REDDUCING SUBSPACES FOR A CLASS OF TOEPLITZ OPERATORS ON WEIGHTED HARDY SPACES OVER BIDISK

Shuhei Kuwahara

Abstract. We consider weighted Hardy spaces on bidisk $D^2$ which generalize the weighted Bergman spaces $A^2_{\alpha}(D^2)$. Let $z, w$ be coordinate functions and $T_{\overline{z^Nw}}$ Toeplitz operator with symbol $\overline{z^Nw}$. In this paper, we study the reducing subspaces of $T_{\overline{z^Nw}}$ on the weighted Hardy spaces.

1. Introduction

Let $X$ be a closed subspace in a Hilbert space. Then $X$ is an invariant subspace of an operator $A$ if $AX \subset X$. In addition, $X$ is a reducing subspace of an operator $A$ if $X$ is an invariant subspace of both $A$ and its adjoint $A^*$. The reducing subspace $X$ is called minimal if $\{0\}$ and $X$ are the only reducing subspaces contained in $X$.

Many mathematicians study the reducing subspaces of operators on Hilbert spaces. For instance, M. Albaseer, Y. Lu and Y. Shi [1] determined the reducing subspaces of Toeplitz operator $T_{\overline{z^N\overline{M}^1}}$ on the Bergman space $A^2(D^2)$, where $N$ and $M$ are positive integers. In this paper, we will study the reducing subspaces of the operator $T_{\overline{z^Nw}}$ on weighted Hardy spaces over bidisk. The weighted Hardy spaces over bidisk is a generalization of the weighted Bergman space $A^2_{\alpha}(D^2)$. The definitions and notations in this paper are as follows.

Let $D$ be the unit disk in the complex plane $\mathbb{C}$. For $j = 1, 2$, let $d\mu_j = d\sigma_j(r)d\theta_j/2\pi$ be the probably measures on the unit disk $\mathbb{D}$. We consider weighted Hardy space $H^2(D^2, d\mu)$ which is the closure of all analytic polynomials in $L^2(D^2, d\mu)$, where $d\mu(z, w) = d\mu_1(z)d\mu_2(w)$. Here $L^2(D^2, d\mu)$ is the Hilbert space of square integrable functions on $D^2$ with the inner product

$$\langle f, g \rangle = \int_{D^2} f(z, w) \overline{g(z, w)} d\mu_1(z)d\mu_2(w).$$
If \( d\mu_1(z) = d\mu_2(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z) \) where \( \alpha > -1 \) and \( dA \) is the normalized area measure on \( \mathbb{D} \), then the weighted Hardy space \( H^2(\mathbb{D}^2, d\mu_1 d\mu_2) \) is the weighted Bergman space \( A^2_\alpha(\mathbb{D}^2) \) over bidisk \( \mathbb{D}^2 \).

Next we will introduce notions of weight sequences. Put

\[
\omega_1(n) = \int_{\mathbb{D}} |z|^{2n} d\mu_1(z), \quad \omega_2(n) = \int_{\mathbb{D}} |w|^{2n} d\mu_2(w).
\]

Throughout this paper, we assume that

\[
\sup_n \frac{\omega_1(n+1)}{\omega_1(n)} < \infty \quad \text{and} \quad \sup_n \frac{\omega_2(n+1)}{\omega_2(n)} < \infty
\]

so that the multiplication operators defined by \( z \) and \( w \) are bounded. Let \( P \) be the orthogonal projection from \( L^2(\mathbb{D}^2, d\mu) \) onto \( H^2(\mathbb{D}^2, d\mu) \). For \( \varphi \in L^\infty \), put

\[
T\varphi f = P(\varphi f) \quad (f \in H^2(\mathbb{D}^2, d\mu))
\]

and then \( T\varphi \) is called a Toeplitz operator. By calculation we have the following lemma.

**Lemma 1.1.** Let \( N_1 \) and \( N_2 \) be natural numbers. The following equalities hold:

\[
T_{z^{N_1} w^{N_2}}(z^k w^l) = \begin{cases} 
\frac{\omega_1(k)}{\omega_1(k-N_1)} z^{-N_1} w^{k-N_1} & (k \geq N_1) \\
0 & (k < N_1)
\end{cases}
\]

and

\[
T_{z^{N_1} w^{N_2}}(z^k w^l) = \begin{cases} 
\frac{\omega_2(l)}{\omega_2(l-N_2)} z^{l-N_2} w^{l-N_2} & (l \geq N_2) \\
0 & (l < N_2)
\end{cases}
\]

for nonnegative integers \( k, l \).

From Lemma 1.1, we show an example of the reducing subspaces of \( T_{z^{N} w^{N}} \).

**Proposition 1.2.** Let \( I_0 = \{(n, 0); 0 \leq n < N\} \) as a subset of multi-indices. A subspace

\[
X_0 = \text{Span} \{ z^n; (n, 0) \in I_0 \}
\]

is contained in the kernel of \( T_{z^{N} w^{N}} \) and \( T_{z^{N} w^{N}} \). Moreover \( X_0 \) is the reducing subspace of \( T_{z^{N} w^{N}} \), where we denote by \( \text{Span} X \) the closed linear span of a subset \( X \) in \( H^2(\mathbb{D}^2, d\mu) \).

In this paper, we study the reducing subspaces contained in \( X_0^\perp \). Fix a natural number \( N \). Our main theorems are as follows. For the definition of transparent polynomials, see Section 2.

**Theorem 1.3.** Let \( X \subset X_0^\perp \) be a reducing subspace of \( T_{z^{N} w^{N}} \) on \( H^2(\mathbb{D}^2, d\mu) \). Then the reducing subspace \( X \) contains the minimal reducing subspace \( X_p \) where \( p \) is a transparent polynomial. Moreover \( X \) is the minimal reducing subspace of \( T_{z^{N} w^{N}} \) if and only if there exists a transparent polynomial \( p \) such that \( X = X_p \).
In Section 2, we will prepare for considering the reducing subspaces of the operator $T_{z^Nw}$ contained in $X_0^\perp$. We note that the statements in Section 2 is valid even if $N = 1$. In Section 3, we will state our main theorem and study examples of the reducing subspaces on concrete function spaces. In this paper, we use the technique in [4, 5, 7] with the similar way.

2. Preliminaries

Let $I$ be a subset of multi-indices such that $I = \{(n,0); n \geq N\}$. We put the order on the set $I$ induced by the set of nonnegative integers; $(m,0) < (n,0)$ if $m < n$.

We say that $(m,0) \in I$ and $(n,0) \in I$ are equivalent if

$$\frac{\omega_1(m)}{\omega_1(m - lN)} = \frac{\omega_1(n)}{\omega_1(n - lN)}$$

for all $l$ satisfying $0 < lN \leq m, 0 < lN \leq n$. In this case, we write $(m,0) \sim (n,0)$.

For a natural number $k$, let $I_k$ be a subset of $I$ such that

$I_k = \{(n,0); kN \leq n \leq (k + 1)N - 1\}$.

If a polynomial $p(z)$ is in a form of

$$p(z) = \sum_{(n,0) \in I_k} b_n z^n,$$

then we say that $p$ is a transparent polynomial when we have $\alpha \sim \beta$ for any two nonzero coefficient $b_\alpha, b_\beta$ of $p$.

We partition the set $I_k$ into equivalent classes and sort them in the order of the minimal multi-index. We denote the sorted equivalent classes by $\Omega_1, \Omega_2, \ldots, \Omega_{\tilde{N}}$ with $\tilde{N} \leq N$.

For a function $p(z) = \sum_{(n,0) \in I_k} a_n z^n$, the decomposition

$$p(z) = \sum_m p_m(z)$$

is called the canonical decomposition of $p$, where

$$p_m(z) = \sum_{(n,0) \in \Omega_m} a_n z^n.$$

We note that each $g_i$ is orthogonal to $g_j$ if $i \neq j$.

Let $S$ be the vector space consisting of all finite linear combinations of finite products of the operators $T_{z^Nw}$ and its adjoint $T_{z^Nw}^\ast$. For any $f \in H^2(\mathbb{D}^2, d\mu)$, we put $Sf = \{Tf; T \in S\}$. Denote the closure of $Sf$ in $H^2(\mathbb{D}^2, d\mu)$ by $X_f$ which we call the reducing subspace generated by $f$. It is easy to see that $X_f$ is the smallest reducing subspace containing $f$. Lemma 2.1 shows the relation between reducing subspaces and transparent polynomials.
Lemma 2.1. If 
\[ p(z, w) = \sum_{(n, 0) \in I_k} b_n z^n \]
is a transparent polynomial, then 
\[ X_p = \text{Span}\{ \sum_{(n, 0) \in I_k} b_n z^{n-lN} w^l; 0 \leq l \leq k \}. \]

Proof. Let 
\[ X = \text{Span}\{ \sum_{(n, 0) \in I_k} b_n z^{n-lN} w^l; 0 \leq l \leq k \}. \]

It is obvious that \( p \in X \subset X_p \). From the definition of \( X_p \), it is enough to show that \( X \) is a reducing subspace of \( T^*_w \). Let \((M, 0)\) be the minimal multi-index of nonzero coefficients of \( p \). For \( 0 \leq l < k \), we compute 
\[ T^*_w \sum_{(n, 0) \in I_k} b_n z^{n-lN} w^l = \sum_{(n, 0) \in I_k} b_n \frac{\omega_1(M-lN)}{\omega_1(M-(l+1)N)} z^{n-(l+1)N} w^{l+1} \]
\[ = \sum_{(n, 0) \in I_k} b_n \frac{\omega_1(M-lN)}{\omega_1(M-(l+1)N)} z^{n-(l+1)N} w^{l+1} \]
\[ = \frac{\omega_1(M-lN)}{\omega_1(M-(l+1)N)} \sum_{(n, 0) \in I_k} b_n z^{n-(l+1)N} w^{l+1} \in X. \]
If \( l = k \), then it is clear that 
\[ T^*_w \sum_{(n, 0) \in I_k} b_n z^{n-lN} w^l = 0. \]

Moreover, for \( l > 0 \), we obtain 
\[ T^*_w \sum_{(n, 0) \in I_k} b_n z^{n-lN} w^l = \sum_{(n, 0) \in I_k} b_n \frac{\omega_2(l)}{\omega_2(l-1)} z^{n-(l-1)N} w^{l-1} \]
\[ = \frac{\omega_2(l)}{\omega_2(l-1)} \sum_{(n, 0) \in I_k} b_n z^{n-(l-1)N} w^{l-1} \in X. \]
If \( l = 0 \), then it is easy to see that 
\[ T^*_w \sum_{(n, 0) \in I_k} b_n z^n = 0. \]

By these computation, we can show that \( X \) is a reducing subspace of \( T^*_w \). \( \Box \)

Proposition 2.2. If \( p \) is a transparent polynomial, then \( X_p \) is the minimal reducing subspace.

Proof. It is clear from Lemma 2.1. \( \Box \)
For $f \in \text{Hol}(\mathbb{D}^2)$, we denote $f^{(k)}(0,0) = \frac{d^k}{dz^k}f(0,0)$. For any subspace $X$ with $X \neq \{0\}$, let $(M,0)$ be minimal multi-index such that there exists some $f \in X$ with $f^{(M)}(0,0) \neq 0$ but $g^{(k)}(0,0) = 0$ for all $g \in X$ and $(k,0) < (M,0)$. We call $(M,0)$ the order of $X$ at the origin.

**Proposition 2.3.** Let $X \subset X^\perp_0$ be a nonzero reducing subspace of $T_{\mathbb{Z}^2}$ and $(M,0)$ the order of $X$ at the origin. Then $X$ has a transparent polynomial containing the term $z^M$.

**Proof.** Throughout the proof of Proposition 2.3, we denote $T = T_{\mathbb{Z}^2}$.

If $f$ is a function in $X$ with Taylor expansion
\[
f(z,w) = \sum_{(n_1,n_2)} a(n_1,n_2)z^{n_1}w^{n_2},
\]
then the mapping from $f$ to $f^{(M)}(0,0)$ is a bounded linear functional on $H^2(\mathbb{D}^2, d\mu)$. By Riesz representation theorem, the extremal problem
\[
\sup \{ \text{Re}f^{(M)}(0,0); f \in X, ||f|| \leq 1 \}
\]
has a unique solution $G$ with $\|G\| = 1$ and $G^{(M)}(0,0) > 0$. At first we prove $T^*G = 0$. Put $g_k = \frac{G+Tf}{\|G+Tf\|}$ for $f \in X$. Since $\text{Re}g^{(M)}(0,0) \leq G^{(M)}(0,0)$, it is easy to see that $\|G+Tf\| \geq 1$ for all $f \in X$. From this inequality we obtain $G \perp Tf$. Since $T^*G \in X$, we obtain $T^*G = 0$. Let $k'$ be a natural number such that $(M,0) \in I_{k'}$. The same argument shows that $T^{k'+1}G = 0$. Therefore the function $G$ is in the form of $G(z) = \sum_{(n,0) \in I_{k'}} b_n z^n$.

Let $G(z) = \sum_{i=1}^N g_i(z)$ be the canonical decomposition of $G$ with $g_i \neq 0$. It is trivial that $g_1$ contains the term $z^M$. Put $(M^{(i)},0)$ the minimal multi-index of $g_i$. We note that if $i < j$, then $(M^{(i)},0)$ and $(M^{(j)},0)$ are not equivalent, and $(M,0) \leq (M^{(i)},0) < (M^{(j)},0)$.

Now we will show that $g_1$ is in $X$. Put $g_j(z) = \sum_{n \geq M^{(j)}} b_n z^n$. Here we see that for $k \leq \frac{M^{(j)}}{N}$,
\[
\begin{align*}
(T^*)^k T^k g_j \\
= (T^*)^k \sum_n \frac{\omega_1(n-(k-1)N) \omega_1(n-(k-2)N) \cdots \omega_1(n) \omega_1(n-kN) b_n z^{n-kN} w^k}{\omega_1(n-kN) \omega_1(n-(k-1)N) \cdots \omega_1(n-kN)} \\
= (T^*)^k \sum_n \frac{\omega_1(n \omega_1(M^{(j)}-kN) b_n z^{n-kN} w^k}{\omega_1(M^{(j)}-kN) b_n z^{n-kN} w^k} \\
= \frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)}-kN)} (T^*_{\mathbb{Z}^2})^k \sum_n b_n z^{n-kN} w^k
\end{align*}
\]
using the definition of $g_j$ and Lemma 1.1. Therefore

$$
(1) \quad \left( \frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)} - k_jN)} \cdot \frac{\omega_2(k)}{\omega_2(0)} - (T^*)^{k_j} T^k \right) g_j = 0.
$$

For each natural number $j = 2, 3, \ldots, \bar{N}$, we choose an integer $k_j$ such that

$$
\frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)} - k_jN)} \neq \frac{\omega_1(M)}{\omega_1(M - k_jN)}
$$

and put

$$
C_j = \frac{\omega_2(k_j)}{\omega_2(0)} \left( \frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)} - k_jN)} - \frac{\omega_1(M)}{\omega_1(M - k_jN)} \right).
$$

We will generate the sequence of functions in $X$ inductively as follows;

$$
G_2 = \left( \frac{\omega_1(M^{(2)})}{\omega_1(M^{(2)} - k_2N)} \cdot \frac{\omega_2(k_2)}{\omega_2(0)} - (T^*)^{k_2} T^k \right) \frac{1}{C_2} G
$$

and

$$
G_j = \left( \frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)} - k_jN)} \cdot \frac{\omega_2(k_j)}{\omega_2(0)} - (T^*)^{k_j} T^k \right) \frac{1}{C_j} G_{j-1}.
$$

For example, we have

$$
G_2 = g_1 + \frac{1}{C_2} \cdot \frac{\omega_2(k_2)}{\omega_2(0)} \sum_{i=3}^{\bar{N}} \left( \frac{\omega_1(M^{(2)})}{\omega_1(M^{(2)} - k_2N)} - \frac{\omega_1(M^{(i)})}{\omega_1(M^{(i)} - k_2N)} \right) g_i.
$$

We note that the function $g_2$ vanishes but the function $g_1$ never vanishes from the equality (1) in this calculation.

More generally, let

$$
A(j, i) = \prod_{i=2}^{j-1} \left( \frac{\omega_1(M^{(i)})}{\omega_1(M^{(i)} - k_iN)} - \frac{\omega_1(M^{(i)})}{\omega_1(M^{(i)} - k_iN)} \right),
$$

and we obtain

$$
G_{j-1} = g_1 + \sum_{i=j}^{\bar{N}} A(j, i) g_i
$$

for $3 \leq j \leq \bar{N}$ and $G_{\bar{N}} = g_1$ which is in $X$. It is obvious that $g_1$ contains the term $z^M$ and is transparent. □
3. Main results

Now we state our main result.

**Theorem 3.1.** Let \( X \subset X_0 \) be a reducing subspace of \( T_{z^w} \) on \( H^2(D^2, d\mu) \). Then the reducing subspace \( X \) contains the minimal reducing subspace \( X_p \) where \( p \) is a transparent polynomial. Moreover \( X \) is the minimal reducing subspace of \( T_{z^w} \) if and only if there exists a transparent polynomial \( p \) such that \( X = X_p \).

**Proof.** Let \( X \) be a reducing subspace of \( T_{z^w} \). From Proposition 2.3, there exists a transparent polynomial \( p \). By Lemma 2.1, \( X_p \) is the smallest reducing subspace containing \( p \) and therefore \( X_p \subset X \). In addition, if \( X \) is minimal, then it is clear that \( X = X_p \). The converse is true from Proposition 2.2. \( \square \)

In the rest of this paper, we will show some examples. First we consider the case of the weighted Bergman space \( A_2^\alpha(D^2) \), where

\[
\omega_1(n) = \omega_2(n) = \frac{n!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n)}
\]

for \( \alpha > -1 \).

**Corollary 3.2.** Let \( X \subset X_0 \) be a reducing subspace of \( T_{z^w} \) on \( A_2^\alpha(D^2) \). The reducing subspace \( X \) is minimal if and only if \( X \) is in the form of

\[
X_n = \text{Span}\{z^n - lNw^l; 0 \leq l \leq \frac{n}{N}\}
\]

for any natural number \( n \geq N \).

**Proof.** It is enough to show that any pair of two distinct multi-index \( k \) in \( I \) is not equivalent. Assume

\[
\frac{\omega_1(m)}{\omega_1(m-k)} = \frac{\omega_1(n)}{\omega_1(n-k)}
\]

for \( 0 < k \leq m \) and \( 0 < k \leq n \). This equality implies

\[
\frac{m!\Gamma(2+\alpha+m-k)}{(m-k)!\Gamma(2+\alpha+m)} = \frac{n!\Gamma(2+\alpha+n-k)}{(n-k)!\Gamma(2+\alpha+n)}.
\]

In particular, when \( k = 1 \), we obtain the equality

\[
\frac{m}{1+\alpha+m} = \frac{n}{1+\alpha+n},
\]

which implies \( m = n \). Thus we conclude that any pair of two distinct multi-index in \( I \) is not equivalent. Therefore every transparent polynomial is a monomial. \( \square \)

Next we consider the case of the abstract Hardy space. Let \( C(\mathbb{D}^2) \) be the algebra of complex-valued continuous functions on \( \mathbb{D}^2 \) and \( A \) a uniform algebra on \( \mathbb{D}^2 \) containing \( |z| \). A probability measure \( m \) on \( \mathbb{D}^2 \) denotes a representing measure for some complex homomorphism. The abstract Hardy space \( H^2 = \)
$H^2(m)$ determined by $A$ is defined to be closure of $A$ in $L^2 = L^2(m)$. In this case, $\int fgdm = \int fdm \int gdm$ holds true for $f, g \in H^2$.

**Lemma 3.3.** Any pair of two multi-indices in $I$ is equivalent if $H^2(\mathbb{D}^2, d\mu)$ is a closed subspace of the abstract Hardy space $H^2(m)$.

**Proof.** Put $r = \int_{\mathbb{D}^2} |z|^2 dm$. Using the equality $\int fgdm = \int fdm \int gdm$, we have

$$\omega_1(n) = \int_{\mathbb{D}^2} |z|^{2n} dm = \left(\int_{\mathbb{D}^2} |z|^2 dm\right)^n = r^n.$$

Therefore for all $m, n$,

$$\frac{\omega_1(m)}{\omega_1(m - kN)} = \frac{r^m}{r^{m-kN}} = r^kN = \frac{r^n}{r^{n-kN}} = \frac{\omega_1(n)}{\omega_1(n - kN)}.$$  

\[\Box\]

It is obvious to see that Corollary 3.4 follows from Lemma 3.3.

**Corollary 3.4.** Every reducing subspace $X \subset X_0^\perp$ of $T_{z^N w}$ in the weighted Hardy space which is a closed subspace of $H^2(m)$ is minimal.

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**References**


Shuhei Kuwahara
Sapporo Seishu High School
Sapporo 064-0916, Japan
E-mail address: s.kuwahara@sapporoseishu.ed.jp