up both sides of the above inequality over all subsets \( \{i, j, t\} \) of \( \{1, 2, 3, 4\} \), we obtain
\[ (3.6) \quad \sum_{i=1}^{4} N(r, \mu_i) \leq \sum_{i=1}^{4} 3(N^{[1]}_{i,j}(r, \nu_{f-a_i}) + N^{[1]}_{i,j}(r, \nu_{g-a_i})) + S(r, f). \]

We set:
- \( c_1 = \frac{a_3-a_4}{a_2-a_3} \), \( c_2 = \frac{a_4-a_3}{a_2-a_3} \), \( c_1' = \frac{b_3-b_4}{b_2-b_3} \), \( c_2' = \frac{b_3-b_4}{b_2-b_3} \),
- \( F_1 = c_1(f-a_1), F_2 = c_2(f-a_2), G_1 = c_1'(g-b_1), G_2 = c_2'(g-b_2), \)
- \( h_1 = \frac{f-a_1}{c_1}, h_2 = \frac{f-a_2}{c_2}, h_3 = \frac{f-a_3}{c_3}, h_4 = \frac{f-a_4}{c_4}, \)
- \( \alpha = \frac{c_1(a_2-a_4)}{c_2(a_4-a_2)} \), \( \beta = \frac{c_1'(b_2-b_4)}{c_2'(b_4-b_2)} \),
- \( h_4 = \frac{F_3(a_2-a_4)}{G_3(a_4-a_2)} = \frac{(a_3-a_2)(b_4-b_2)}{(a_4-a_2)(b_4-b_2)} \).

We define \( h_4 = \frac{F_3(a_2-a_4)}{G_3(a_4-a_2)} = \frac{(a_3-a_2)(b_4-b_2)}{(a_4-a_2)(b_4-b_2)} \).
It is easy to see that $c_1 \neq c_2$, $c_1' \neq c_2'$, $\alpha \neq 1$, $\beta \neq 1$ and all $c_i, c_i'$ ($1 \leq i \leq 2$) are small with respect to $f$ and

\[
(3.7) \quad N^{[i]}(r, \nu^0_{c_i}) + N^{[i]}(r, \nu^\infty_{c_i}) = N^{[i]}(r, \mu_i) + S(r, f) \quad (1 \leq i \leq 4).
\]

From the definition of functions $F_i, G_i$ ($1 \leq i \leq 2$), we have the following equations system:

\[
\begin{align*}
F_1 - h_1G_1 &= 0, \\
F_2 - h_2G_2 &= 0, \\
F_1 - F_2 - h_3G_1 + h_3G_2 &= 0, \\
F_1 - \alpha F_2 - h_4G_1 + h_4\beta G_2 &= 0.
\end{align*}
\]

This implies that

\[
\det \begin{pmatrix} 1 & 0 & -h_1 & 0 \\ 0 & 1 & 0 & -h_2 \\ 1 & -1 & -h_3 & h_3 \\ 1 & -\alpha & -h_4 & h_4\beta \end{pmatrix} = 0.
\]

Then

\[
(3.8) \quad (1 - \alpha)h_1h_2 - h_1h_3 + \beta h_1h_4 + \alpha h_2h_3 - h_2h_4 + (1 - \beta)h_3h_4 = 0.
\]

Denote by $\mathcal{I}$ the set of all subsets $I = \{i, j\}$ of the set $\{1, 2, 3, 4\}$. For $I \in \mathcal{I}$, we define the function $h_I$ as follows:

\[
h_{\{1, 2\}} = (1 - \alpha)h_1h_2, \quad h_{\{1, 3\}} = -h_1h_3, \quad h_{\{1, 4\}} = \beta h_1h_4, \\
h_{\{2, 3\}} = \alpha h_2h_3, \quad h_{\{2, 4\}} = -h_2h_4, \quad h_{\{3, 4\}} = (1 - \beta)h_3h_4.
\]

Then we have

\[
\sum_{I \in \mathcal{I}} h_I = 0.
\]

Take a meromorphic function $d$ on $\mathbb{C}$ such that $dh_I$ ($I \in \mathcal{I}$) are all holomorphic functions on $\mathbb{C}$ without common zero. Then it is easy to see that

\[
\sum_{I \in \mathcal{I}} N^{[i]}(r, dh_I) \leq 3 \sum_{i=1}^{4} (N^{[i]}(r, \nu^0_{c_i}) + N^{[i]}(r, \nu^\infty_{c_i})) + S(r, f)
\]

\[
= 3 \sum_{i=1}^{4} N^{[i]}(r, \mu_i) + S(r, f).
\]

Take $I_0 \in \mathcal{I}$. Then

\[
dh_{I_0} = - \sum_{I \neq I_0} dh_I.
\]

Denote by $t$ the minimum number satisfying the following: There exist $t$ elements $I_1, \ldots, I_t \in \mathcal{I}$ and $t$ nonzero constants $b_v \in \mathbb{C}$ ($1 \leq v \leq t$) such that $dh_{I_0} = \sum_{v=1}^{t} b_v dh_{I_v}$.

By the minimality of $t$, then the family $\{dh_{I_1}, \ldots, dh_{I_t}\}$ is linearly independent over $\mathbb{C}$. 
Case 1. \( t = 1 \). Then \( h_{I_0} \in \mathbb{C} \setminus \{0\} \).

Case 2. \( t \geq 2 \). Consider the linearly non-degenerate holomorphic mapping \( h : \mathbb{C} \to \mathbb{P}^{t-1}(\mathbb{C}) \) with the representation \( h = (dh_{I_1} : \cdots : dh_{I_t}) \). Applying Theorem 2.5, we have

\[
T_h(r) \leq \sum_{i=1}^{t} N^{[t-1]}_{dh_{I_i}}(r) + N^{[t-1]}_{dh_{I_0}}(r) + S(r, f) \\
\leq (t - 1) \sum_{i=1}^{t} N^{[t]}_{dh_{I_i}}(r) + (t - 1) N^{[t]}_{dh_{I_0}}(r) + S(r, f)
\]

(3.9)

We define the following rational functions:

\[
H_1(X) = \frac{c_1(X - a_1)}{c_1'(X - b_1)}, \quad H_2(X) = \frac{c_2(X - a_2)}{c_2'(X - b_2)}, \\
H_3(X) = \frac{b_2 - b_1}{a_3 - a_1} \cdot \frac{X - a_3}{X - b_3}, \\
H_4(X) = \frac{(a_3 - a_2)(b_4 - b_2)}{(a'_3 - a'_2)(b'_4 - b'_2)} \cdot \frac{X - a_4}{X - b_4}.
\]

For each \( I \subset \{1, \ldots, 4\} \), put \( I^c = \{1, \ldots, 4\} \setminus I \). For \( 0 \leq u, v \leq t \), \( u \neq v \) and \( i \in ((I_v \cup I_u) \setminus (I_u \cap I_v))^c \), we see that

\[
T\left(r, \frac{h_{I_u}}{h_{I_v}}\right) = T\left(r, \prod_{j \in I_u} h_{I_j} \right) + S(r, f) \\
\geq N\left(r, \nu_0^{\prod_{j \in I_u \setminus I_v} h_{I_j}} \right) + S(r, f) \\
\geq N^{[1]}_{\leq k}(r, \nu^0_{f-a_i}) + S(r, f).
\]

Similarly, we have

\[
T\left(r, \frac{h_{I_v}}{h_{I_u}}\right) \geq N^{[1]}_{\leq k}(r, \nu^0_{g-b_i}) + S(r, f).
\]

Therefore

\[
T\left(r, \frac{h_{I_u}}{h_{I_v}}\right) \geq \frac{1}{2} \left( N^{[1]}_{\leq k}(r, \nu^0_{f-a_i}) + N^{[1]}_{\leq k}(r, \nu^0_{g-b_i}) \right) + S(r, f).
\]
Since \((I_0 \cup I_1 \setminus (I_0 \cap I_1))^c \cup (I_1 \cup I_2 \setminus (I_1 \cap I_2))^c \cup (I_2 \cup I_0 \setminus (I_2 \cap I_0))^c = \{1, \ldots, 4\}\), we have

\[
3T(r, h) \geq T\left(r, \frac{h_{I_0}}{h_{I_1}}\right) + T\left(r, \frac{h_{I_1}}{h_{I_2}}\right) + T\left(r, \frac{h_{I_2}}{h_{I_0}}\right) \\
\geq \frac{1}{2} \left(N_{\leq k}^{[i]}(r, \nu_{f-a_i}^0) + N_{\leq k}^{[i]}(r, \nu_{g-b_i}^0)\right) + S(r, f) \quad (1 \leq i \leq 4).
\]

Thus we have

\[
\sum_{i=1}^{4} \left( N_{\leq k}^{[i]}(r, \nu_{f-a_i}^0) + N_{\leq k}^{[i]}(r, \nu_{g-b_i}^0)\right) \leq 24T(r, h) + S(r, f) \\
\leq 288 \sum_{i=1}^{4} N(r, \mu_i) + S(r, f) \\
\leq 864 \sum_{i=1}^{4} \left( N_{> k}^{[i]}(r, \nu_{f-a_i}^0) + N_{> k}^{[i]}(r, \nu_{g-b_i}^0)\right) + S(r, f).
\]

By Yamanoi’s second main theorem (Theorem 2.1), for every \(\epsilon > 0\) we have

\[
(2 - \epsilon)T(r) \leq \sum_{i=1}^{4} \sum_{u=f-a_i, g-b_i} N^{[i]}(r, \nu_u^0) + S(r, f) \\
= \sum_{i=1}^{4} \sum_{u=f-a_i, g-b_i} \left( N_{\leq k}^{[i]}(r, \nu_u^0) + N_{\geq k}^{[i]}(r, \nu_u^0)\right) + S(r, f) \\
\leq \sum_{i=1}^{4} \sum_{u=f-a_i, g-b_i} \left( \frac{865}{k + 865} N_{\leq k}^{[i]}(r, \nu_u^0) \right) \\
+ \left( \frac{k}{k+865} \right) \left( N_{> k}^{[i]}(r, \nu_u^0) \right) + S(r, f) \quad \text{(by the above inequality)} \\
\leq \sum_{i=1}^{4} \sum_{u=f-a_i, g-b_i} \left( \frac{865}{k + 865} \right) \left( N_{\leq k}^{[i]}(r, \nu_u^0) + N_{> k}^{[i]}(r, \nu_u^0)\right) + S(r, f) \\
\leq \sum_{i=1}^{4} \sum_{u=f-a_i, g-b_i} \left( \frac{865}{k + 865} \right) N(r, \nu_u^0) + S(r, f) \\
\leq 4 \cdot \frac{865}{k + 865} T(r) + S(r, f).
\]

Letting \(r \to +\infty\), we get

\[
2 - \epsilon \leq \frac{4 \cdot 865}{k + 865}.
\]
Since the above inequality holds for every \( \epsilon > 0 \), letting \( \epsilon \to 0 \) we get

\[
2 \leq \frac{4 \cdot 865}{k + 865}, \quad \text{i.e., } k \leq 865.
\]

This is a contradiction.

Then from Case 1 and Case 2, it follows that for each \( I \in \mathcal{I} \), there is \( J \in \mathcal{I} \setminus \{I\} \) such that \( \frac{h_I}{h_J} \in \mathbb{C} \setminus \{0\} \). We consider the following two cases:

**Case a.** There exist \( I = \{i, j\} \), \( J = \{i, l\} \), \( j \neq l \), \( \frac{h_I}{h_J} = \text{constant} \). Then

\[
h_j = ah_l \quad \text{with} \quad a \text{ is a nonzero meromorphic function in } \mathcal{R}_f.
\]

Therefore, \( f \) is a quasi-M"obius transformation of \( g \). This contradicts the supposition that \( f \) is not a quasi-M"obius transformation of \( g \).

**Case b.** There exist nonzero constants \( b, c \) such that \( h_{\{1, 2\}} = bh_{\{3, 4\}} \) and \( h_{\{1, 3\}} = ch_{\{2, 4\}} \), i.e.,

\[
(1 - \alpha)h_1h_2 = b(1 - \beta)h_3h_4 \quad \text{and} \quad h_1h_3 = ch_2h_4.
\]

Then

\[
\left(\frac{h_1}{h_4}\right)^2 = \frac{bc(1-\beta)}{1-\alpha} \in \mathcal{R}_f.
\]

This implies that \( \frac{h_1}{h_4} \in \mathcal{R}_f \). Hence \( f \) is a quasi-M"obius transformation of \( g \). This is a contradiction.

From the above two cases, we get the contradiction to the supposition. Hence \( f \) is a quasi-M"obius transformation of \( g \). With the help of Lemma 3.2, we have the desired conclusion of the theorem. \( \square \)

**References**


TWO MEROMORPHIC FUNCTIONS

Van An Nguyen
Division of Mathematics
Banking Academy
12-Chua Boc, Dong Da, Hanoi, Vietnam
E-mail address: an0883@gmail.com

Duc Quang Si
Department of Mathematics
Hanoi National University of Education
136-Xuan Thuy, Cau Giay, Hanoi, Vietnam
and
Thang Long Institute of Mathematics and Applied Sciences
Ngheem Xuan Yem, Hoang Mai, Hanoi, Vietnam
E-mail address: quangsd@hnue.edu.vn